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ON HOMOTOPY NILPOTENCY

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Abstract. We review established and recent results on the homotopy nilpotence of spaces. In particular, the homotopy nilpotency of the loop spaces $\Omega(G/K)$ of homogenous spaces G/K for a compact Lie group G and its closed homotopy nilpotent subgroup $K < G$ is discussed.

Introduction

In group theory, if we consider only nilpotent groups, the nilpotency class is the one which measures a distance from commutativity. Already G. Whitehead ([35]) had the insight that the (J.H.C.) Whitehead products satisfy identities which reflect commutator identities for groups. Berstein and Ganea ([4]) adapted the nilpotency to H-spaces as follows. Let X be an H-space, $\varphi_{X,1} = \mathrm{id}_X$ and $\varphi_{X,2} : X^2 \to X$ the commutator map. Put $\varphi_{X,n+1} = \varphi_{X,1} \circ (\mathrm{id}_X \times \varphi_{X,n})$ for $(n+1)$-fold commutator map of X with $n \geq 2$. An H-space X is called homotopy nilpotent of class n if $\varphi_{X,n+1} \simeq *$, is null homotopic but $\varphi_{X,n}$ is not ([4]). In this case, we write $\operatorname{nil} X = n$.

Then, Berstein and Ganea ([4]) introduced a concept of the homotopy nilpotency of a pointed space by means of its loop space (resp. suspension space). Its fibrewise version has been studied by James ([22]). In particular, the m-iterated Samelson products vanish in the loop space $\Omega(X)$, or equivalently, the m-iterated Whitehead products vanish in X provided $m > \operatorname{nil}\Omega(X)$.

The homotopy nilpotency classes $\operatorname{nil} X$ of associative H-spaces X has been extensively studied as well as their homotopy commutativity. Work

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of Hopkins ([19]) drew renewed attention to such problems by relating this classical nilpotency notion with the nilpotence theorem of Devinatz, Hopkins, and Smith ([9]). In particular, Hopkins ([19]) made substantial progress by giving cohomological criteria for homotopy associative finite H-spaces to be homotopy nilpotent. For example, he showed that if a homotopy associative finite H-space has no torsion in the integral homology, then it is homotopy nilpotent. Later, Rao ([29]) showed that the converse of the above criterion is true in the case of groups $\mathrm{Spin}(n)$ and $SO(n)$. Eventually, Yagita ([37]) proved that, when G is a compact, simply connected Lie group, its p-localization $G_{(p)}$ is homotopy nilpotent if and only if G has no torsion in the integral homology. Finally, Rao ([30]) showed that a connected compact Lie group is homotopy nilpotent if and only if it has no torsion in homology.

Crabb, Sutherland and Zhang in [8] got surprisingly low bounds for the homotopy nilpotency class of gauge groups when the bundles are stages of the Milnor construction of the classifying space for a compact Lie group G, even when G itself is not homotopy nilpotent. Furthermore, they have proved an equivariant version of Hopkin's result for unitary groups. Although many results on the homotopy nilpotency are obtained as above and others, e.g., [25] and [33], the homotopy nilpotency classes have not been determined in almost all cases.

Before describing our purpose, it is worthwhile to mention a few recent contributions along these lines. Berger and Bourn ([3]) study nilpotency in the context of exact Mal'tsev categories taking central extensions as the primitive notion. This yields a nilpotency tower which is analysed from the perspective of Goodwillie's functor calculus. They show in particular that the reflection into the subcategory of n-nilpotent objects is the universal endofunctor of degree n if and only if every n-nilpotent object is n-folded. In the special context of a semi-Abelian category, an object is n-folded precisely when its Higgins commutator of length $(n+1)$ vanishes.

Biedermann and Dwyer ([5]) study the connection between the Goodwillie tower of the identity and the lower central series of the loop group on connected spaces. They define homotopy nilpotent groups as homotopy algebras over certain simplicial algebraic theories. This notion interpolates between infinite loop spaces and loop spaces, but backwards. Then, the relation to ordinary nilpotent groups is studied. It is proved that n-excisive functors of the form $\Omega(F)$ factor over the category of homotopy n-nilpotent groups.

Kaji and Kishimoto ([24]) consider the problem: how far from being homotopy commutative is a loop space having the homotopy type of the p-completion of a product of finite numbers of spheres? They determine the homotopy nilpotency of those loop spaces as an answer to this problem.

The paper is a survey of known results, some being older and some recent. In Section 1, we set stages for developments to come. This introductory section is devoted to a general discussion and establishes notations on the homotopy nilpotency of H-spaces used in the rest of the paper.

Section 2 makes use of [4] to review established and recent results on the homotopy nilpotence of spaces.

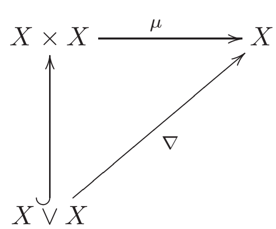
Section 3, based mainly on [15], takes up the systematic study of the homotopy nilpotency of homogeneous spaces G/K for a Lie group G and its closed subgroup $K < G$. The homotopy nilpotency of the loop spaces $\Omega(G_{n,m}(\mathbb{K}))$, $\Omega(F_{n;n_1,\ldots,n_k}(\mathbb{K}))$, and $\Omega(V_{n,m}(\mathbb{K}))$ of Grassmann $G_{n,m}(\mathbb{K})$, flag $F_{n;n_1,\ldots,n_k}(\mathbb{K})$ and Stiefel $V_{n,m}(\mathbb{K})$ manifolds for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions is studied. In particular, the homotopy nilpotency $\operatorname{nil}\Omega(\mathbb{H}P^n) < \infty$ is shown for the quaternionic projective space $\mathbb{H}P^n$ with $n \geq 1$ which does not appear in the literature known to the author.

1. Prerequisites

All spaces and maps in this note are assumed to be connected and based with the homotopy type of CW-complexes unless we assume otherwise. We also do not distinguish notationally between a continuous map and its homotopy class. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the loop (resp. suspension) space on a space X and $[X, Y]$ for the set of homotopy classes of maps $X \to Y$.

Given a space X, we use the customary notations $X \vee X$ and $X \wedge X$ for the *wedge* and the *smash square* of X, respectively.

Recall that an *H-space* is a pair (X, μ), where X is a space and $\mu : X \times X \to X$ is a map such that the diagram



commutes up to homotopy, where $\nabla : X \vee X \to X$ is the folding map. We call μ a *multiplication* or an *H-structure* for X. Two examples of H-spaces come in mind: topological groups and the loop spaces $\Omega(X)$. In the sequel, we identify an H-space (X, μ) with the space X.

An H-space X is called a *group-like space* if X satisfies all the axioms of groups up to homotopy. Recall that a homotopy associative H-CW-complex always has a homotopy inverse. More precisely, according to [39, 1.3.2. Corollary] (see also [2, Proposition 8.4.4]), we have the following statement.

Proposition 1.1. *If X is a homotopy associative H-CW-complex then X is a group-like space.*

From now on, we assume that any H-space X is group-like.

Given spaces $X_1, \dots, X_n$, we use the customary notations $X_1 \times \cdots \times X_n$ for their Cartesian and $T_m(X_1, \dots, X_n)$ for the subspace of $X_1 \times \cdots \times X_n$ consisting of those points with at least m coordinates at base points with $m = 0, 1, \dots, n$. Then, $T_0(X_1, \dots, X_n) = X_1 \times \cdots \times X_n$, $T_1(X_1, \dots, X_n)$ is the so called the *fat wedge* of spaces $X_1, \dots, X_n$ and $T_{n-1}(X_1, \dots, X_n) = X_1 \vee \cdots \vee X_n$, the wedge products of spaces $X_1, \dots, X_n$. We write $j_m(X_1, \dots, X_n) : T_m(X_1, \dots, X_n) \to X_1 \times \cdots \times X_n$ for the inclusion map with $m = 0, 1, \dots, n$ and $X_1 \wedge \cdots \wedge X_n = X_1 \times \cdots \times X_n / T_1(X_1, \dots, X_n)$ for the *smash product* of spaces $X_1, \dots, X_n$.

Let $f_m : (X_m, \star_m) \to (Y_m, *_m)$ be continuous maps of pointed topological spaces for $m = 1, \dots, n$. The map $f_1 \times \cdots \times f_n : (X_1 \times \cdots \times X_n, (\star_1, \dots, \star_n)) \to (Y_1 \times \cdots \times Y_n, (*_1, \dots, *_n))$ sends the point $(x_1, \dots, x_n)$ into $(f_1(x_1), \dots, f_n(x_n))$ for $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ and restricts to maps $T_m(f_1, \dots, f_n) : T_m(X_1, \dots, X_n) \to T_m(Y_1, \dots, Y_n)$ with $m = 0, 1, \dots, n$. If $X_m = X$ and $f_m = f$ for $m = 1, \dots, n$ then we write $X^n = X_1 \times \cdots \times X_n$, $X^{\vee n} = X_1 \vee \cdots \vee X_n$, $X^{\wedge n} = X_1 \wedge \cdots \wedge X_n$, $f^n = f_1 \times \cdots \times f_n$ and $f^{\vee n} = f_1 \vee \cdots \vee f_n$. The identity map of a space X involved is consistently denoted by ι_X.

Given an H-group X, the functor $[-, X]$ takes its values in the category of groups. One may then ask when those functors take their values in various subcategories of groups. For example, X is homotopy commutative if and only if $[Y, X]$ is Abelian for all Y.

Given an H-space X, we write $\varphi_{1,X} = \iota_X$, $\varphi_{2,X} : X^2 \to X$ for the basic commutator map and $\varphi_{n+1,X} = \varphi_{2,X} \circ (\varphi_{n,X} \times \iota_X)$ for $n \geq 2$.

1.1. *The nilpotency class.* The *nilpotency class* $\operatorname{nil}(X, \mu)$ of an H-space (X, μ) is the least integer $n \geq 0$ for which the map $\varphi_{n+1,X} \simeq *$ is nullhomotopic and we call the homotopy associative H-space X homotopy nilpotent. If no such integer exists, we put $\operatorname{nil}(X, \mu) = \infty$. In the sequel, we simply write $\operatorname{nil} X$ for the nilpotency class of an H-space X. Thus, $\operatorname{nil} X = 0$ if and only if X is contractible and, as is easily seen, $\operatorname{nil} X \leq 1$ if and only if X is homotopy commutative.

The set $\pi_0(X)$ of all path-components of an H-space X is known to be a group. The following result is easy to prove.

LEMMA 1.2. *If X is an H-space and the path component of the base-point $\star \in X$ is contractible then* $\operatorname{nil} \pi_0(X) = \operatorname{nil} X$.

The definition of the nilpotency classes may be extended to maps. The nilpotency class $\operatorname{nil} f$ of an H-map $f : X_1 \to X_2$ is the least integer $n \geq 0$ for which the map $f \circ \varphi_{n+1,X} : X_1^{n+1} \to X_2$ is nullhomotopic; if no such integer exists, we put $\operatorname{nil} f = \infty$.

In the sequel we need the following result.

Lemma 1.3. *If X is an H-space then the composite map*

$$T_1(X,\dots,X) \stackrel{j_1(X,\dots,X)}{\longrightarrow} X^n \stackrel{\varphi_{n,X}}{\longrightarrow} X$$

is nullhomotopic.

Since the space $X^{\wedge n}$, the n-th smash power of X is the homotopy cofiber of the map $j_1(X,\dots,X) : T_1(X,\dots,X) \to X^n$, the result above implies an existence of a map $\bar{\varphi}_{n,X} : X^{\wedge n} \to X$ for $n \geq 1$ with $\bar{\varphi}_{1,X} = \varphi_{1,X}$.

Next, notice that [15, Lemma 1.2] (cf. [36, Theorem 2.10]) leads to the following lemma.

Lemma 1.4. *If X is a group-like CW-space and Y a finite dimensional CW-complex with $\dim Y = n$ then the group $[Y,X]$ is nilpotent with the nilpotency class at most n.*

Proof. First, recall that given an H-CW-space X, in view of [23], all its m-th Postnikov stages P_mX are also an H-space and the canonical maps $X \to P_mX$ are H-maps for $m \geq 1$. Since the map $X \to P_nX$ is an $(n-1)$-homotopy equivalence and Y is a CW-complex with $\dim Y = n$, there is an isomorphism $[Y,X] \approx [Y,P_nX]$ of groups determined by the map $X \to P_nX$. Then, the map $\bar{\varphi}_{P_nX,n+1}(f_1\wedge\cdots\wedge f_{n+1}) : Y^{\wedge(n+1)} \to P_nX$ is homotopy trivial for any maps $f_1,\dots,f_{n+1} : Y \to P_nX$ since the space $Y^{\wedge(n+1)}$ is n-connected. Consequently, $\operatorname{nil}[Y,X] = \operatorname{nil}[Y,P_nX] \leq n$ and the proof follows. □

Next, any CW-complex Y can be expressed as

$$Y = \lim_{\rightarrow} Y_\alpha,$$

where Y_α are finite CW-complexes. Given a group-like space X, this leads to the short exact sequence

$$1 \to \lim_{\leftarrow}{}^1 [\Sigma Y_\alpha, X] \longrightarrow [Y,X] \longrightarrow \lim_{\leftarrow}[Y_\alpha, X] \to 1,$$

where the group $\lim_{\leftarrow}[Y_\alpha,X]$, by means of Lemma 1.4, is visibly pro-nilpotent. By the proof of [19, Proposition 1.2], the intersection $\Gamma_d \cap (\lim_{\leftarrow}{}^1[\Sigma Y_\alpha, X]) = 0$ for $\dim X = d < \infty$, where Γ_d stands for the d-th member of the lower central series of the group $[Y,X]$. This leads, by means of [19, Proposition 1.1], to pro-nilpotency of the group $[Y,X]$ provided X is a finite group-like CW-space.

Thus, in view of Lemma 1.4, we obtain the following result.

Proposition 1.5. *If X is a finite group-like CW-space then the group $[Y,X]$ is pro-nilpotent for any CW-complex Y.*

It is well known that the quotient map $X^n \to X^{\wedge n}$ has a right homotopy inverse after suspending for $n \geq 1$, and the fact that X is an H-space means that the suspension map $[Y,X] \to [\Sigma Y, \Sigma X]$ is a monomorphism for any space Y. Thus, we may state the proposition.

Proposition 1.6. *Let X be an H-space. Then $\varphi_{n,X} \simeq *$ if and only if $\bar{\varphi}_{n,X} \simeq *$ for $n \geq 1$.*

Then, [4, 2.7. Theorem] and Proposition 1.6 lead to the following theorem.

Theorem 1.7. *If X is an H-space then*

$$\operatorname{nil} X = \sup_m \operatorname{nil}[X^m, X] = \sup_m \operatorname{nil}[X^{\wedge m}, X] = \sup_Y \operatorname{nil}[Y, X],$$

where m ranges over all integers and Y over all topological spaces.

Furthermore, in view of [39, Lemma 2.6.1], we may state the following.

Corollary 1.8. *A connected H-space X is homotopy nilpotent if and only if the functor $[-, X]$ on the category of all spaces is nilpotent group valued.*

Proof. Certainly, the homotopy nilpotency of a connected associative H-space X implies that the functor $[-, X]$ on the category of all pointed spaces is nilpotent group valued.

Now, suppose that the functor $[-, X]$ is nilpotent groups valued and $\operatorname{nil}[\prod_1^\infty X, X] < n$. Then, for the projection map $\prod_1^\infty X \to X^n$ on the first n factors, the composite map

$$\prod_1^\infty X \to X^n \xrightarrow{\varphi_{n,X}} X$$

is null-homotopic. Since, the projection $\prod_1^\infty X \to X^n$ has a retraction, we deduce that the map $\varphi_{n,X} : X^n \to X$ is also null-homotopic and the proof is complete. ∎

1.2. *The nilpotency class of three and seven spheres.* The nilpotency class of the topological group $\mathbb{S}^3$, the 3-sphere has been calculated by Porter ([28]) for the standard multiplication on $\mathbb{S}^3$. We make use of Proposition 1.6, to present another proof of that result.

Proposition 1.9. $\operatorname{nil} \mathbb{S}^3 = 3$.

Proof. First, we notice that all calculations of compositions of homotopy groups of spheres can be found in [34]. Next, James ([21, p. 176]) proves that $\bar{\gamma}_{\mathbb{S}^3,2}$ generates $\pi_6(\mathbb{S}^3) \approx \mathbb{Z}_{12}$ so that in Toda's notation ([34]) we have $\bar{\gamma}_{\mathbb{S}^3,2} = \nu' + \alpha_1(3)$. Now, $\bar{\gamma}_{\mathbb{S}^3,3} = \bar{\gamma}_{\mathbb{S}^3,2} \circ (\bar{\gamma}_{\mathbb{S}^3,1} \wedge \bar{\gamma}_{\mathbb{S}^3,2}) = \bar{\gamma}_{\mathbb{S}^3,2} \circ \Sigma^3(\bar{\gamma}_{\mathbb{S}^3,2}) \in \pi_9(\mathbb{S}^3) \approx \mathbb{Z}_3$ generated by $(\alpha_1(3)+\nu')\circ(\alpha_1(6)+\Sigma^3(\nu')) = \alpha_1(3)\circ\alpha_1(6)$. Next, $\bar{\gamma}_{\mathbb{S}^3,4} = \bar{\gamma}_{\mathbb{S}^3,2} \circ (\bar{\gamma}_{\mathbb{S}^3,1} \wedge \bar{\gamma}_{\mathbb{S}^3,3}) = \bar{\gamma}_{\mathbb{S}^3,2} \circ \Sigma^3(\bar{\gamma}_{\mathbb{S}^3,3}) = \bar{\gamma}_{\mathbb{S}^3,2} \circ \alpha_1(6) \circ \alpha_1(12) = 0$ and the proof follows. ∎

We point out that by Arkowitz-Curjel ([1, Lemma 2]) and James ([21]) that up to homotopy the sphere $\mathbb{S}^3$ admits twelve distinct H-structures which can written as $\mu_t = \mu_0 + \gamma^t_{\mathbb{S}^3,2}$ for $t = 0, \ldots, 11$, where the exponent and juxtaposition are taken with respect to the standard multiplication μ_0 on $\mathbb{S}^3$.

Denote by $\gamma^{(t)}_{n,\mathbb{S}^3}$ (resp. $\bar{\gamma}^{(t)}_{n,\mathbb{S}^3}$) the n-fold (resp. smash) commutator map on $\mathbb{S}^3$ with respect to the H-structure μ_t. Arkowitz-Curjel ([1]) have calculated the homotopy nilpotency class $\operatorname{nil}(\mathbb{S}^3,\mu_t)$ of $\mathbb{S}^3$ for all of its twelve H-structures. More precisely, in view of [1, Lemma 4], we have:

(1) $\gamma^{(t)}_{n,\mathbb{S}^3} = \gamma^{2t+1}_{n,\mathbb{S}^3}$,
(2) $\bar{\gamma}^{(t)}_{n,\mathbb{S}^3} = \bar{\gamma}^{2t+1}_{n,\mathbb{S}^3}$.

Then, [1, Theorem B] states the following.

Theorem 1.10. *If* $t = 1, 4, 7$ *or* 10*, then* $\operatorname{nil}(\mathbb{S}^3,\mu_t) = 2$. *If* $t = 0, 2, 3, 5, 6, 8, 9$ *or* 11*, then* $\operatorname{nil}(\mathbb{S}^3,\mu_t) = 3$.

Furthermore, recall that the Cayley multiplication μ_0 on the seven sphere $\mathbb{S}^7$ is not associative but is diassociative, i.e., any two elements generate an associative subalgebra. Gilbert ([14]) make a choice in bracketing to define inductively the n-fold commutator map $\varphi_{n,\mathbb{S}^7} : (\mathbb{S}^7)^n \to \mathbb{S}^7$ for $n \geq 1$ and show that

$$\operatorname{nil}(\mathbb{S}^7,\mu_0) = 3.$$

There are 120 different homotopy classes of multiplications on $\mathbb{S}^7$ and, as in [1, Lemma 2], it can be shown that they can be written additively in the form

$$\mu_t = \mu_0 + t\varphi_{2,\mathbb{S}^7}$$

for $t = 0, 1, \ldots, 119$. Then, in [14] it is deduced

$$\operatorname{nil}(\mathbb{S}^7,\mu_t) = 3$$

for $t = 0, 1, \ldots, 119$.

2. Properties of the homotopy nilpotency

We mainly make use of [4] to present known and state some new results on the homotopy nilpotency of loop spaces.

With any based space X, we associate the integer $\operatorname{nil}\Omega(X)$ called the *nilpotency class* of X.

Evidently, $\operatorname{nil}\pi_1(X) \leq \operatorname{nil}\Omega(X)$. We give an extension of this result involving Whitehead products, generally denoted by $[\alpha_1,\alpha_2] \in \pi_{m_1+m_2-1}(X)$ if $\alpha_i \in \pi_{m_i}(X)$ for $m_i \geq 1$ with $i = 1, 2$.

We define $(n+1)$-fold Whitehead products $[\alpha_1,\ldots,\alpha_{n+1}]$ as

$$[[\alpha_1,\ldots,\alpha_n],\alpha_{n+1}],$$

if $\alpha_i \in \pi_{m_i}(X)$ for $m_i \geq 1$ with $i = 1,\ldots,n+1$ agreeing that, for $n = 0$, $[\alpha] = \alpha$.

Recall that W-length X, the *Whitehead length* of a space X is the least integer $n \geq 0$ such that $[\alpha_1,\ldots,\alpha_{n+1}] = 0$ for all $\alpha_i \in \pi_{m_i}(X)$, $m_i \geq 1$; if no such integer exists, we put W-length $X = \infty$.

Then, according to [4, 4.6. Theorem], we have the following result.

Theorem 2.1. W-length $X \leq \operatorname{nil}\Omega(X)$.

Example 2.2. (1) It is well-known that

$$\text{W-length}\,\mathbb{S}^n = \operatorname{nil}\Omega(\mathbb{S}^n) = \begin{cases} 3, & \text{for } n \text{ even with } n \neq 2, \\ 2, & \text{for } n \text{ odd with } n \neq 1,3,7 \text{ or } n = 2, \\ 1, & \text{for } n = 1,3,7. \end{cases}$$

(2) The Whitehead lengths W-length $\mathbb{K}P^n$ of the n-th projective spaces $\mathbb{K}P^n$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions have been computed in [17].

The concept of a nilpotent space is due to E. Dror ([10]). Recall that a pointed path-connected space X is said to be *nilpotent* if its fundamental group $\pi_1(X)$ acts nilpotently on the higher homotopy groups $\pi_n(X)$ for $n \geq 1$. But, the action of $\pi_1(X)$ on $\pi_n(X)$ for $n \geq 1$ may be written in terms of Whitehead products. Then, by Theorem 2.1, the space X is nilpotent if $\Omega(X)$ is homotopy nilpotent.

But, not every space $\Omega(X)$ is homotopy nilpotent if X is nilpotent or even simply connected. For the wedge $\mathbb{S}^m \vee \mathbb{S}^n$ of two spheres with $m, n \geq 2$, there is an iterated nontrivial Whitehead product of any weight. Therefore, by Theorem 2.1, we conclude that

$$\operatorname{nil}\Omega(\mathbb{S}^m \vee \mathbb{S}^n) = \infty.$$

We now proceed to find upper bounds for $\operatorname{nil}\Omega(X)$. First, let X be a connected aspherical CW-complex. Then $\pi_m(X) = 0$ for all $m > 1$. Therefore, Lemma 1.2 yields the following statement.

Proposition 2.3. *If X is a connected aspherical CW-complex, then*

$$\operatorname{nil}\pi_1(X) = \operatorname{nil}\Omega(X).$$

For further reference, we state the easily proved lemma.

Lemma 2.4. $\operatorname{nil}(X_1 \times X_2) = \max\{\operatorname{nil} X_1, \operatorname{nil} X_2\}$.

We notice that applying the principal refinement of the Postnikov system of a path-connected nilpotent space, [4, 4.11. Theorem] yields the following.

Theorem 2.5. *Let X be a path-connected nilpotent CW-complex. Suppose the invariants $k^{n+2}(X)$ of a Postnikov system for X are trivial for almost all values of n. Then* $\operatorname{nil}\Omega(X) < \infty$.

In particular, if X is a path-connected nilpotent CW-complex with a finite Postnikov system then $\operatorname{nil}\Omega(X) < \infty$.

Corollary 2.6. *Let X be a path-connected nilpotent CW-complex.*

(1) *If $\pi_n(X) = 0$ for almost all values of n, then* $\operatorname{nil}\Omega(X) < \infty$.
(2) *If the invariants $k^{n+2}(X)$ vanish for almost all values of n, then* W–length $X < \infty$.

Recall that a space X is said to dominate a space Y provided there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \iota_Y$, the composite map $f \circ g$ is homotopic to ι_Y.

Next, in view of [12], $\text{co-cat}\, X$, the *co-category* of a topological space X is a (possibly infinite) strictly positive integer as given by:

$\text{co-cat}\, X = 1$ if and only if X is contractible.

Let $n \geq 1$ be given and suppose the phrase $\text{co-cat}\, Y = m$ has been defined for any space Y and all integers m satisfying $1 \leq m \leq n$; then, $\text{co-cat}\, X = n+1$ provided:

(i) $\text{co-cat}\, X \neq m$ for any m satisfying $1 \leq m \leq n$, and
(ii) there exists a fibration $Q \to Y \to B$ such that Q dominates X and $\text{co-cat}\, Y = n$.

If $\text{co-cat}\, X \neq n$ for all $n \geq 1$, we put $\text{co-cat}\, X = \infty$.

Then, by [12, Theorem 2.12], we have the following theorem.

THEOREM 2.7. $\text{nil}\, \Omega(X) \leq \text{co-cat}\, X - 1$.

We close this section by reviewing some recent works related to the subject above.

The Lusternik-Schnirelmann category of a space X, $\text{LS-cat}\, X$ is the minimal number of open sets needed to cover X which are contractible in X. This was originally defined for manifolds, and is a lower bound for the number of critical points of a function on X. The definition was broadened to arbitrary spaces, and later definitions include an inductive version, ind LS-cat by Ganea ([12]), and symmetric version, symm LS-cat, by Hopkins ([18]).

Co-category is much less well understood than category. The first attempt to define co-category was made by Ganea ([12]). Different possible definitions of co-category exist depending on which feature of the classical LS-category is dualized.

Hopkins ([18]) presents new formulations of category and co-category closer in spirit to the original definition of category ([12]). One byproduct of these formulations is a new characterization of iterated loop spaces and a dual characterization of iterated suspensions.

Eldred ([11]) uses constructions in Goodwillie's calculus of homotopy functors to reformulate Hopkins's definition ([18]) of symmetric LS-co-cat and applies it to spaces determined by functors associated to reduced homotopy endofunctors of spaces. Her result is concluded by the following inequalities

$$\text{W-length}\, X \leq \text{nil}\, \Omega(X) \leq \text{ind LS-co-cat}\, X \leq \text{sym LS-co-cat}\, X.$$

Hovey ([20]) introduces a new definition of a co-category that has some advantages over the previous two definitions. It dualizes Whitehead's definition of a category, so it is defined by a map making a suitable diagram commutative. Another advantage of Hovey's co-category is that in [33] it was shown that the weak (Hovey) co-category of X coincides with $\operatorname{nil}\Omega(X)$ if X is simply-connected.

Murillo and Viruel ([27]) present a new approach to the Lusternik - Schnirelmann co-category of a space. This approach is based on a dual of the Whitehead definition of category. Using this new definition they are able to prove all the classical properties satisfied by Ganea's original concept of co-category.

Yau ([38]) introduces the Clapp - Puppe type generalized Lusternik - Schnirelmann co-category in a Quillen model category, establishes some of their basic properties and gives various characterizations of them. As an application of these characterizations, it is shown that the generalized co-category is invariant under Quillen modelization equivalences. In particular, generalized co-category of spaces and simplicial sets coincide. Another application of these characterizations is to define and study rational co-category.

3. The homotopy nilpotency of some homogeneous spaces

We mainly make use of [15] to review the homotopy nilpotency $\Omega(G/K)$ of loop spaces for some Lie groups G and their closed subgroups $K < G$.

The homotopy nilpotency of H-spaces has been extensively studied as well as the homotopy commutativity. The first major advance was made by Hopkins ([19, Theorem 2.1]) and completed by Rao ([30, Theorem 0.2]). He showed that a finite H-space X is homotopy nilpotent if and only if for sufficiently large n, the $\varphi_{X,n}$'s induce trivial homomorphism in complex bordism. This is the same as asking that the $\varphi_{X,n}$'s induce trivial homomorphisms in all Morava K-theories. We point out that [19, Theorem 2.1] has been proved by reducing the problem to one in stable homotopy theory and applying the nilpotence theorem ([9]). Then, in [19, Corollary 2.2], it was deduced the following corollary.

Corollary 3.1. *If X is a finite associative H-space and the integral homology $H_*(X,\mathbb{Z})$ is torsion free then X is homotopy nilpotent.*

This corollary implies:

$$\operatorname{nil} U(n) < \infty \ \text{ and } \ \operatorname{nil} \mathrm{Sp}(n) < \infty.$$

Because of H-homotopy equivalences $O(n) \simeq SO(n)\times\mathbb{Z}_2$ and $U(n) \simeq SU(n)\times \mathbb{S}^1$, we derive:

$$\operatorname{nil} SO(n) = \operatorname{nil} O(n) \ \text{ and } \ \operatorname{nil} SU(n) = \operatorname{nil} U(n) < \infty.$$

Next, write $O = \varinjlim O(n)$, $U = \varinjlim U(n)$ and $Sp = \varinjlim \mathrm{Sp}(n)$. Then, notice that by Bott periodicity theorem: $\Omega^8(O) \simeq O$, $\Omega^2(U) \simeq U$ and $\Omega^8(Sp) \simeq Sp$, we get

$$\operatorname{nil} O = \operatorname{nil} U = \operatorname{nil} Sp = 1.$$

But, the classical homotopy nilpotency does not capture the algebraic nilpotency of a topological group.

REMARK 3.2. Since the commutators $[SO(3), SO(3)] = SO(3)$ and $[SU(2), SU(2)] = SU(2)$ and: $SO(3) \subseteq SO(n) \subseteq O(n)$ for $n \geq 3$, $SU(2) \subseteq SU(n) \subseteq U(n)$ for $n \geq 2$ and $SU(2) = \mathrm{Sp}(1) \subseteq \mathrm{Sp}(n)$ for $n \geq 1$, we derive that the groups:

(1) $SO(n)$ and $O(n)$ are not nilpotent for $n \geq 3$,
(2) $U(n)$ and $SU(n)$ are not nilpotent for $n \geq 2$,
(3) $\mathrm{Sp}(n)$ is not nilpotent for $n \geq 1$.

Recall that Biedermann and Dwyer ([5]) define homotopy n-nilpotent groups as homotopy algebras over certain simplicial algebraic theories to study the connection between the Goodwillie tower of the identity and the lower central series of the loop group on connected spaces. This notion interpolates between infinite loop spaces and loop spaces, but backwards. They study the relation to ordinary nilpotent groups and prove that n-excisive functors of the form $\Omega(F)$ factor over the category of homotopy n-nilpotent groups.

Nilpotency for discrete groups can be defined in terms of central extensions. Costoya, Scherer, and Viruel ([7]) study the analogous definition for spaces in terms of principal fibrations having infinite loop spaces as fibers, yielding a new invariant they compare with classical co-category, but also with the more recent notion of homotopy nilpotency introduced by Biedermann and Dwyer ([5]). This allows them to characterize finite homotopy nilpotent loop spaces in the spirit of Hubbuck's Torus Theorem and corresponding results for p-compact groups and p-Noetherian groups.

Rao ([29]) showed that the converse of the criterion from Corollary 3.1 is true in the case of $\mathrm{Spin}(n)$ and $SO(n)$ by showing that $\mathrm{Spin}(n)$, $SO(n)$, $n \geq 7$ and $SO(3)$, $SO(4)$ are not homotopy nilpotent.

Let now $G_{(p)}$ stand for the p-localization in the sense of [6] of a compact Lie group G. Then, Yagita ([37, Theorem]) has shown the following theorem.

THEOREM 3.3. *Let G be a simply connected Lie group. Then for each prime p, the p-localization $G_{(p)}$ is homotopy nilpotent if and only if the cohomology $H^*(G, \mathbb{Z})$ has no p-torsion.*

Next, Rao ([30, Theorem 0.2]) has generalized Theorem 3.3 as follows.

THEOREM 3.4. *Let G be a compact connected Lie group and let p be a prime. Then $G_{(p)}$ is homotopy nilpotent if and only if the homology $H_*(G, \mathbb{Z}_{(p)})$ is torsion-free.*

Recall that an H-map $f : X \to Y$ of H-spaces is called *central* provided $\bar{\varphi}_{2,Y}(f \wedge \iota_Y) \simeq *$. Then, in view of [39, Lemma 2.6.6], we have the following result.

LEMMA 3.5. *Let* $F \xrightarrow{i} E \xrightarrow{q} B$ *be an* H*-fibration, i.e.,* $F \xrightarrow{i} E \xrightarrow{q} B$ *is a fibration,* F, E *and* B *are* H*-spaces and the maps* $i : F \to E$*, and* $q : E \to B$ *are* H*-maps.*

(1) *If* $\operatorname{nil} q \leq n$ *and* $i : F \to E$ *is central then* $\operatorname{nil} E \leq n + 1$.
(2) *If* $\Omega(Y) \xrightarrow{i} E \xrightarrow{q} X$ *is the induced* H*-fibration by an* H*-map* $f : X \to Y$ *then the map* $i : \Omega(Y) \to E$ *is central.*

Next, recall that if a topological group G acts freely on a paracompact space X then there is a homeomorphism

$$X/G \approx X \times_G EG,$$

where $G \to EG \to BG$ is the universal principal G-bundle.

Since the connecting map $\partial_X : \Omega(X/G) \to G$, in view of [32, Theorem 8.6] (see also [16, Corollary 3.4]), is an H-map, the fibration $X \to X/G \to EG/G \approx BG$ leads to the H-fibration

$$\Omega(X) \longrightarrow \Omega(X/G) \xrightarrow{\partial_X} \Omega(BG) \simeq G.$$

Now, let G be a compact Lie group and $K < G$ its closed subgroup. Then, the quotient space G/K is a manifold and the quotient map $q : G \to G/K$ is a submersion. Hence, $q : G \to G/K$ has a local section at the point $q(e) = K$ for the unit $e \in G$. This certainly implies that the map $q : G \to G/K$ has a local section at any point $q(g)$ for any $g \in G$. Consequently, the quotient map $q : G \to G/K$ is a fiber bundle with the fiber K as a principal H-bundle. Thus, the fibration $G \to G/K \to BK$ leads to the H-fibration

$$\Omega(G) \longrightarrow \Omega(G/K) \xrightarrow{\partial_G} \Omega(BK) \simeq K.$$

Since this fibration is induced by the inclusion map $K \hookrightarrow G$ which is certainly an H-map, Lemma 3.5 yields the following statement.

PROPOSITION 3.6. *If* G *is a compact Lie group and* $K < G$ *its closed subgroup with* $\operatorname{nil} K < \infty$ *then* $\operatorname{nil} \Omega(G/K) < \infty$.

3.1. *Grassmannians.* Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then, we set:

$$U_{\mathbb{K}}(n) = \begin{cases} O(n), & \text{if } \mathbb{K} = \mathbb{R}, \\ U(n), & \text{if } \mathbb{K} = \mathbb{C}, \\ \mathrm{Sp}(n), & \text{if } \mathbb{K} = \mathbb{H}, \end{cases} \quad \text{and } SU_{\mathbb{K}}(n) = \begin{cases} SO(n), & \text{if } \mathbb{K} = \mathbb{R}, \\ SU(n), & \text{if } \mathbb{K} = \mathbb{C}, \\ \mathrm{Sp}(n), & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

Write $G_{n,m}(\mathbb{K})$ (resp. $G^+_{n,m}(\mathbb{K})$) for the (resp. oriented) Grassmannian of m-dimensional subspaces in the n-dimensional $\mathbb{K}$-vector space. For example, the set of lines $G_{n+1,1}(\mathbb{K}) = \mathbb{K}P^n$ is the projective n-space over $\mathbb{K}$.

The homotopy nilpotency of the loop spaces $\Omega(\mathbb{K}P^n)$ has been first studied by Ganea ([13]), Snaith ([31]) and then their p-localization $\Omega((\mathbb{K}P^n)_{(p)})$ by Meier [26]. Recall that by Ganea ([13, Propositions 1.3-1.5]) and Snaith ([31, Corollaries 2.6 and 2.13]), we have the following result.

PROPOSITION 3.7.

(1) $\operatorname{nil}\Omega(\mathbb{R}P^{2n+1}) = \begin{cases} 2, & \textit{for } n \geq 0, \\ 1, & \textit{if and only if } n = 0, 1 \textit{ or } 3. \end{cases}$
(2) $\operatorname{nil}\Omega(\mathbb{R}P^{2n}) = \infty$ *for* $n \geq 1$.
(3) $\operatorname{nil}\Omega(\mathbb{C}P^{2n+1}) = \begin{cases} 2, & \textit{for any odd } n \geq 1, \\ 1, & \textit{if and only if } n = 1. \end{cases}$
(4) $4 \leq \operatorname{nil}\Omega(\mathbb{C}P^{2n}) \leq 6$ *for* $n \geq 1$.
(5) $3 \leq \operatorname{nil}\Omega(\mathbb{H}P^{2n+1})$ *for any* $n \geq 1$.
(6) $4 \leq \operatorname{nil}\Omega(\mathbb{H}P^{2n})$ *for any* $n \geq 1$.
(7) $\operatorname{nil}\Omega(\mathbb{H}P^{n}) = 3$ *if* $n \equiv -1 \pmod{24}$.

REMARK 3.8. In view of [31, Propositions 2.11, 2.12], we have that $\bar{\varphi}_{7,\Omega(\mathbb{C}P^{2n})} = 0$. Hence, $\operatorname{nil}\Omega(\mathbb{C}P^{2n}) \leq 6$. But, in [31, Corollary 2.13] it is stated that $\operatorname{nil}\Omega(\mathbb{C}P^{2n}) \leq 7$.

Then, Meier ([26, Theorem 5.4]) has shown some results on the homotopy nilpotency of p-localized projective spaces.

THEOREM 3.9. *Let p be an odd prime and $n \geq 2$ a natural number. Then:*
(1) $\operatorname{nil}\Omega(\mathbb{C}P^{n}_{(p)}) = 1$,
(2) $\operatorname{nil}\Omega(\mathbb{H}P^{n}_{(p)}) = 1$ *if* $p > 3$,
(3) $3 \leq \operatorname{nil}\Omega(\mathbb{H}P^{n}_{(3)}) \leq 4$,
(4) $\operatorname{nil}\Omega(\mathbb{H}P^{n}_{(3)}) = 3$ *if* $n \equiv 2 \pmod 3$.

Since the space $\mathbb{R}P^{2n+1}$ is simple, there is its p-localization $\mathbb{R}P^{2n+1}_{(p)}$ for any prime $p \geq 2$. It it also easy to see that $\operatorname{nil}\Omega(\mathbb{R}P^{2n+1}_{(p)}) = 1$ for any odd prime p and $n \geq 0$. But, the nilpotency $\operatorname{nil}\Omega(\mathbb{H}P^{n})$ for any $n \geq 2$ does not appear in the literature known to the author.

It is well known that $G_{n,m}(\mathbb{K})$ (resp. $G^{+}_{n,m}(\mathbb{K})$) are smooth manifolds with diffeomorphisms

$$G_{n,m}(\mathbb{K}) \approx U_{\mathbb{K}}(n)/(U_{\mathbb{K}}(m) \times U_{\mathbb{K}}(n-m))$$

and

$$G^{+}_{n,m}(\mathbb{K}) \approx SU_{\mathbb{K}}(n)/(SU_{\mathbb{K}}(m) \times SU_{\mathbb{K}}(n-m)),$$

for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

Since the homomorphism $\pi_1(SU(m)_{\mathbb{K}}) \to \pi_1(SU(n)_{\mathbb{K}})$ of fundamental groups determined by the inclusion map $SU(m)_{\mathbb{K}} \hookrightarrow SU(n)_{\mathbb{K}}$ for $2 \leq m \leq n$ is an epimorphism, we derive that the spaces $G^{+}_{n,m}(\mathbb{K})$ are simply connected for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{H}$. Next, there is the universal covering map

$$\mathbb{Z}_2 \longrightarrow G^{+}_{n,m}(\mathbb{R}) \longrightarrow G_{n,m}(\mathbb{R})$$

and the fibre bundle

$$\mathbb{S}^1 \longrightarrow G^+_{n,m}(\mathbb{C}) \longrightarrow G_{n,m}(\mathbb{C}).$$

Now, recall that the classifying

$$BU_{\mathbb{K}}(m) = \lim_{\to} G_{n,m}(\mathbb{K}) = G_{\infty,m}(\mathbb{K}).$$

Since the cohomology $H^*(SO(n),\mathbb{Z})$ has only 2-torsion and the cohomologies $H^*(U(n),\mathbb{Z})$, $H^*(\mathrm{Sp}(n),\mathbb{Z})$ are torsion free, the fibration

$$\Omega(SU_{\mathbb{K}}(n)) \longrightarrow \Omega(G^+_{n,m}(\mathbb{K})) \longrightarrow SU_{\mathbb{K}}(m) \times SU_{\mathbb{K}}(n-m),$$

the homotopy equivalence $\Omega(BU_{\mathbb{K}}(m)) \simeq U_{\mathbb{K}}(m)$, Corollary 3.1, Theorem 3.4 and Proposition 3.6 lead to the following result.

PROPOSITION 3.10. *If* $1 \le m < n \le \infty$ *then:*

(1) $\operatorname{nil}\Omega(G^+_{n,m}(\mathbb{R})_{(p)}) < \infty$ *for* $p > 2$,
(2) $\operatorname{nil}\Omega(G_{n,m}(\mathbb{K})) < \infty$ *and* $\operatorname{nil}\Omega(G^+_{n,m}(\mathbb{K})) < \infty$ *for* $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

In particular, $\operatorname{nil}\Omega(\mathbb{H}P^n) < \infty$.

We do not mention above any result on the p-localization of $G_{n,m}(\mathbb{R})$ because we are not sure of its existence.

REMARK 3.11. The (resp. oriented) flag manifold $F_{n;n_1,\dots,n_k}(\mathbb{K})$ (resp. $F^+_{n;n_1,\dots,n_k}(\mathbb{K})$) with $1 \le n_1 < \cdots < n_k \le n-1$ in the n-dimensional $\mathbb{K}$-vector space is smooth with a diffeomorphism $F_{n;n_1,\dots,n_k}(\mathbb{K}) \approx U_{\mathbb{K}}(n)/(U_{\mathbb{K}}(n_1) \times U_{\mathbb{K}}(n_1-n_2) \times \cdots \times U_{\mathbb{K}}(n_{k-1}-n_k) \times U_{\mathbb{K}}(n-n_k))$ (resp. $F^+_{n;n_1,\dots,n_k}(\mathbb{K}) \approx SU_{\mathbb{K}}(n)/(SU_{\mathbb{K}}(n_1) \times SU_{\mathbb{K}}(n_1-n_2) \times \cdots \times SU_{\mathbb{K}}(n_{k-1}-n_k) \times SU_{\mathbb{K}}(n-n_k)))$. Furthermore, there is the universal covering map

$$(\mathbb{Z}_2)^k \to F^+_{n;n_1,\dots,n_k}(\mathbb{R}) \to F_{n;n_1,\dots,n_k}(\mathbb{R})$$

and a fibre bundle

$$(\mathbb{S}^1)^k \to F^+_{n;n_1,\dots,n_k}(\mathbb{C}) \to F_{n;n_1,\dots,n_k}(\mathbb{C}).$$

Consequently, Corollary 3.1, Theorem 3.4 and Proposition 3.6 lead to:

(1) $\operatorname{nil}\Omega(F^+_{n;n_1,\dots,n_k}(\mathbb{R})_{(p)}) < \infty$ for $p > 2$,
(2) $\operatorname{nil}\Omega(F_{n;n_1,\dots,n_k}(\mathbb{K})) < \infty$ and $\operatorname{nil}\Omega(F^+_{n;n_1,\dots,n_k}(\mathbb{K})) < \infty$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$.

As above, we do not mention above any result on the p-localization of $F_{n;n_1,\dots,n_k}(\mathbb{R})$ because we are not sure of its existence.

3.2. *Stiefel manifolds.* The Stiefel manifold $V_{n,m}(\mathbb{K})$ is the set of all orthonormal m-frames in the vector space $\mathbb{K}^n$. That is, it is the set of ordered orthonormal m-tuples of vectors in $\mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

It is well known that $V_{n,m}(\mathbb{K})$ is a smooth manifold and there are diffeomorphisms:

(1) $V_{n,m}(\mathbb{R}) = V_{n,m} \approx O(n)/O(n-m) \approx SO(n)/SO(n-m)$,
(2) $V_{n,m}(\mathbb{C}) = W_{n,m} \approx U(n)/U(n-m) \approx SU(n)/SU(n-m)$,

(3) $V_{n,m}(\mathbb{H}) = X_{n,m} \approx \mathrm{Sp}(n)/\mathrm{Sp}(n-m)$.

Since the homomorphism $\pi_1(SU(m)_{\mathbb{K}}) \to \pi_1(SU(n)_{\mathbb{K}})$ of fundamental groups determined by the inclusion map $SU(m)_{\mathbb{K}} \hookrightarrow SU(n)_{\mathbb{K}}$ for $2 \leq m \leq n$ is an epimorphism, we derive that the spaces $V_{n,m}(\mathbb{K})$ are simply connected for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Then, Corollary 3.1, Theorem 3.4 and Proposition 3.6 lead to the following proposition.

PROPOSITION 3.12. *If* $1 \leq m \leq n$ *then:*

(1) $\operatorname{nil}\Omega(V_{n,m})_{(p)} < \infty$ *for* $p > 2$,
(2) $\operatorname{nil}\Omega(W_{n,m}) < \infty$,
(3) $\operatorname{nil}\Omega(X_{n,m}) < \infty$.

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