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REGULARITY OF A WEAK SOLUTION TO A LINEAR FLUID-COMPOSITE STRUCTURE INTERACTION PROBLEM

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Abstract. In this manuscript, we deal with the regularity of a weak solution to the fluid-composite structure interaction problem introduced in [12]. The problem describes a linear fluid-structure interaction between an incompressible, viscous fluid flow, and an elastic structure composed of a cylindrical shell supported by a mesh-like elastic structure. The fluid and the mesh-supported structure are coupled via the kinematic and dynamic boundary coupling conditions describing continuity of velocity and balance of contact forces at the fluid-structure interface. In [12], it is shown that there exists a weak solution to the described problem. By using the standard techniques from the analysis of partial differential equations we prove that such a weak solution possesses an additional regularity in both time and space variables for initial and boundary data satisfying the appropriate regularity and compatibility conditions imposed on the interface.

1. Introduction

Fluid-structure interaction (FSI) problems are multi-physics problems which arise in many applications. The most known examples are aeroelasticity and biomedicine. They are often too complex to be solved analytically so they have to be analyzed by means of various advanced mathematical tools and challenging numerical simulations. In FSI problems which involve an elastic structure, there are generally two different scenarios: the first one in which an elastic solid is fully immersed in a fluid, and the second one where a fluid is flowing inside a container with elastic walls. The benchmark problem we study in this manuscript corresponds the second scenario and is motivated by

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the interaction between the blood flow in a coronary artery treated with a vascular stent.

We consider a fluid-structure interaction problem between the flow of a viscous, incompressible fluid in a 3D cylindrical domain, and an elastic, composite structure. The fluid flow is modeled by the time-dependent Stokes equations, while the elastic structure consists of a cylindrical shell supported by a mesh-like structure, the latter one being a 3D elastic body consisting of a collection of slender elastic rod-like components. The structure displacement is assumed to be small, modeled by a system of linear Koiter shell equations allowing displacement in all three spatial directions. A 1D hyperbolic net model consisting of a collection of linearly elastic curved rods is used to model the elastodynamics of the mesh-like structure. The fluid and the composite structure are coupled via kinematic and dynamic coupling conditions evaluated along a linearized fluid-structure interface, which coincides with the fixed fluid domain boundary.

In a recent work [12], the existence of a weak solution to the corresponding problem was proven, and here we focus on the other fundamental mathematical issue, the regularity of the obtained weak solution. Mathematical analysis of solutions to the coupled fluid-mesh-shell interaction problem is challenging due to the *parabolic-hyperbolic-hyperbolic nature* with the coupling taking place at the fluid-structure interface. Nevertheless it is of great practical relevance since mathematical modeling and numerical simulations have been proven to be an indispensable tool in guiding optimal stent design and performance, see e.g. [10, 11, 49] and references therein.

Let us now mention a few references which are most closely related to the present work in the area of analysis [16, 17, 18, 24, 25, 26, 29, 31, 33, 39, 40, 41] and numerical simulations [2, 4, 7, 9, 32]. The fluid-structure interaction problems with composite structure were studied in [8, 42]. The only works in which analysis of an FSI problem including an approximation of a stent-supported vessel were considered are [6, 12, 15]. In [12], a stent was modeled as a separate mesh-like structure, while in the other two papers, the presence of a stent was modeled by the jump in the elasticity coefficients of a shell.

In [13], the authors extended the weak solution existence result of [12] to a nonlinear problem by considering the nonlinear flow modeled by the Navier-Stokes equations, and by coupling the fluid to the mesh-supported shell along the current, deformed interface, giving rise to a strong geometric nonlinearity.

We also mention that the 1D hyperbolic net model considered here was first introduced in [46] as an alternative to computationally expensive engineering approaches in which a stent is modeled as a single 3D elastic body and approximated using 3D based finite elements. Although the model is one-dimensional, it provides 3D information about the stent struts' deformation in all three spatial directions.

When considering well-posedness (including regularity) of fluid equations coupled with elastic structure equations, the authors in [23] and [24] proved the local-in-time existence and uniqueness of a regular solution between 3D incompressible, viscous fluid and a 3D linear/quasilinear structure. Furthermore, in series of papers [35, 36, 37, 38], the authors dealt with the existence and uniqueness of a strong solution to the FSI problems in which they considered Navier-Stokes equations coupled to a linear second order hyperbolic equation. Some other works that deal with the strong solutions of various FSI problems are [3, 5, 44].

It is important to emphasize that all of the mentioned papers which deal with the regularity of solutions have the same mathematical obstacle in common which is a *mismatch between parabolic and hyperbolic regularity*, mostly pronounced at the interface. In the present manuscript, we establish regularity of a weak solution to the stated fluid-mesh-shell interaction problem, for initial and boundary data satisfying the appropriate regularity as well as compatibility conditions imposed on the interface. The regularity result is valid up to the boundary, i.e. up to mesh vertices.

The article is organized as follows. In Sections 2 and 3, we describe the fluid-mesh-shell interaction problem in consideration, and state the existence theorem proven in [12]. In Section 4, the main result is stated, and in Sections 5 and 6, we derive the formal estimates, as well as rigorous justification of the time and space regularity, respectively.

2. The problem description

2.1. *The fluid.* We consider the flow of an incompressible, viscous fluid through a cylindrical domain, denoted by Ω:

$$\Omega = \{(z, x, y) \in \mathbb{R}^3 : z \in (0, L), \sqrt{x^2 + y^2} < R\}.$$

The fluid domain boundary consists of three parts: the lateral boundary Γ, which is a cylinder of radius R, the inlet boundary Γ_{in}, which is a circular area of radius R located at $z = 0$ and the outlet boundary Γ_{out}, which is a circular area of radius R located at $z = L$, see Figure 1.

The time-dependent Stokes equations for an incompressible, viscous fluid are used to model the flow in Ω:

$$\left.\begin{array}{rcl} \rho_F \partial_t \mathbf{u} & = & \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}\right\} \text{ in } \Omega, \quad t \in (0, T), \tag{2.1}$$

where ρ_F denotes the fluid density, $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\sigma} = -pI + 2\mu_F \mathbf{D}(\mathbf{u})$ is the fluid Cauchy stress tensor, p is the fluid pressure, μ_F is the dynamic viscosity coefficient, and $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$. At the inlet and outlet we prescribe the pressure, with the tangential

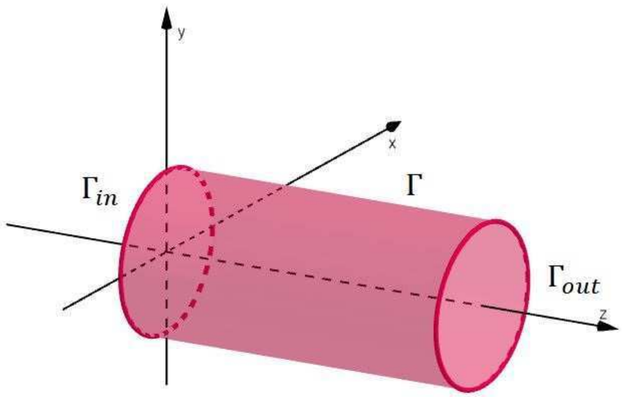

FIGURE 1. A sketch of the fluid domain

fluid velocity equal to zero (see [22]):

$$\left.\begin{array}{c} p = P_{in/out}(t) \\ \mathbf{u} \times \mathbf{e}_z = 0 \end{array}\right\} \text{ on } \Gamma_{in/out},$$

where $P_{in/out}$ are given. Therefore, the fluid flow is driven by the pressure drop, and the fluid flow is orthogonal to the inlet and outlet boundary.

The fluid velocity will be assumed to belong to the following classical function space

$$V_F = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \times \mathbf{e}_z = 0 \text{ on } \Gamma_{in/out}\}. \tag{2.2}$$

2.2. *The shell.* The lateral boundary of the fluid domain will be assumed elastic, and modeled by the cylindrical Koiter shell equations. The shell thickness will be denoted by $h > 0$, the length by L, and its reference radius of the middle surface by R. We consider a clamped cylindrical shell. This reference configuration, which we denote by Γ, can be parameterized by

$$\boldsymbol{\varphi} : \omega \to \mathbb{R}^3, \quad \boldsymbol{\varphi}(z, \theta) = (z, R\cos\theta, R\sin\theta),$$

where $\omega = (0, L) \times (0, 2\pi)$, and $R > 0$, thus:

$$\Gamma = \{(z, R\cos\theta, R\sin\theta) : z \in (0, L), \theta \in (0, 2\pi)\}.$$

Under loading, the Koiter shell is displaced from its reference configuration Γ by a displacement $\boldsymbol{\eta} = \boldsymbol{\eta}(t, z, \theta) = (\eta_z, \eta_r, \eta_\theta)$, where η_z, η_r, and η_θ denote the tangential, radial and azimuthal components of displacement. The end points of the shell will be assumed to be clamped, whereas the boundary conditions at $\theta = 0, 2\pi$, will be periodic. For the details, see [12].

Let V_K denote the following function space:

$$V_K = \{\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in H^1(\omega) \times H^2(\omega) \times H^1(\omega) : \\ \boldsymbol{\eta}(t,z,\theta) = \partial_z \eta_r(t,z,\theta) = 0, z \in \{0, L\}, \theta \in (0, 2\pi), \\ \boldsymbol{\eta}(t,z,0) = \boldsymbol{\eta}(t,z,2\pi), \partial_\theta \eta_r(t,z,0) = \partial_\theta \eta_r(t,z,2\pi), z \in (0,L)\}, \tag{2.3}$$

equipped with the corresponding norm:

$$\|\boldsymbol{\eta}\|_k^2 = \|\eta_z\|_{H^1(\omega)}^2 + \|\eta_r\|_{H^2(\omega)}^2 + \|\eta_\theta\|_{H^1(\omega)}^2.$$

We are interested in weak solutions $\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in V_K$ satisfying the following elastodynamics problem for a cylindrical Koiter shell (see [21, 34]): find $\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in V_K$ such that

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle = \int_\omega \mathbf{f} \cdot \boldsymbol{\psi} R, \quad \forall \boldsymbol{\psi} \in V_K, \tag{2.4}$$

where ρ_K is the shell density, $\mathbf{f}$ is the outside loading, and $\mathcal{L}$ is the linear operator associated with the Koiter elastic energy:

$$\langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle = h \int_\omega \mathcal{A}\boldsymbol{\gamma}(\boldsymbol{\eta}) : \boldsymbol{\gamma}(\boldsymbol{\psi}) R + \frac{h^3}{12} \int_\omega \mathcal{A}\boldsymbol{\varrho}(\boldsymbol{\eta}) : \boldsymbol{\varrho}(\boldsymbol{\psi}) R,$$

where $\mathcal{A}$ is the shell elasticity tensor, $\boldsymbol{\gamma}$ is the linearized change of metric tensor and $\boldsymbol{\varrho}$ is the linearized change of curvature tensor:

$$\boldsymbol{\gamma}(\boldsymbol{\eta}) = \begin{pmatrix} \partial_z \eta_z & \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) \\ \frac{1}{2}(\partial_\theta \eta_z + R\partial_z \eta_\theta) & R\partial_\theta \eta_\theta + R\eta_r \end{pmatrix}, \tag{2.5}$$

$$\boldsymbol{\varrho}(\boldsymbol{\eta}) = \begin{pmatrix} -\partial_{zz} \eta_r & -\partial_{z\theta} \eta_r + \partial_z \eta_\theta \\ -\partial_{z\theta} \eta_r + \partial_z \eta_\theta & -\partial_{\theta\theta} \eta_r + 2\partial_\theta \eta_\theta + \eta_r \end{pmatrix}. \tag{2.6}$$

We emphasize that from Theorem 2.6-4 in [20], we get the coercivity of the operator $\mathcal{L}$, i.e.

$$\langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\eta} \rangle \geq c\|\boldsymbol{\eta}\|_k^2, \ \forall \boldsymbol{\eta} \in V_K.$$

The differential form of the cylindrical Koiter shell elastodynamics problem on $(0,T) \times \omega$ is then given by:

$$\rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} = \mathbf{f} R. \tag{2.7}$$

2.3. *The elastic mesh.* An elastic mesh is a three-dimensional elastic body defined as a union of three-dimensional slender components called struts [14, 46]. Since struts are slender or "thin", meaning that the ratio between the thickness of each strut versus its length is small, 1D (reduced) models can be used to approximate their elastodynamic properties. In particular, we will be using a 1D curved rod model to approximate the elastodynamic properties of slender mesh struts. The one space dimension corresponds to the parameterization of the middle line of curved rod. For the i-th curved rod, the middle line is parameterized via

$$\mathbf{P}_i : [0, l_i] \to \boldsymbol{\varphi}(\overline{\omega}), \quad i = 1, \ldots, n_E,$$

where n_E denotes the number of curved rods in a mesh. By using $s \in (0, l_i)$ to denote the location along the middle line, and $\mathbf{d}_i(t,s)$ to denote the displacement of the middle line from its reference configuration, $\mathbf{w}_i(t,s)$ the infinitesimal rotation of cross-sections, $\mathbf{q}_i(t,s)$ the contact moment, and $\mathbf{p}_i(t,s)$ the contact force, the following system of equations will be used to model the elastodynamics of 1D curved rods:

$$\begin{aligned} \rho_S A_i \partial_t^2 \mathbf{d}_i &= \partial_s \mathbf{p}_i + \mathbf{f}_i, \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &= \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i, \\ 0 &= \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i, \\ 0 &= \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i. \end{aligned} \tag{2.8}$$

Here, ρ_S is the strut's material density, A_i is the cross-sectional area of the i-th rod, M_i is the matrix related to the moments of inertia of the cross-sections, $\mathbf{f}_i$ is the line force density acting on the i-th rod, and $\mathbf{t}_i$ is the unit tangent on the middle line of the i-th rod. Matrix H_i is a positive definite matrix which describes the elastic properties and the geometry of the cross section, while matrix $Q_i \in SO(3)$ represents the local basis at each point of the middle line of the i-th rod (see [1] for more details). The first two equations describe the linear impulse-momentum law and the angular impulse-momentum law, respectively, while the last two equations describe a constitutive relation for a curved, linearly elastic rod, and the condition of inextensibility and unshearability, respectively.

System (2.8) is defined on a graph domain, determined by the geometry and topology of the mesh structure. The graph consists of a set of vertices $\mathcal{V}$ (points where the middle lines meet), and a set of edges $\mathcal{E}$ (pairing of vertices). The ordered pair $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ defines the mesh net topology.

To define the weak solution space, we first introduce a function space consisting of all the H^1-functions $(\mathbf{d}, \mathbf{w})$ defined on the entire net $\mathcal{N}$, such that they satisfy the kinematic coupling conditions at each vertex $V \in \mathcal{V}$:

$$\begin{aligned} H^1(\mathcal{N}; \mathbb{R}^6) = \{(\mathbf{d}, \mathbf{w}) = ((\mathbf{d}_1, \mathbf{w}_1), \ldots, (\mathbf{d}_{n_E}, \mathbf{w}_{n_E})) \in \prod_{i=1}^{n_E} H^1(0, l_i; \mathbb{R}^6) : \\ \mathbf{d}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{d}_j(\mathbf{P}_j^{-1}(V)), \mathbf{w}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{w}_j(\mathbf{P}_j^{-1}(V)), \\ \forall V \in \mathcal{V}, V = e_i \cap e_j,\ i, j = 1, \ldots, n_E\}. \end{aligned}$$

The solution space is defined to contain the conditions of inextensibility and unshearability as follows:

$$V_S = \{(\mathbf{d}, \mathbf{w}) \in H^1(\mathcal{N}; \mathbb{R}^6) : \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, i = 1, \ldots, n_E\}. \tag{2.9}$$

For a function $(\mathbf{d}, \mathbf{w}) \in V_S$, we consider the following norms on $H^1(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{H^1(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{H^1(0,l_i;\mathbb{R}^3)},$$

and the following norms on $L^2(\mathcal{N}; \mathbb{R}^3)$:

$$\|\mathbf{d}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{d}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}, \quad \|\mathbf{w}\|^2_{L^2(\mathcal{N};\mathbb{R}^3)} := \sum_{i=1}^{n_E} \|\mathbf{w}_i\|^2_{L^2(0,l_i;\mathbb{R}^3)}.$$

The weak formulation for the mesh net problem is obtained by summing up all the weak formulations for each local mesh component (i.e. curved rod, or strut) and it reads: find $(\mathbf{d}, \mathbf{w}) \in V_S$ such that

$$\begin{aligned} \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \partial_t^2 \mathbf{d}_i \cdot \boldsymbol{\xi}_i + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \partial_t^2 \mathbf{w}_i \cdot \boldsymbol{\zeta}_i \\ + \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i = \sum_{i=1}^{n_E} \int_0^{l_i} \mathbf{f}_i \cdot \boldsymbol{\xi}_i, \end{aligned} \tag{2.10}$$

for all test functions $(\boldsymbol{\xi}, \boldsymbol{\zeta}) = ((\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1), \dots, (\boldsymbol{\xi}_{n_E}, \boldsymbol{\zeta}_{n_E})) \in V_S$.

Remark 2.1. We will be assuming that the elastic mesh is always confined to the shell surface so that the following holds:

$$\bigcup_{i=1}^{n_E} \mathbf{P}_i([0, l_i]) \subset \Gamma = \boldsymbol{\varphi}(\overline{\omega}).$$

Since $\boldsymbol{\varphi}$ is injective on ω, functions $\boldsymbol{\pi}_i$, denoting the reparameterizations of the stent struts:

$$\boldsymbol{\pi}_i = \boldsymbol{\varphi}^{-1} \circ \mathbf{P}_i : [0, l_i] \to \overline{\omega}, \quad i = 1, \dots, n_E,$$

are well defined. The reference configuration of the mesh defined on ω will be denoted by

$$\omega_S = \bigcup_{i=1}^{n_E} \boldsymbol{\pi}_i([0, l_i]),$$

and of the mesh defined on Γ will be denoted by

$$\Gamma_S = \bigcup_{i=1}^{n_E} \mathbf{P}_i([0, l_i]).$$

See Figure 2.

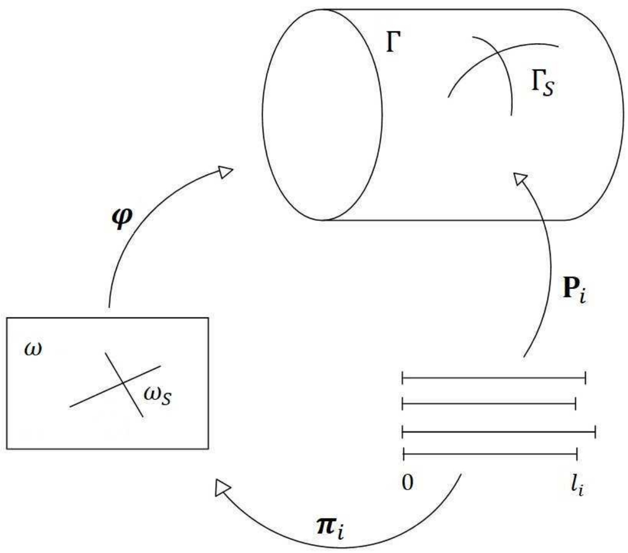

FIGURE 2. Parameterization of the mesh struts

2.4. *The fluid-composite structure coupling.* By imposing the kinematic and dynamic coupling conditions, which describe the continuity of velocity and the balance of contact forces respectively, at the fluid-structure interface, the fluid-composite structure interaction problem reads:

PROBLEM 1. Find $(\mathbf{u}, p, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ such that:

$$\left.\begin{aligned} \rho_F \partial_t \mathbf{u} &= \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}\right\} \text{ in } (0,T) \times \Omega, \tag{2.11}$$

$$\left.\begin{aligned} \partial_t \boldsymbol{\eta} &= \mathbf{u} \circ \boldsymbol{\varphi} \\ \rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{||\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}||} \delta_{J_i} &= -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}) \end{aligned}\right\} \text{ on } (0,T) \times \omega, \tag{2.12}$$

$$\left.\begin{aligned} \rho_S A_i \partial_t^2 \mathbf{d}_i &= \partial_s \mathbf{p}_i + \mathbf{f}_i \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &= \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i \\ 0 &= \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i \\ 0 &= \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i \end{aligned}\right\} \text{ on } (0,T) \times (0, l_i). \tag{2.13}$$

Problem (2.11)-(2.13) is supplemented with the following set of boundary and initial conditions:

$$\left\{\begin{array}{rll} p &= P_{in/out}(t), & \text{on } (0,T)\times\Gamma_{in/out}, \\ \mathbf{u}\times\mathbf{e}_z &= 0, & \text{on } (0,T)\times\Gamma_{in/out}, \\ \boldsymbol{\eta}(t,0,\theta) &= \boldsymbol{\eta}(t,L,\theta)=0, & \text{on } (0,T)\times(0,2\pi), \\ \partial_z\eta_r(t,0,\theta) &= \partial_z\eta_r(t,L,\theta)=0, & \text{on } (0,T)\times(0,2\pi), \\ \boldsymbol{\eta}(t,z,0) &= \boldsymbol{\eta}(t,z,2\pi), & \text{on } (0,T)\times(0,L), \\ \partial_\theta\eta_r(t,z,0) &= \partial_\theta\eta_r(t,z,2\pi), & \text{on } (0,T)\times(0,L), \end{array}\right. \tag{2.14}$$

$$\begin{gathered} \mathbf{u}(0)=\mathbf{u}_0,\ \boldsymbol{\eta}(0)=\boldsymbol{\eta}_0,\ \partial_t\boldsymbol{\eta}(0)=\mathbf{v}_0, \\ \mathbf{d}_i(0)=\mathbf{d}_{0i},\ \partial_t\mathbf{d}_i(0)=\mathbf{k}_{0i},\ \mathbf{w}_i(0)=\mathbf{w}_{0i},\ \partial_t\mathbf{w}_i(0)=\mathbf{z}_{0i}, \end{gathered} \tag{2.15}$$

where we introduced the following notation for the Koiter shell velocity, the mesh velocity and the rotation velocity, respectively:

$$\mathbf{v}=\partial_t\boldsymbol{\eta},\ \mathbf{k}=\partial_t\mathbf{d},\ \mathbf{z}=\partial_t\mathbf{w}. \tag{2.16}$$

3. Existence of a weak solution

We define the following evolution spaces associated with the fluid problem, the Koiter shell problem, the mesh problem and the coupled mesh-shell problem:

- $V_F(0,T)=L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V_F)$, where V_F is defined by (2.2),
- $V_K(0,T)=W^{1,\infty}(0,T;L^2(R;\omega))\cap L^\infty(0,T;V_K)$, where V_K is defined by (2.3),
- $V_S(0,T)=W^{1,\infty}(0,T;L^2(\mathcal{N}))\cap L^\infty(0,T;V_S)$, where V_S is defined by (2.9),
- $V_{KS}(0,T)=\{(\boldsymbol{\eta},\mathbf{d},\mathbf{w})\in V_K(0,T)\times V_S(0,T):\boldsymbol{\eta}|_{\omega_S}\circ\boldsymbol{\pi}=\mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\}$.

The solution space for the coupled fluid-mesh-shell interaction problem involves the kinematic coupling condition:

$$\mathcal{V}(0,T)=\{(\mathbf{u},\boldsymbol{\eta},\mathbf{d},\mathbf{w})\in V_F(0,T)\times V_{KS}(0,T):\mathbf{u}|_\Gamma\circ\boldsymbol{\varphi}=\partial_t\boldsymbol{\eta} \text{ on } \omega\}.$$

The associated test space is given by:

$$\mathcal{Q}(0,T)=\{(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in C_c^1([0,T);V_F\times V_{KS}):\boldsymbol{\upsilon}|_\Gamma\circ\boldsymbol{\varphi}=\boldsymbol{\psi} \text{ on } \omega\},$$

where $V_{KS}=\{(\boldsymbol{\eta},\mathbf{d},\mathbf{w})\in V_K\times V_S:\boldsymbol{\eta}|_{\omega_S}\circ\boldsymbol{\pi}=\mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\}$.

We now state a definition of weak solutions of our fluid-mesh-shell interaction problem, with the fluid flow in Ω.

Definition 3.1. *We say that* $(\mathbf{u},\boldsymbol{\eta},\mathbf{d},\mathbf{w})\in\mathcal{V}(0,T)$ *is a weak solution of Problem 1 if for all test functions* $(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in\mathcal{Q}(0,T)$ *the following equality* *holds:*

$$
\begin{aligned}
&-\rho_F \int_0^T \int_\Omega \mathbf{u} \cdot \partial_t \boldsymbol{\upsilon} + 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\upsilon}) - \rho_K h \int_0^T \int_\omega \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\psi} R \\
&+ \int_0^T a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) - \rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_t \mathbf{d}_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_t \mathbf{w}_i \cdot \partial_t \boldsymbol{\zeta}_i \\
&+ \int_0^T a_S(\mathbf{w}, \boldsymbol{\zeta}) = \int_0^T \langle F(t), \boldsymbol{\upsilon} \rangle_{\Gamma_{in/out}} + \rho_F \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\upsilon}(0) + \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R \\
&+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0),
\end{aligned}
$$

where

$$
a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) = \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle,
$$

$$
a_S(\mathbf{w}, \boldsymbol{\zeta}) = \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i,
$$

and

$$
\langle F(t), \boldsymbol{\upsilon} \rangle_{\Gamma_{in/out}} = P_{in}(t) \int_{\Gamma_{in}} \upsilon_z - P_{out}(t) \int_{\Gamma_{out}} \upsilon_z.
$$

As we already mentioned, the authors in [12] showed the existence of such a weak solution. For the sake of completeness, we state here the existence theorem.

THEOREM 3.2 ([12]). *Let* $\mathbf{u}_0 \in L^2(\Omega)$, $\boldsymbol{\eta}_0 \in H^1(\omega)$, $\mathbf{v}_0 \in L^2(R;\omega)$, $(\mathbf{d}_0, \mathbf{w}_0) \in V_S$, $(\mathbf{k}_0, \mathbf{z}_0) \in L^2(\mathcal{N}; \mathbb{R}^6)$ *be such that*

$$
\nabla \cdot \mathbf{u}_0 = 0,\ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi}) \cdot \mathbf{e}_r = (\mathbf{v}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.
$$

Furthermore, let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ *and* $A_i > 0, \forall i = 1, \dots, n_E$, *and let* $P_{in/out} \in L^2_{loc}(0, \infty)$. *Then for every* $T > 0$ *there exists a weak solution to Problem 1 in the sense of Definition 3.1.*

The proof method is based on a semi-discretization approach, where the coupled problem is discretized in time, and at the same time split into a fluid and a structure subproblem using the so called Lie operator splitting strategies. The constructed weak solutions are shown to satisfy an energy inequality, uniform estimates are obtained and existence of weak and weak* convergent subsequences established. Since the problem is linear, weak and weak* convergent subsequences are then shown to satisfy the weak formulation of the coupled continuous FSI problem.

We also mention that one could use a slightly different definition of a weak solution which is equivalent to Definition 3.1.

DEFINITION 3.3. *We say that* $(\mathbf{u},\boldsymbol{\eta},\mathbf{d},\mathbf{w}) \in \mathcal{V}(0,T)$ *is a weak solution of Problem 1 if for all test functions* $(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta}) \in \mathcal{V}(0,T)$ *the following equality holds:*

$$\int_0^T \langle(\partial_t\mathbf{u},\partial_{tt}\boldsymbol{\eta},\partial_{tt}\mathbf{d},\partial_{tt}\mathbf{w}),(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\rangle + 2\mu_F\int_0^T\int_\Omega \mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{\upsilon})$$
$$+\int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi}) + \int_0^T a_S(\mathbf{w},\boldsymbol{\zeta}) = \int_0^T \langle F(t),\boldsymbol{\upsilon}\rangle_{\Gamma_{in/out}},$$

where $\langle\cdot,\cdot\rangle$ *denotes the dual product on space* V :

$$V = \Big\{(\mathbf{u},\boldsymbol{\eta},\mathbf{d},\mathbf{w}) \in V_F\times V_K\times V_S : \mathbf{u}|_\Gamma\circ\boldsymbol{\varphi} = \partial_t\boldsymbol{\eta} \text{ on } \omega,$$
$$\boldsymbol{\eta}|_{\omega_S}\circ\boldsymbol{\pi} = \mathbf{d} \text{ on } \prod_{i=1}^{n_E}(0,l_i)\Big\}.$$

We use the previous definition to obtain the regularity of the time derivatives in the dual space V^*. More precisely,

$$\begin{aligned}
&\|(\partial_t\mathbf{u},\partial_{tt}\boldsymbol{\eta},\partial_{tt}\mathbf{d},\partial_{tt}\mathbf{w})\|_{L^2(0,T;V^*)}\\
&= \sup_{\substack{(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in L^2(0,T;V)\\ \|(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|=1}} \Bigg|\rho_F\int_0^T\int_\Omega \partial_t\mathbf{u}\cdot\boldsymbol{\upsilon} + \rho_K h\int_0^T\int_\omega \partial_{tt}\boldsymbol{\eta}\cdot\boldsymbol{\psi}\\
&\quad +\rho_S\int_0^T\sum_{i=1}^{n_E} A_i\int_0^{l_i}\partial_{tt}\mathbf{d}_i\cdot\boldsymbol{\xi}_i + \rho_S\int_0^T\sum_{i=1}^{n_E}\int_0^{l_i} M_i\partial_{tt}\mathbf{w}_i\cdot\boldsymbol{\zeta}_i\Bigg|\\
&= \sup_{\substack{(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in L^2(0,T;V)\\ \|(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|=1}} \Bigg|\int_0^T P_{in}(t)\int_{\Gamma_{in}}\upsilon_z - \int_0^T P_{out}(t)\int_{\Gamma_{out}}\upsilon_z\\
&\quad -2\mu_F\int_0^T\int_\Omega \mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{\upsilon}) - \int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi}) - \int_0^T a_S(\mathbf{w},\boldsymbol{\zeta})\Bigg|\\
&\le \sup_{\substack{(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\in L^2(0,T;V)\\ \|(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})\|=1}} \big(\|P_{in/out}\|_{L^2(0,T)}\|\boldsymbol{\upsilon}\|_{L^2(0,T;H^1(\Omega))}\\
&\quad +2\mu_F\|\mathbf{D}(\mathbf{u})\|_{L^2(0,T;L^2(\Omega))}\|\mathbf{D}(\boldsymbol{\upsilon})\|_{L^2(0,T;L^2(\Omega))}\\
&\quad +\|\boldsymbol{\eta}\|_{L^2(0,T;H^1(\omega))}\|\boldsymbol{\psi}\|_{L^2(0,T;H^1(\omega))}\\
&\quad +\|\mathbf{w}\|_{L^2(0,T;H^1(\mathcal{N}))}\|\boldsymbol{\zeta}\|_{L^2(0,T;H^1(\mathcal{N}))}\big).
\end{aligned}$$

Since the right-hand side is bounded, we see that

$$(\partial_t\mathbf{u},\partial_{tt}\boldsymbol{\eta},\partial_{tt}\mathbf{d},\partial_{tt}\mathbf{w}) \in L^2(0,T;V^*)$$

which implies

$$(\mathbf{u},\partial_t\boldsymbol{\eta},\partial_t\mathbf{d},\partial_t\mathbf{w}) \in H^1(0,T;V^*).$$

Remark 3.4. In particular, by taking $(\mathbf{u}, 0, 0, 0)$, where $\mathbf{u} \in H_0^1(\Omega)$ as a test function, one easily obtains that $\partial_t \mathbf{u} \in L^2(0, T; H^{-1}(\Omega))$.

Proposition 3.5. *A weak solution to Problem 1, whose existence is guaranteed by Theorem 3.2, is unique.*

Remark 3.6. We omit the uniqueness proof and emphasize that one could easily prove it by using the regularization techniques (mollifying the functions in the time variable) which was done in Theorem 4.2. in [28] for the Navier-Stokes equations, and in [43, 45] for FSI problems between incompressible, viscous fluid and rigid body/elastic plate, respectively. For the related result in the context of FSI with compressible fluid see [48]. All the mentioned papers, however, encounter certain difficulties connected with the fact that the fluid domain boundary is moving, and, due to the presence of the convective term in the Navier-Stokes equations, they could only prove a weak-strong uniqueness result. The uniqueness proof in our case is straightforward since we do not have to deal with any of the mentioned difficulties.

4. Main regularity result

After short introduction of the problem, we are now ready to address the question whether a weak solution to problem (2.11)-(2.15) is in fact *more regular* provided the initial and boundary data are so. The following main theorem of this manuscript gives the answer to the raised question.

Theorem 4.1. *Let $P_{in/out} \in H^1_{loc}(0, \infty)$ and let initial conditions*

$$\mathbf{u}_0 \in H^2(\Omega),\ \boldsymbol{\eta}_0 \in H^2(\omega),\ \mathbf{v}_0 \in V_K,\ (\mathbf{d}_0, \mathbf{w}_0) \in H^2(\mathcal{N}; \mathbb{R}^6),\ (\mathbf{k}_0, \mathbf{z}_0) \in V_S,$$

be such that

$$\nabla \cdot \mathbf{u}_0 = 0,\ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi}) \cdot \mathbf{e}_r = (\partial_t \boldsymbol{\eta}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.$$

Furthermore, let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ and $A_i > 0, \forall i = 1, \dots, n_E$.

- *The weak solution to Problem 1, whose existence is guaranteed by Theorem 3.2, belongs to the following function spaces:*

$$\begin{aligned} &\mathbf{u} \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; V_F), \\ &\boldsymbol{\eta} \in W^{2,\infty}(0, T; L^2(R; \omega)) \cap W^{1,\infty}(0, T; V_K), \\ &(\mathbf{d}, \mathbf{w}) \in W^{2,\infty}(0, T; L^2(\mathcal{N})) \cap W^{1,\infty}(0, T; V_S). \end{aligned}$$

- *For each $i = 1, \dots, n_E$, fix any open set $I_i \subset\subset (0, l_i)$ and choose an open set J_i such that $I_i \subset\subset J_i \subset\subset (0, l_i)$. Then select a smooth cut-off function χ_i which satisfies $0 \le \chi_i \le 1$ and*

$$\begin{cases} \chi_i = 1 \ \text{on } I_i, \\ \chi_i = 0 \ \text{on } \mathbb{R} \backslash J_i. \end{cases}$$

Set $\chi := \prod_{i=1}^{n_E} \chi_i$ *and let* $\bar{\chi}$ *be the extension of* $\chi \circ \boldsymbol{\pi}^{-1}$ *in the interior of the shell domain* ω *such that* $\bar{\chi} = 0$ *on* $\partial\omega$ *and let* $\tilde{\chi}$ *be the extension of* $\bar{\chi} \circ \boldsymbol{\varphi}^{-1}$ *in the interior of the fluid domain* Ω *such that* $\tilde{\chi} = 0$ *on* $\Gamma_{in/out}$*. Then the weak solution to Problem 1, provided by Theorem 3.2, possesses an additional regularity in s-direction, where* $s \in (0, l_i), i = 1, \dots, n_E$*, in the following sense:*

$$\tilde{\chi}\partial_{ss}\mathbf{u} \in L^2(0,T;L^2(\Omega)),\ \bar{\chi}\partial_s\boldsymbol{\eta} \in L^\infty(0,T;V_K),$$
$$\chi\partial_{ss}\mathbf{d} \in L^\infty(0,T;L^2(\mathcal{N})),\ \chi\partial_{ss}\mathbf{w} \in L^\infty(0,T;L^2(\mathcal{N})),$$

where $\partial_{ss}\mathbf{u}$ *and* $\partial_s\boldsymbol{\eta}$ *are given by* (6.11) *and* (6.12)*, respectively.*

- *Fix any open set* $\omega_0 \subset\subset \omega\backslash(\partial\omega \cup \omega_S)$ *and choose an open set* ω_1 *such that* $\omega_0 \subset\subset \omega_1 \subset\subset \omega\backslash(\partial\omega \cup \omega_S)$*. Then select a smooth function* χ *satisfying* $0 \le \chi \le 1$ *and*

$$\begin{cases} \chi = 1 \ \text{on } \omega_0, \\ \chi = 0 \ \text{on } \mathbb{R}^2\backslash\omega_1. \end{cases}$$

Next, let $\tilde{\chi}$ *be the extension of* $\chi \circ \boldsymbol{\varphi}^{-1}$ *in the interior of the fluid domain and such that* $\tilde{\chi} = 0$ *on* $\Gamma_{in/out}$*. Then the fluid velocity as well as the shell displacement have an additional regularity in all directions:*

$$\tilde{\chi}\Delta\mathbf{u} \in L^2(0,T;L^2(\Omega)),$$
$$\chi\nabla\boldsymbol{\eta} \in L^\infty(0,T;V_K).$$

The following sections are dedicated to proving this theorem. First we deal with the time regularity, and after that with the space regularity.

5. Time regularity

5.1. *Motivation and formal estimates.* Formal energy estimates show that taking $(\mathbf{u}, \partial_t\boldsymbol{\eta}, \partial_t\mathbf{d}, \partial_t\mathbf{w})$ as a test function in the full, coupled problem, leads to the following regularity of the solution:

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_F),$$
$$\boldsymbol{\eta} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V_K),$$
$$(\mathbf{d},\mathbf{w}) \in W^{1,\infty}(0,T;L^2(\mathcal{N})) \cap L^\infty(0,T;H^1(\mathcal{N})).$$

To obtain an additional time regularity, the natural step would be to take $(\partial_t\mathbf{u}, \partial_{tt}\boldsymbol{\eta}, \partial_{tt}\mathbf{d}, \partial_{tt}\mathbf{w})$ as a test function. For the fluid, we have the unsteady Stokes equations for the incompressible viscous fluid:

$$\begin{cases} \rho_F\partial_t\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0. \end{cases}$$

Multiplying the first equation by $\partial_t\mathbf{u}$ and integrating over Ω, we obtain estimates on $\|\partial_t\mathbf{u}\|^2_{L^2(\Omega)}$ and on $\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}$. Notice that we are not considering boundary terms (which arise after integration by parts). Next, take $\partial_{tt}\boldsymbol{\eta}$ as a test function in the shell equation

$$\rho_K h \partial_{tt}\boldsymbol{\eta} + \mathcal{L}\boldsymbol{\eta} = \mathbf{f}$$

and integrate over ω. From the first term, we get the estimate on $\|\partial_{tt}\boldsymbol{\eta}\|^2_{L^2(R;\omega)}$, while from the second term on the left-hand side we obtain:

$$\int_\omega \mathcal{L}\boldsymbol{\eta} \cdot \partial_{tt}\boldsymbol{\eta} = -\int_\omega \mathcal{L}\partial_t\boldsymbol{\eta} \cdot \partial_t\boldsymbol{\eta} = -\langle \mathcal{L}\partial_t\boldsymbol{\eta}, \partial_t\boldsymbol{\eta}\rangle \leq -\|\partial_t\boldsymbol{\eta}\|_k,$$

where in the last inequality we used the coercivity of the operator $\mathcal{L}$.

The problem we encounter here is the "wrong sign" in front of the elastic term so we can not get the bound on $\|\partial_t\boldsymbol{\eta}\|_k$. This mismatch arises due to *parabolic-hyperbolic nature* of the fluid and the shell coupling, namely, even though the pair $(\partial_t\mathbf{u}, \partial_{tt}\boldsymbol{\eta})$ is an admissible test function, it is not appropriate to get the wanted estimates. Analogously, due to the hyperbolic nature of the mesh equations, we can not get the right bounds on $\|\partial_t\mathbf{d}\|_{H^1(\mathcal{N})}$ and $\|\partial_t\mathbf{w}\|_{H^1(\mathcal{N})}$.

In order to justify the following calculations, we assume that our solution is smooth enough, and first differentiate the fluid, the shell and the mesh equations with respect to t, multiply the obtained equations by $\partial_t\mathbf{u}, \partial_{tt}\boldsymbol{\eta}, \partial_{tt}\mathbf{d}, \partial_{tt}\mathbf{w}$ and integrate over Ω, ω and $(0, l_i), i = 1, \ldots, n_E$, respectively. Integrating by parts, and enforcing the kinematic and dynamic boundary conditions on ω, we obtain:

$$\begin{aligned}
&\frac{\rho_F}{2}\frac{d}{dt}\|\partial_t\mathbf{u}\|^2_{L^2(\Omega)} + 2\mu_F\|\mathbf{D}(\partial_t\mathbf{u})\|^2_{L^2(\Omega)} + \frac{\rho_K h}{2}\frac{d}{dt}\|\partial_{tt}\boldsymbol{\eta}\|^2_{L^2(R;\omega)} \\
&+ \frac{1}{2}\frac{d}{dt}\langle \mathcal{L}\partial_t\boldsymbol{\eta}, \partial_t\boldsymbol{\eta}\rangle + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E} A_i\|\partial_{tt}\mathbf{d}_i\|^2_{L^2(0,l_i)} + \frac{\rho_S}{2}\frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_{tt}\mathbf{w}_i\|^2_m \\
&+ \frac{d}{dt}\sum_{i=1}^{n_E}\|\partial_t\mathbf{w}_i\|^2_r = \int_{\Gamma_{in}} \partial_t p \; \partial_t u_z - \int_{\Gamma_{out}} \partial_t p \; \partial_t u_z,
\end{aligned} \tag{5.1}$$

where $\|\mathbf{w}\|_m$ and $\|\mathbf{w}\|_r$ denote the following norms associated with the elastic energy of the elastic mesh:

$$\|\mathbf{w}\|^2_m := \sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_m = \sum_{i=1}^{n_E}\int_0^{l_i} M_i\mathbf{w}_i \cdot \mathbf{w}_i,$$

$$\|\mathbf{w}\|^2_r := \sum_{i=1}^{n_E}\|\mathbf{w}_i\|^2_r = \sum_{i=1}^{n_E}\int_0^{l_i} Q_i H_i Q_i^T \partial_s\mathbf{w}_i \cdot \partial_s\mathbf{w}_i,$$

and $\|\boldsymbol{\eta}\|_{L^2(R;\omega)}$ denotes the weighted L^2-norm on ω, with the weight R associated with the geometry of the Koiter shell (Jacobian):

$$\|\boldsymbol{\eta}\|^2_{L^2(R;\omega)} := \int_\omega |\boldsymbol{\eta}|^2 R \; d\omega.$$

We can estimate the right-hand side of (5.1) by using the trace theorem, Korn's inequality and Cauchy inequality with ε, and then integrate from 0 to T to see that the left-hand side is bounded by the norms of initial data and inlet/outlet pressure. Provided that initial and boundary data have the needed regularity, we conclude that our solution indeed possesses an additional regularity in time variable:

$$\begin{aligned}
&\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;V_F),\\
&\boldsymbol{\eta} \in W^{2,\infty}(0,T;L^2(R;\omega)) \cap W^{1,\infty}(0,T;V_K),\\
&(\mathbf{d},\mathbf{w}) \in W^{2,\infty}(0,T;L^2(\mathcal{N})) \cap W^{1,\infty}(0,T;H^1(\mathcal{N})).
\end{aligned}$$

Previous formal calculations do not really constitute a proof since our solution is not smooth enough to be used as a test function, but they suggest that the weak solution may indeed be more regular provided initial and boundary data are so. Next section is devoted to rigorous justification of those estimates.

5.2. *Estimates by difference quotients.* We begin by recalling the definition of the difference quotients. Let $u : U \to \mathbb{R}$ be locally summable function and $V \subset\subset U$.

Definition 5.1 ([27]). *The i-th* difference quotient *of size h is*

$$D_i^h u(x) = \frac{u(x+he_i)-u(x)}{h},\ i = 1,\dots,n,$$

for $x \in V$ *and* $h \in \mathbb{R}, 0 < |h| < \mathrm{dist}(V,\partial U)$. *We set* $D^h u := (D_1^h u,\dots,D_n^h u)$.

Theorem 5.2 ([27]). (i) *Suppose* $1 \le p < \infty$ *and* $u \in W^{1,p}(U)$. *Then for each* $V \subset\subset U$

$$\|D^h u\|_{L^p(V)} \le C\|\nabla u\|_{L^p(U)}$$

for some constant C *and all* $0 < |h| < \frac{1}{2}\mathrm{dist}(V,\partial U)$.

(ii) *Assume* $1 < p < \infty, u \in L^p(V)$ *and there exists a constant* C *such that*

$$\|D^h u\|_{L^p(V)} \le C$$

for all $0 < |h| < \frac{1}{2}\mathrm{dist}(V,\partial U)$. *Then*

$$u \in W^{1,p}(V),\ \text{with } \|\nabla u\|_{L^p(V)} \le C.$$

It is also useful to state some properties of the difference quotients which will be used frequently throughout the rest of the manuscript:

$$\text{(i)}\quad \int_U u D_k^{-h} v\, dx = -\int_U (D_k^h u) v\, dx,\ \text{where } v \in C_c^\infty(U), \tag{5.2}$$

$$\text{(ii)}\quad D_k^h(uv) = u^h D_k^h v + v D_k^h u,\ \text{where } u^h(x) = u(x+he_k). \tag{5.3}$$

In this section we are dealing with the time regularity of our weak solution so we define the time difference quotients in the following way:

$$D^{\Delta t}\mathbf{u}(t,\mathbf{x}) = \frac{\mathbf{u}(t+\Delta t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})}{\Delta t}, \tag{5.4}$$

where $\Delta t > 0$. Having (5.4) in mind, we define the test functions for our fluid-composite structure interaction problem as follows:

$$\begin{aligned} \boldsymbol{v} = -D^{-\Delta t}(D^{\Delta t}\mathbf{u}), \quad \boldsymbol{\psi} = -D^{-\Delta t}(D^{\Delta t}\partial_t\boldsymbol{\eta}), \\ \boldsymbol{\xi} = -D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{d}), \quad \boldsymbol{\zeta} = -D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{w}), \end{aligned} \tag{5.5}$$

where $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ is a weak solution to Problem 1.

Remark 5.3. Unfortunately, $\partial_t\boldsymbol{\eta}, \partial_t\mathbf{d}$ and $\partial_t\mathbf{w}$ are not regular enough to be used as test functions. For that reason, we follow the *artificial viscoelasticity approach* to justify the formal estimates. Namely, we add viscoelastic terms $\varepsilon\Delta\partial_t\boldsymbol{\eta}, \varepsilon\partial_{ss}\partial_t\mathbf{d}$ and $\varepsilon\partial_{ss}\partial_t\mathbf{w}$ in the shell and mesh equations, with ε being a regularization parameter. From the energy estimates we then obtain that $\partial_t\boldsymbol{\eta}_\varepsilon \in L^2(0,T;V_K)$ and $(\partial_t\mathbf{d}_\varepsilon, \partial_t\mathbf{w}_\varepsilon) \in L^2(0,T;V_S)$ so the finite differences involving these solutions can be used as test functions. The obtained regularity estimates are uniform in the viscoelasticity parameter and therefore are also valid for the limiting solution. Finally, from the uniqueness (Proposition 3.5), we can conclude that regularity estimates hold for the weak solution to the original problem.

We multiply the fluid equation by $\boldsymbol{v}$ defined above, and integrate over Ω and $(0,T)$ to see that:

$$\begin{aligned} \rho_F \int_0^T \int_\Omega \partial_t\mathbf{u}\cdot\boldsymbol{v} &= -\rho_F \int_0^T \int_\Omega \partial_t\mathbf{u}\cdot D^{-\Delta t}(D^{\Delta t}\mathbf{u}) \\ &= \rho_F \int_0^T \int_\Omega D^{\Delta t}\partial_t\mathbf{u}\cdot D^{\Delta t}\mathbf{u} = \rho_F \int_0^T \int_\Omega \frac{1}{2}\frac{d}{dt}|D^{\Delta t}\mathbf{u}|^2 \\ &= \frac{\rho_F}{2}\int_0^T \frac{d}{dt}\|D^{\Delta t}\mathbf{u}\|^2_{L^2(\Omega)}, \end{aligned}$$

$$\begin{aligned} 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v}) &= -2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}):\mathbf{D}(D^{-\Delta t}(D^{\Delta t}\mathbf{u})) \\ &= 2\mu_F \int_0^T \int_\Omega D^{\Delta t}\mathbf{D}(\mathbf{u}):D^{\Delta t}\mathbf{D}(\mathbf{u}) \\ &= 2\mu_F \int_0^T \|D^{\Delta t}\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}, \end{aligned}$$

where we have used

$$D^{\Delta t}\partial_t \mathbf{u}(t,\mathbf{x}) = \partial_t D^{\Delta t}\mathbf{u}(t,\mathbf{x}),$$
$$D^{\Delta t}\mathbf{D}(\mathbf{u}(t,\mathbf{x})) = \mathbf{D}(D^{\Delta t}\mathbf{u}(t,\mathbf{x})).$$

Recall that $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$.

We next multiply the shell equation by $\boldsymbol{\psi}$ defined in (5.5), and integrate over ω and $(0,T)$ to see that:

$$\begin{aligned}\rho_K h\int_0^T\int_\omega \partial_{tt}\boldsymbol{\eta}\cdot\boldsymbol{\psi}R &= -\rho_K h\int_0^T\int_\omega \partial_{tt}\boldsymbol{\eta}\cdot D^{-\Delta t}(D^{\Delta t}\partial_t\boldsymbol{\eta})R\\ &= \rho_K h\int_0^T\int_\omega \partial_t D^{\Delta t}\partial_t\boldsymbol{\eta}\cdot D^{\Delta t}\partial_t\boldsymbol{\eta}R\\ &= \frac{\rho_K h}{2}\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\boldsymbol{\eta}\|^2_{L^2(R;\omega)},\end{aligned}$$

$$\begin{aligned}\int_0^T a_K(\boldsymbol{\eta},\boldsymbol{\psi}) = \int_0^T\langle\mathcal{L}\boldsymbol{\eta},\boldsymbol{\psi}\rangle &= \int_0^T\langle\mathcal{L}\boldsymbol{\eta}, -D^{-\Delta t}(D^{\Delta t}\partial_t\boldsymbol{\eta})\rangle\\ = \int_0^T\langle\mathcal{L}(D^{\Delta t}\boldsymbol{\eta}),\partial_t(D^{\Delta t}\boldsymbol{\eta})\rangle &= \frac{1}{2}\int_0^T\frac{d}{dt}\langle\mathcal{L}(D^{\Delta t}\boldsymbol{\eta}), D^{\Delta t}\boldsymbol{\eta}\rangle.\end{aligned}$$

Remark 5.4. The coercivity of the operator $\mathcal{L}$ will enable us to get a bound on $\|D^{\Delta t}\boldsymbol{\eta}\|_k$.

Next, we deal with the inertial term involving mesh displacement:

$$\begin{aligned}\rho_S\sum_{i=1}^{n_E}A_i\int_0^T\int_0^{l_i}\partial_{tt}\mathbf{d}_i\cdot\boldsymbol{\xi}_i &= -\rho_S\sum_{i=1}^{n_E}A_i\int_0^T\int_0^{l_i}\partial_{tt}\mathbf{d}_i\cdot D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{d}_i)\\ &= \rho_S\sum_{i=1}^{n_E}A_i\int_0^T\int_0^{l_i}\partial_t(D^{\Delta t}\partial_t\mathbf{d}_i)\cdot D^{\Delta t}\partial_t\mathbf{d}_i\\ &= \frac{\rho_S}{2}\sum_{i=1}^{n_E}A_i\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\mathbf{d}_i\|^2_{L^2(0,l_i)},\end{aligned}$$

as well as the inertial term involving infinitesimal rotation:

$$\rho_S\sum_{i=1}^{n_E}\int_0^T\int_0^{l_i}M_i\partial_{tt}\mathbf{w}_i\cdot\boldsymbol{\zeta}_i = \frac{\rho_S}{2}\sum_{i=1}^{n_E}\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\mathbf{w}_i\|^2_m.$$

Finally, the elastic part of the mesh equations can be rewritten as follows:

$$\begin{aligned}\int_0^T a_S(\mathbf{w},\boldsymbol{\zeta}) &= \sum_{i=1}^{n_E}\int_0^T\int_0^{l_i} Q_iH_iQ_i^T\partial_s\mathbf{w}_i\cdot\partial_s\boldsymbol{\zeta}_i\\ &= -\sum_{i=1}^{n_E}\int_0^T\int_0^{l_i} Q_iH_iQ_i^T\partial_s\mathbf{w}_i\cdot\partial_s D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{w}_i)\\ &= \sum_{i=1}^{n_E}\int_0^T\int_0^{l_i} Q_iH_iQ_i^T D^{\Delta t}\partial_s\mathbf{w}_i\cdot\partial_t(D^{\Delta t}\partial_s\mathbf{w}_i)\\ &= \frac{1}{2}\sum_{i=1}^{n_E}\int_0^T\frac{d}{dt}\|D^{\Delta t}\mathbf{w}_i\|_r^2.\end{aligned}$$

By enforcing the kinematic and dynamic boundary conditions on Γ (the details can be found in [12]), one obtains the following equality:

$$\begin{aligned}&\frac{\rho_F}{2}\int_0^T\frac{d}{dt}\|D^{\Delta t}\mathbf{u}\|_{L^2(\Omega)}^2 + 2\mu_F\int_0^T\|D^{\Delta t}\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}^2\\ &+\frac{\rho_K h}{2}\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\boldsymbol{\eta}\|_{L^2(R;\omega)}^2 + \frac{1}{2}\int_0^T\frac{d}{dt}\langle\mathcal{L}(D^{\Delta t}\boldsymbol{\eta}),D^{\Delta t}\boldsymbol{\eta}\rangle\\ &+\frac{\rho_S}{2}\sum_{i=1}^{n_E}A_i\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\mathbf{d}_i\|_{L^2(0,l_i)}^2 + \frac{\rho_S}{2}\sum_{i=1}^{n_E}\int_0^T\frac{d}{dt}\|D^{\Delta t}\partial_t\mathbf{w}_i\|_m^2\\ &+\frac{1}{2}\sum_{i=1}^{n_E}\int_0^T\frac{d}{dt}\|D^{\Delta t}\mathbf{w}_i\|_r^2 = \int_0^T\int_{\Gamma_{in}}D^{\Delta t}pD^{\Delta t}u_z - \int_0^T\int_{\Gamma_{out}}D^{\Delta t}pD^{\Delta t}u_z.\end{aligned}$$

The right-hand side is equal to:

$$\int_0^T\int_{\Gamma_{in}}D^{\Delta t}P_{in}(t)D^{\Delta t}u_z - \int_0^T\int_{\Gamma_{out}}D^{\Delta t}P_{out}(t)D^{\Delta t}u_z,$$

which we estimate using the trace theorem, Korn's inequality (Theorem 6.3-4 in [19]) and Cauchy inequality:

$$\begin{aligned}\left|\int_{\Gamma_{in/out}}D^{\Delta t}P_{in/out}D^{\Delta t}u_z\right| &\le C\left|D^{\Delta t}P_{in/out}\right|\|D^{\Delta t}\mathbf{u}\|_{H^1(\Omega)}\\ &\le C|\partial_t P_{in/out}|\|\mathbf{D}(D^{\Delta t}\mathbf{u})\|_{L^2(\Omega)}\\ &\le \frac{C}{2\varepsilon}|\partial_t P_{in/out}|^2 + \frac{C\varepsilon}{2}\|D^{\Delta t}\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}^2.\end{aligned}$$

Evaluating all the integrals in time and using the above estimate for the right-hand side, one obtains:

$$\begin{aligned}
&\rho_F\|D^{\Delta t}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} + 4\mu_F\|D^{\Delta t}\mathbf{D}(\mathbf{u})\|_{L^2(0,T;L^2(\Omega))}\\
&\quad + \rho_K h\|D^{\Delta t}\partial_t\boldsymbol{\eta}\|_{L^\infty(0,T;L^2(R;\omega))} + \|D^{\Delta t}\boldsymbol{\eta}\|_{L^\infty(0,T;V_K)}\\
&\quad + \rho_S\|D^{\Delta t}\partial_t\mathbf{d}\|_{L^\infty(0,T;L^2(\mathcal{N}))} + \rho_S\|D^{\Delta t}\partial_t\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))}\\
&\quad + \|D^{\Delta t}\partial_s\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))}\\
&\le \rho_F\|D^{\Delta t}\mathbf{u}_0\|^2_{L^2(\Omega)} + \rho_K h\|D^{\Delta t}\mathbf{v}_0\|^2_{L^2(R;\omega)} + \|D^{\Delta t}\boldsymbol{\eta}_0\|^2_k\\
&\quad + \rho_S\|D^{\Delta t}\mathbf{k}_0\|^2_{L^2(\mathcal{N})} + \rho_S\|D^{\Delta t}\mathbf{z}_0\|^2_{L^2(\mathcal{N})} + \|D^{\Delta t}\partial_s\mathbf{w}_0\|^2_{L^2(\mathcal{N})}\\
&\quad + \frac{C}{\varepsilon}\|P_{in/out}\|^2_{H^1(0,T)} + C\varepsilon\|D^{\Delta t}\mathbf{D}(\mathbf{u})\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

By choosing ε such that $C\varepsilon \le 4\mu_F$, the symmetrized gradient term can be absorbed into the left-hand side. Using the property (ii) of difference quotients stated in Theorem 5.2, we obtain:

$$\begin{aligned}
&\|D^{\Delta t}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \le C \Longrightarrow \|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \le C,\\
&\|D^{\Delta t}\mathbf{D}(\mathbf{u})\|_{L^2(0,T;L^2(\Omega))} \le C \Longrightarrow \|\partial_t\mathbf{D}(\mathbf{u})\|_{L^2(0,T;L^2(\Omega))} \le C,\\
&\|D^{\Delta t}\partial_t\boldsymbol{\eta}\|_{L^\infty(0,T;L^2(R;\omega))} \le C \Longrightarrow \|\partial_{tt}\boldsymbol{\eta}\|_{L^\infty(0,T;L^2(R;\omega))} \le C,\\
&\|D^{\Delta t}\boldsymbol{\eta}\|_{L^\infty(0,T;V_K)} \le C \Longrightarrow \|\partial_t\boldsymbol{\eta}\|_{L^\infty(0,T;V_K)} \le C,\\
&\|D^{\Delta t}\partial_t\mathbf{d}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C \Longrightarrow \|\partial_{tt}\mathbf{d}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C,\\
&\|D^{\Delta t}\partial_t\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C \Longrightarrow \|\partial_{tt}\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C,\\
&\|D^{\Delta t}\partial_s\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C \Longrightarrow \|\partial_t\partial_s\mathbf{w}\|_{L^\infty(0,T;L^2(\mathcal{N}))} \le C,
\end{aligned} \tag{5.6}$$

provided that initial data satisfy the following:

$$\mathbf{u}_0 \in H^2(\Omega),\ \boldsymbol{\eta}_0 \in V_K,\ \mathbf{v}_0 \in V_K,\ (\mathbf{d}_0, \mathbf{w}_0) \in V_S,\ (\mathbf{k}_0, \mathbf{z}_0) \in V_S$$

together with the compatibility conditions:

$$\nabla\cdot\mathbf{u}_0 = 0,\ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi})\cdot\mathbf{e}_r = (\mathbf{v}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.$$

For the inlet and outlet pressure we demand $P_{in/out} \in H^1_{loc}(0,\infty)$. Notice how we did not obtain the estimate on $\partial_t\partial_s\mathbf{d}$. In order to achieve that, we use the condition of inextensibility and unshearability:

$$\|\partial_t\partial_s\mathbf{d}\|_{L^2(\mathcal{N})} = \|-\partial_t(\mathbf{t}\times\mathbf{w})\|_{L^2(\mathcal{N})} \le C(\|\mathbf{w}\|_{L^2(\mathcal{N})} + \|\partial_t\mathbf{w}\|_{L^2(\mathcal{N})}).$$

We now summarize the time regularity results obtained in this section:

$$\begin{aligned}
&\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;V_F),\\
&\boldsymbol{\eta} \in W^{2,\infty}(0,T;L^2(R;\omega)) \cap W^{1,\infty}(0,T;V_K),\\
&(\mathbf{d},\mathbf{w}) \in W^{2,\infty}(0,T;L^2(\mathcal{N})) \cap W^{1,\infty}(0,T;V_S).
\end{aligned} \tag{5.7}$$

6. Spatial regularity

6.1. *Formal estimates.*

Remark 6.1. For the simplicity of notation, from now on, $\mathbf{u}$ will represent the fluid velocity function written in cylindrical coordinates (z, r, θ).

Just like in the previous section, we begin by derivation of formal estimates. One could naively take $(-\Delta\mathbf{u}, -\Delta\partial_t\boldsymbol{\eta}, -\Delta\partial_t\mathbf{d}, -\Delta\partial_t\mathbf{w})$ as a test function, where

$$\begin{aligned}\Delta\mathbf{u}(z,r,\theta) &= (\Delta u_z(z,r,\theta), \Delta u_r(z,r,\theta), \Delta u_\theta(z,r,\theta))\\ &= (\partial_{zz}u_z + \partial_{rr}u_z + \partial_{\theta\theta}u_z, \partial_{zz}u_r + \partial_{rr}u_r + \partial_{\theta\theta}u_r,\\ &\quad \partial_{zz}u_\theta + \partial_{rr}u_\theta + \partial_{\theta\theta}u_\theta)\end{aligned}$$

and

$$\begin{aligned}\Delta\partial_t\boldsymbol{\eta}(z,\theta) &= (\Delta\partial_t\eta_z(z,\theta), \Delta\partial_t\eta_r(z,\theta), \Delta\partial_t\eta_\theta(z,\theta))\\ &= (\partial_{zz}\partial_t\eta_z + \partial_{\theta\theta}\partial_t\eta_z, \partial_{zz}\partial_t\eta_r + \partial_{\theta\theta}\partial_t\eta_r, \partial_{zz}\partial_t\eta_\theta + \partial_{\theta\theta}\partial_t\eta_\theta).\end{aligned}$$

The problem that we encounter here is non-compatibility of the test functions, i.e. $\Delta\mathbf{u} \neq \Delta\partial_t\boldsymbol{\eta}$ on Γ. In what follows, we develop ideas to overcome those difficulties, which will then be justified by rigorous calculations using difference quotients.

Step 1: Fluid interior regularity

For the fluid test function we take $-\chi\Delta\mathbf{u}$, where χ is a smooth cut-off function which has support in the interior of the fluid domain. Its purpose is to restrict all expressions to the subset $\Omega_0 \subseteq \Omega$ which has a positive distance from $\partial\Omega$. Taking $(-\chi\Delta\mathbf{u}, 0, 0, 0)$ as a test function for the full, coupled problem, meaning that in this first step we exclude the elastic, composite structure which coincides with the fluid domain boundary, and integrating over Ω and $(0, T)$ we obtain the estimates on

$$\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega_0))} \text{ and } \|\Delta\mathbf{u}\|_{L^2(0,T;L^2(\Omega_0))},$$

i.e. we get, provided that $\mathbf{u}_0 \in H^1(\Omega)$, an additional interior regularity for the fluid velocity

$$\mathbf{u} \in L^\infty(0,T;H^1(\Omega_0)) \cap L^2(0,T;H^2(\Omega_0)).$$

Step 2: Shell interior regularity

Recall that ω_S denotes the reference configuration of the mesh defined on ω, namely $\omega_S = \bigcup_{i=1}^{n_E} \boldsymbol{\pi}_i([0, l_i])$. Following the ideas from the first step, we now exclude the mesh from calculations by taking $\boldsymbol{\psi} = -\chi\Delta\partial_t\boldsymbol{\eta}$, where χ is a smooth cut-off function which has support in $\omega\backslash(\partial\omega \cup \omega_S)$, as a test function in the shell equation, and $-\tilde{\chi}\Delta\mathbf{u}$, where $\tilde{\chi}$ is the extension of $\chi \circ \boldsymbol{\varphi}^{-1}$ in the interior of the fluid domain, as a test function in the fluid equations. As we already noticed these two test functions are non-compatible, since the shell displacement does not depend on r. For that reason, we have to take slightly modified test function for the fluid part, namely:

$$\boldsymbol{v} = \tilde{\chi}(-\partial_{zz}u_{zz} - \partial_{\theta\theta}u_{zz}, -\partial_{zz}u_{rr} - \partial_{\theta\theta}u_{rr}, -\partial_{zz}u_{\theta\theta} - \partial_{\theta\theta}u_{\theta\theta}).$$

Notice how this test function will give us an additional fluid regularity only in the tangential and azimuthal direction; for the radial direction we will get no information.

Finally, taking $(\boldsymbol{v}, \boldsymbol{\psi}, 0, 0)$ as a test function in the full coupled problem, we obtain the following:

- from the inertial term in the Koiter shell equations we get the estimate on $\|\nabla\partial_t\boldsymbol{\eta}\|_{L^\infty(0,T;L^2(\omega_0))}$, i.e. $\boldsymbol{\eta} \in W^{1,\infty}(0,T;H^1(\omega_0))$, where $\omega_0 \subset\subset \omega$;
- the elastic term can be rewritten as

  $$a_K(\boldsymbol{\eta}, -\Delta\partial_t\boldsymbol{\eta}) = a_K(\nabla\boldsymbol{\eta}, \partial_t\nabla\boldsymbol{\eta}) = \frac{d}{dt}a_K(\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\eta}) = \frac{d}{dt}\langle\mathcal{L}\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\eta}\rangle$$

  so by using the coercivity of the elastic operator $\mathcal{L}$ one easily obtains that

  $$\boldsymbol{\eta} \in L^\infty(0,T;H^2(\omega_0));$$
- for the fluid velocity, we obtain an additional regularity in z-direction and in θ-direction;
- an additional regularity of the fluid velocity in radial direction is obtained by using the Stokes equation.

Notice how we tacitly used the additional regularity of the initial data, namely, none of these estimates would hold if we had no the following assumptions:

$$\mathbf{u}_0 \in H^1(\Omega),\ \boldsymbol{\eta}_0 \in H^2(\omega),\ \mathbf{v}_0 \in V_K.$$

Step 3: Mesh interior regularity

In this step we calculate the mesh interior regularity (by excluding the mesh vertices), i.e. we calculate the coupled fluid-shell problem up to the boundary, i.e. up to the mesh vertices. As before, we do not have the compatibility of the test functions. The mesh consists of n_E struts, and on each strut we have the pairs of the test functions

$$(-\Delta\partial_t\mathbf{d}_i, -\Delta\partial_t\mathbf{w}_i) = (-\partial_{ss}\partial_t\mathbf{d}_i, -\partial_{ss}\partial_t\mathbf{w}_i),\ i = 1, \dots, n_E.$$

Since on $\prod_{i=1}^{n_E}(0, l_i)$ the shell displacement is equal to the mesh displacement, i.e. $\boldsymbol{\eta}|_{\omega_S} \circ \boldsymbol{\pi}_i = \mathbf{d}_i$, it follows that the corresponding test function for the shell problem should be $-\partial_{ss}\partial_t\boldsymbol{\eta}$, which we can explicitly calculate:

$$\begin{aligned}\partial_{ss}\partial_t\boldsymbol{\eta} &= \partial_{ss}\partial_t\boldsymbol{\eta}(t,z,\theta) = \partial_{ss}\partial_t\boldsymbol{\eta}(t,\boldsymbol{\pi}_i(s)) = \partial_{ss}\partial_t\boldsymbol{\eta}\cdot(\boldsymbol{\pi}_i')^2 + \partial_s\partial_t\boldsymbol{\eta}\cdot\boldsymbol{\pi}_i'' \\ &= \Delta\partial_t\boldsymbol{\eta}\cdot(\boldsymbol{\pi}_i'\circ\boldsymbol{\pi}_i^{-1})^2 + \nabla\partial_t\boldsymbol{\eta}\cdot(\boldsymbol{\pi}_i''\circ\boldsymbol{\pi}_i^{-1}).\end{aligned}$$

Similarly, by using the kinematic coupling condition at the fluid-shell interface $\mathbf{u}|_\Gamma \circ \boldsymbol{\varphi} = \partial_t \boldsymbol{\eta}$, we can calculate the fluid test function $-\partial_{ss}\mathbf{u}$:

$$\begin{aligned}\partial_{ss}\mathbf{u} &= \partial_{ss}\mathbf{u}(t,z,r,\theta) = \partial_{ss}\partial_t\boldsymbol{\eta}(t,\boldsymbol{\varphi}(z,\theta)) = \partial_{ss}\partial_t\boldsymbol{\eta}(t,\boldsymbol{\varphi}(\boldsymbol{\pi}_i(s))) \\ &= \Delta\mathbf{u}\cdot((\boldsymbol{\varphi}(\boldsymbol{\pi}_i))'\circ\boldsymbol{\pi}_i^{-1}(\boldsymbol{\varphi}^{-1}))^2 + \nabla\mathbf{u}\cdot((\boldsymbol{\varphi}(\boldsymbol{\pi}_i))''\circ\boldsymbol{\pi}_i^{-1}(\boldsymbol{\varphi}^{-1})).\end{aligned}$$

Just like in Step 1 and Step 2, we have to move away from the boundary, i.e. mesh vertices, so we multiply the test functions with appropriate smooth cut-off functions (details will be presented in the next section).

Finally, by using $(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta})$, where

$$\boldsymbol{\upsilon} = -\tilde{\chi}\partial_{ss}\mathbf{u}, \quad \boldsymbol{\psi} = -\bar{\chi}\partial_{ss}\partial_t\boldsymbol{\eta}, \quad \boldsymbol{\xi} = -\chi\partial_{ss}\partial_t\mathbf{d}, \quad \boldsymbol{\zeta} = -\chi\partial_{ss}\partial_t\mathbf{w}.$$

as a test function in the full, coupled problem, we obtain:

- an additional fluid velocity regularity in s-direction;
- an additional shell displacement regularity in s-direction;
- $\mathbf{d}_i \in L^\infty(0,T;H^2(I_i))$, $\mathbf{w}_i \in L^\infty(0,T;H^2(I_i))$, where $I_i \subset\subset (0,l_i)$, $i = 1,\ldots,n_E$.

The ideas we presented here as well as the formal estimates we obtained do not really constitute a proof since they have to be rigorously justified.

6.2. *Estimates by difference quotients.*

Step 1: Fluid interior regularity

Fix any open set $\Omega_0 \subset\subset \Omega$ and choose an open set Ω_1 such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. Then select a smooth function χ satisfying $0 \le \chi \le 1$ and

$$\begin{cases} \chi = 1 \text{ on } \Omega_0, \\ \chi = 0 \text{ on } \mathbb{R}^3\backslash\Omega_1. \end{cases}$$

The purpose of a cut-off function χ is to restrict all expressions to the subset Ω_1 which is a positive distance away from $\partial\Omega$. Let $|h| > 0$ be small and set

$$\boldsymbol{\upsilon} = -D_k^{-h}(\chi^2 D_k^h\mathbf{u}),$$

where $k \in \{z,r,\theta\}$ and

$$D_k^h\mathbf{u}(t,\mathbf{x}) = \frac{\mathbf{u}(t,\mathbf{x}+h\mathbf{e}_k)-\mathbf{u}(t,\mathbf{x})}{h},$$

where $\mathbf{x} = (z,r,\theta)$ and $\mathbf{e}_k$ are the basis vectors in cylindrical coordinates. The test function for the full, coupled problem is then

$$(\boldsymbol{\upsilon},\boldsymbol{\psi},\boldsymbol{\xi},\boldsymbol{\zeta}) = (-D_k^{-h}(\chi^2 D_k^h\mathbf{u}),0,0,0).$$

In this step, we treat all three directions simultaneously since we are away from the boundary (which will not be the case in other two steps). The problem is that our fluid test function is not divergence-free. Namely,

$$\begin{aligned}\nabla\cdot\boldsymbol{v}&=\nabla\cdot(-D_k^{-h}(\chi^2D_k^h\mathbf{u}))=-D_k^{-h}\nabla\cdot(\chi^2D_k^h\mathbf{u})\\&=-D_k^{-h}(\nabla\chi^2)D_k^h\mathbf{u}=-(\nabla\chi^2)D_k^{-h}D_k^h\mathbf{u},\end{aligned}$$

where in the third equality we used the fact that $\nabla\cdot\mathbf{u}=0$.

In order to be able to use $\boldsymbol{v}$ as a test function, we need to look for its correction. The correction is designed by using Lemma III.3.1. from [30], which deals with the problem of finding a vector field $\mathbf{v}\in W_0^{1,p}(U)$ such that

$$\nabla\cdot\mathbf{v}=\mathbf{f}\ \text{in}\ U, \tag{6.1}$$

where $\mathbf{f}\in L^p(U)$ satisfies

$$\int_U\mathbf{f}=0. \tag{6.2}$$

We look for the velocity "correction" $\boldsymbol{v}^c$ which will ensure the solenoidality of the corrected test function $\boldsymbol{v}+\boldsymbol{v}^c$, namely we need to find a function $\boldsymbol{v}^c$ such that $\nabla\cdot(\boldsymbol{v}+\boldsymbol{v}^c)=0$, i.e.

$$\nabla\cdot\boldsymbol{v}^c=-\nabla\cdot\boldsymbol{v}=(\nabla\chi^2)D_k^{-h}D_k^h\mathbf{u}. \tag{6.3}$$

The compatibility condition (6.2) in this case reads $\int_\Omega\nabla\cdot\boldsymbol{v}=\int_{\partial\Omega}\boldsymbol{v}\cdot\mathbf{n}=0$ and it is automatically satisfied since $\boldsymbol{v}$ is, by definition, zero on $\partial\Omega$. Our domain is star-shaped with respect to every point of the domain, so we can use Lemma III.3.1. from [30] to see that there exists $\boldsymbol{v}^c$ which satisfies (6.3) and the following estimate holds

$$\|\boldsymbol{v}^c\|_{H^1(\Omega)}\le c\|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)}. \tag{6.4}$$

Now we take $(\boldsymbol{v}+\boldsymbol{v}^c,0,0,0)$ as a test function in the full, coupled problem and estimate each side of the equality separately. The left-hand side is equal to:

$$\begin{aligned}&\rho_F\int_0^T\int_\Omega\partial_t\mathbf{u}\cdot(\boldsymbol{v}+\boldsymbol{v}^c)+2\mu_F\int_0^T\int_\Omega\mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v}+\boldsymbol{v}^c)\\&=\rho_F\int_0^T\int_\Omega\partial_t\mathbf{u}\cdot\boldsymbol{v}+2\mu_F\int_0^T\int_\Omega\mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v})\\&\quad+\rho_F\int_0^T\int_\Omega\partial_t\mathbf{u}\cdot\boldsymbol{v}^c+2\mu_F\int_0^T\int_\Omega\mathbf{D}(\mathbf{u}):\mathbf{D}(\boldsymbol{v}^c)\\&=:I_1+I_2+I_3+I_4.\end{aligned}$$

From the first integral

$$\begin{aligned}I_1&=-\rho_F\int_0^T\int_\Omega\partial_t\mathbf{u}\cdot(D_k^{-h}(\chi^2D_k^h\mathbf{u}))\\&=\rho_F\int_0^T\int_\Omega\partial_tD_k^h\mathbf{u}\cdot\chi^2D_k^h\mathbf{u}=\frac{\rho_F}{2}\int_0^T\frac{d}{dt}\int_\Omega|\chi(D_k^h\mathbf{u})|^2\end{aligned}$$

we see that we will not get any new information regarding regularity of the fluid velocity. This is why we focus on the dissipative term:

$$\begin{aligned}
I_2 &= -2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(D_k^{-h}(\chi^2 D_k^h \mathbf{u})) \\
&= 2\mu_F \int_0^T \int_\Omega D_k^h(\mathbf{D}(\mathbf{u})) : \mathbf{D}(\chi^2 D_k^h \mathbf{u}) \\
&= 2\mu_F \int_0^T \int_\Omega D_k^h(\mathbf{D}(\mathbf{u})) : (2\mathbf{D}(\chi)\chi D_k^h \mathbf{u} + \chi^2 \mathbf{D}(D_k^h \mathbf{u})) \\
&= 2\mu_F \int_0^T \int_\Omega D_k^h(\mathbf{D}(\mathbf{u})) : 2\mathbf{D}(\chi)\chi D_k^h \mathbf{u} + 2\mu_F \int_0^T \int_\Omega D_k^h \mathbf{D}(\mathbf{u}) : \chi^2 D_k^h \mathbf{D}(\mathbf{u}) \\
&= 2\mu_F \int_0^T \int_\Omega D_k^h(\mathbf{D}(\mathbf{u})) : 2\mathbf{D}(\chi)\chi D_k^h \mathbf{u} + 2\mu_F \int_0^T \int_\Omega |\chi D_k^h \mathbf{D}(\mathbf{u})|^2.
\end{aligned}$$

Notice that there are no boundary terms since $\boldsymbol{v} = 0$ on $\partial\Omega$. The first integral in I_2 is estimated by using Cauchy inequality with ε

$$\begin{aligned}
\int_\Omega D_k^h(\mathbf{D}(\mathbf{u})) : 2\mathbf{D}(\chi)\chi D_k^h \mathbf{u} &\leq C\|D_k^h(\mathbf{D}(\mathbf{u}))\|_{L^2(\Omega)}\chi\|D_k^h \mathbf{u}\|_{L^2(\Omega)} \\
&\leq \frac{C}{\varepsilon}\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + C\varepsilon\chi^2\|D_k^h \mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}.
\end{aligned}$$

Furthermore, the integral I_3, which involves the correction term $\boldsymbol{v}^c$, is estimated as follows:

$$\begin{aligned}
\left|\int_\Omega \partial_t \mathbf{u} \cdot \boldsymbol{v}^c\right| &\leq \|\partial_t \mathbf{u} \cdot \boldsymbol{v}^c\|_{L^1(\Omega)} \leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|\boldsymbol{v}^c\|_{L^2(\Omega)} \\
&\leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|\boldsymbol{v}^c\|_{H^1(\Omega)} \leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \\
&= C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|D_k^{-h}(\chi^2 D_k^h \mathbf{u})\|_{L^2(\Omega)} \\
&\leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|\nabla(\chi^2 D_k^h \mathbf{u})\|_{L^2(\Omega)} \\
&\leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\left(\|2(\nabla\chi)\chi D_k^h \mathbf{u}\|_{L^2(\Omega)} + \|\chi^2\nabla(D_k^h \mathbf{u})\|_{L^2(\Omega)}\right) \\
&\leq C\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\left(\chi\|D_k^h \mathbf{u}\|_{L^2(\Omega)} + \chi\|\nabla D_k^h \mathbf{u}\|_{L^2(\Omega)}\right) \\
&\leq C\chi\|\partial_t \mathbf{u}\|_{L^2(\Omega)}\|\nabla \mathbf{u}\|_{L^2(\Omega)} + \frac{C}{\varepsilon}\|\partial_t \mathbf{u}\|^2_{L^2(\Omega)} \\
&\quad + C\varepsilon\chi^2\|\nabla(D_k^h \mathbf{u})\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used the equality (6.3), the estimate (6.4) as well as Cauchy inequality with ε. The integral I_4 is estimated in a similar way:

$$\begin{aligned}
\left|\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{v}^c)\right| &\leq \|\mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{v}^c)\|_{L^1(\Omega)} \leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\|\mathbf{D}(\boldsymbol{v}^c)\|_{L^2(\Omega)} \\
&\leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\|\boldsymbol{v}^c\|_{H^1(\Omega)} \leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \\
&= \|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\|D_k^{-h}(\chi^2 D_k^h \mathbf{u})\|_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\|\nabla(\chi^2 D_k^h\mathbf{u})\|_{L^2(\Omega)}\\
&\leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\big(\|2(\nabla\chi)\chi D_k^h\mathbf{u}\|_{L^2(\Omega)}+\|\chi^2\nabla(D_k^h\mathbf{u})\|_{L^2(\Omega)}\big)\\
&\leq C\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}\left(\chi\|D_k^h\mathbf{u}\|_{L^2(\Omega)}+\chi\|\nabla(D_k^h\mathbf{u})\|_{L^2(\Omega)}\right)\\
&\leq C\chi\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+\frac{C}{\varepsilon}\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}+C\varepsilon\chi^2\|\nabla(D_k^h\mathbf{u})\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Since $\boldsymbol{v}=0$ on $\partial\Omega$, the right-hand side in the weak formulation is equal to zero so, by summing up all the previous calculations, we obtain the following inequality:

$$
\begin{aligned}
&\frac{\rho_F}{2}\int_0^T\frac{d}{dt}\|\chi(D_k^h\mathbf{u})\|^2_{L^2(\Omega)}+2\mu_F\int_0^T\|\chi D_k^h\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}\\
&\leq\frac{C}{\varepsilon}\int_0^T\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+C\varepsilon\int_0^T\chi^2\|D_k^h\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}\\
&\quad+C\int_0^T\chi\|\partial_t\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\\
&\quad+\frac{C}{\varepsilon}\int_0^T\|\partial_t\mathbf{u}\|^2_{L^2(\Omega)}+C\varepsilon\int_0^T\chi^2\|\nabla(D_k^h\mathbf{u})\|^2_{L^2(\Omega)}\\
&\quad+C\int_0^T\chi\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}+\frac{C}{\varepsilon}\int_0^T\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega)}+C\varepsilon\int_0^T\chi^2\|\nabla(D_k^h\mathbf{u})\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Since $\mathbf{u}\in H^1(\Omega)$, we have the boundedness of the terms involving $\|\nabla\mathbf{u}\|_{L^2(\Omega)}$ and $\|\mathbf{D}(\mathbf{u})\|_{L^2(\Omega)}$ while the time regularity of the fluid velocity obtained in the previous section (see (5.7)) provides that $\|\partial_t\mathbf{u}\|_{L^2(\Omega)}\leq C$. Furthermore, since

$$\|\nabla(D_k^h\mathbf{u})\|_{L^2(\Omega)}\leq C\|\mathbf{D}(D_k^h\mathbf{u})\|_{L^2(\Omega)}=C\|D_k^h(\mathbf{D}(\mathbf{u}))\|_{L^2(\Omega)},$$

we can absorb terms involving $\|D_k^h(\mathbf{D}(\mathbf{u}))\|_{L^2(\Omega)}$ into the left-hand side by choosing ε such that $C\varepsilon\leq 2\mu_F$. Finally, we see that, provided $\mathbf{u}_0\in H^1(\Omega)$, the following estimate holds true:

$$\chi D_k^h\mathbf{D}(\mathbf{u})\in L^2(0,T;L^2(\Omega)).$$

LEMMA 6.2. *Let* $\mathbf{u}\in L^2(\Omega;\mathbb{R}^3)$ *and let* χ *be a smooth cut-off function such that* $0\leq\chi\leq 1$ *and*

$$\begin{cases}\chi=1 \text{ on } \Omega_0,\\ \chi=0 \text{ on } \mathbb{R}^3\backslash\Omega_1,\end{cases}$$

where $\Omega_0\subset\subset\Omega_1\subset\subset\Omega$. *Then,*

$$\chi D_k^h\mathbf{u}\in L^2(\Omega)\implies\chi\nabla\mathbf{u}\in L^2(\Omega).\tag{6.5}$$

PROOF. It is clear that we can rewrite $D_k^h\mathbf{u}$ as follows:

$$D_k^h(\chi\mathbf{u})=D_k^h(\chi)\mathbf{u}+\chi D_k^h\mathbf{u}.$$

Since $\chi D_k^h \mathbf{u} \in L^2(\Omega)$ and χ is a smooth function, we see that $D_k^h(\chi\mathbf{u}) \in L^2(\Omega)$. From Theorem 5.2 (ii), it follows that $\nabla(\chi\mathbf{u}) \in L^2(\Omega)$ which implies $\chi\nabla\mathbf{u} \in L^2(\Omega)$. ◻

By using Lemma 6.2, we conclude that

$$\chi\Delta\mathbf{u} \in L^2(0,T;L^2(\Omega)).$$

To sum up, in this step we obtained an additional regularity of the fluid velocity only in the interior of the fluid domain. In the next two steps, we will investigate what happens on the fluid domain boundary, which coincides with the fluid-structure interface.

Step 2: Shell interior regularity

In this step we restrict ourselves to the fluid equations coupled with the Koiter shell equations (without the mesh). To do so, we need to construct special test functions which do not see the mesh and which satisfy the coupling conditions at the boundary. Fix any open set $\omega_0 \subset\subset \omega\backslash(\partial\omega \cup \omega_S)$ and choose an open set ω_1 such that $\omega_0 \subset\subset \omega_1 \subset\subset \omega\backslash(\partial\omega \cup \omega_S)$. Recall that

$$\omega_S = \bigcup_{i=1}^{n_E} \boldsymbol{\pi}_i([0,l_i])$$

is the reference configuration of the mesh defined on ω. Then select a smooth function χ satisfying $0 \le \chi \le 1$ and

$$\begin{cases} \chi = 1 \text{ on } \omega_0, \\ \chi = 0 \text{ on } \mathbb{R}^2\backslash\omega_1. \end{cases}$$

The purpose of a cut-off function χ is to restrict all expressions defined on the shell domain ω to a subset ω_0 which is a positive distance away from $\partial\omega$ and ω_S, i.e. we "move away" from the shell boundary as well as from the area occupied by the mesh. Next, let $\tilde{\chi}$ be the extension of $\chi \circ \boldsymbol{\varphi}^{-1}$ in the interior of the fluid domain and such that $\tilde{\chi} = 0$ on $\Gamma_{in/out}$.

As we already announced, we prove the spatial shell regularity separately for each direction.

Regularity in tangential direction. First we deal with the regularity in tangential direction. For the shell equation, take

$$\boldsymbol{\psi} = -D_z^{-h}(\chi^2 D_z^h \partial_t \boldsymbol{\eta})$$

as a test function, and for the fluid part take

$$\boldsymbol{\upsilon} = -D_z^{-h}(\tilde{\chi}^2 D_z^h \mathbf{u}).$$

It is clear that we have $\boldsymbol{\upsilon} = \boldsymbol{\psi}$ on ω, i.e. kinematic coupling condition is satisfied.

Unfortunately, we again encounter the same problem as in Step 1, namely $\boldsymbol{v}$ is not divergence-free. We have to look for the fluid velocity "correction" which consists of two parts: $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$, so that the velocity $\boldsymbol{v}$, corrected by $\boldsymbol{v}^1 + \boldsymbol{v}^2$ is divergence-free, and has all the desired properties. Finally, we test the coupled problem with $(\boldsymbol{v} + \boldsymbol{v}^1 + \boldsymbol{v}^2, \boldsymbol{\psi}, 0, 0)$ and provided initial and boundary data are smooth enough, we get the additional regularity of the fluid velocity, as well as the shell displacement, in the tangential direction.

Let us now carry out this procedure in details. The part $\boldsymbol{v}^2$ is introduced so that condition (6.2) can be satisfied when (6.1) is solved for $\boldsymbol{v}^1$, where $\boldsymbol{v}^1$ is such that

$$\nabla \cdot \boldsymbol{v}^1 = -\nabla \cdot (\boldsymbol{v} + \boldsymbol{v}^2). \tag{6.6}$$

The resulting corrected fluid velocity $\boldsymbol{v} + \boldsymbol{v}^1 + \boldsymbol{v}^2$ is divergence-free. More precisely, we want to construct $\boldsymbol{v}^2$ such that it does not change the trace of the fluid velocity on the boundary Γ, such that condition (6.2) is satisfied for $\mathbf{f} = -\nabla \cdot (\boldsymbol{v} + \boldsymbol{v}^2)$ and such that the H^1-norm of $\boldsymbol{v}^2$ is controlled by the H^1-norm of $\boldsymbol{v}$:

1. $\boldsymbol{v}^2|_\Gamma = 0$,
2. $\int_\Gamma (\boldsymbol{v} + \boldsymbol{v}^2) \cdot \mathbf{n} = 0$,
3. $\|\boldsymbol{v}^2\|_{H^1(\Omega)} \leq C\|\boldsymbol{v}\|_{H^1(\Omega)}$.

The first condition will ensure that $(\boldsymbol{v} + \boldsymbol{v}^2)|_\Gamma = \boldsymbol{v}$ while the second condition is the compatibility condition corresponding to the fact that the integral of the right-hand side of problem (6.6) has to be zero. To satisfy the second condition, we can for example take:

$$\boldsymbol{v}^2 := -\alpha \mathbf{g}, \quad \text{where} \quad \alpha = \int_\Gamma \boldsymbol{v} \cdot \mathbf{n}, \quad \text{and} \quad \int_\Gamma \mathbf{g} \cdot \mathbf{n} = 1.$$

To obtain the desired H^1-estimate from the third condition, we can choose $\mathbf{g}$ independent of $\boldsymbol{v}$ and $\mathbf{n}$.

The existence of correction $\boldsymbol{v}^1$ follows directly from Lemma III.3.1. in [30], namely, there exists $\boldsymbol{v}^1$ such that

$$\nabla \cdot \boldsymbol{v}^1 = -\nabla \cdot (\boldsymbol{v} + \boldsymbol{v}^2)$$

with

$$\|\boldsymbol{v}^1\|_{H^1(\Omega)} \leq C\|\boldsymbol{v} + \boldsymbol{v}^2\|_{H^1(\Omega)}.$$

Therefore, we have "corrected" the test function for the fluid velocity, and now we can use $(\boldsymbol{v} + \boldsymbol{v}^1 + \boldsymbol{v}^2, \boldsymbol{\psi}, 0, 0)$ as a test function in the weak formulation of our fluid-mesh-shell interaction problem and take care of each term separately. In the fluid part, the only difference with the previous step is that we now have additional correction $\boldsymbol{v}^2$ of the test function, but we know how to estimate it since $\|\boldsymbol{v}^2\|_{H^1(\Omega)} \leq C\|\boldsymbol{v}\|_{H^1(\Omega)}$. That is the reason why we omit the calculations of the terms involving fluid velocity since they can be estimated analogously as in Step 1. We now focus on the inertial and elastic term in shell equations:

$$\begin{aligned}\rho_K h \int_0^T \int_\omega \partial_{tt}\boldsymbol{\eta} \cdot \boldsymbol{\psi} R &= -\rho_K h \int_0^T \int_\omega \partial_{tt}\boldsymbol{\eta} \cdot D_z^{-h}(\chi^2 D_z^h \partial_t \boldsymbol{\eta}) R \\ &= \rho_K h \int_0^T \int_\omega D_z^h \partial_{tt}\boldsymbol{\eta} \cdot (\chi^2 D_z^h \partial_t \boldsymbol{\eta}) R \\ &= \frac{\rho_K h}{2} \int_0^T \frac{d}{dt} \|\chi D_z^h \partial_t \boldsymbol{\eta}\|^2_{L^2(R;\omega)},\end{aligned}$$

$$\begin{aligned}\int_0^T a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) &= -\int_0^T a_K(\boldsymbol{\eta}, D_z^{-h}(\chi^2 D_z^h \partial_t \boldsymbol{\eta})) \\ &= \int_0^T a_K(D_z^h \boldsymbol{\eta}, \chi^2 D_z^h \partial_t \boldsymbol{\eta}) = \int_0^T \frac{d}{dt} a_K(\chi D_z^h \boldsymbol{\eta}, \chi D_z^h \boldsymbol{\eta}).\end{aligned}$$

Recall that we chose cut-off function $\tilde{\chi}$ such that it satisfies $\tilde{\chi} = 0$ on $\Gamma_{in/out}$. For that reason, the right-hand side of the weak formulation is equal to zero. Having in mind that we can estimate the correction terms from the fluid velocity test functions just like in Step 1, we easily obtain that the following holds true:

$$\begin{aligned}&\tilde{\chi} D_z^h \mathbf{D}(\mathbf{u}) \in L^2(0,T;L^2(\Omega)), \\ &\chi D_z^h \partial_t \boldsymbol{\eta} \in L^\infty(0,T;L^2(R;\omega)), \\ &\chi D_z^h \boldsymbol{\eta} \in L^\infty(0,T;V_K),\end{aligned} \tag{6.7}$$

which implies (see Lemma 6.2):

$$\begin{aligned}&\tilde{\chi} \partial_z \mathbf{D}(\mathbf{u}) \in L^2(0,T;L^2(\Omega)), \\ &\chi \partial_z \partial_t \boldsymbol{\eta} \in L^\infty(0,T;L^2(R;\omega)), \\ &\chi \partial_z \boldsymbol{\eta} \in L^\infty(0,T;V_K).\end{aligned} \tag{6.8}$$

Regularity in azimuthal direction. In the azimuthal direction, namely θ-direction, we apply the same procedure as we did in tangential direction and obtain that:

$$\begin{aligned}&\tilde{\chi} \partial_\theta \mathbf{D}(\mathbf{u}) \in L^2(0,T;L^2(\Omega)), \\ &\chi \partial_\theta \partial_t \boldsymbol{\eta} \in L^\infty(0,T;L^2(R;\omega)), \\ &\chi \partial_\theta \boldsymbol{\eta} \in L^\infty(0,T;V_K).\end{aligned} \tag{6.9}$$

Regularity in radial direction. To obtain an additional regularity of the fluid velocity in the radial direction, we go back to the Stokes equation

(omitting the constants which are positive)

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p = 0$$

which can be rewritten as

$$-\Delta \mathbf{u} + \nabla p = -\partial_t \mathbf{u}.$$

Since $\partial_t \mathbf{u} \in L^2(\Omega)$, from the regularity theory for the Stokes equation (see [47]) we know that $\nabla p \in L^2(\Omega)$. Next, we write

$$\partial_{rr} \mathbf{u} = \partial_t \mathbf{u} - \partial_{zz} \mathbf{u} - \partial_{\theta\theta} \mathbf{u} + \nabla p.$$

Since $\tilde{\chi}\partial_{zz}\mathbf{u}, \tilde{\chi}\partial_{\theta\theta}\mathbf{u} \in L^2(0,T;L^2(\Omega))$, we get $\tilde{\chi}\partial_{rr}\mathbf{u} \in L^2(0,T;L^2(\Omega))$.

Results obtained in this step are summarized here:

$$\begin{aligned} &\tilde{\chi}\Delta \mathbf{u} \in L^2(0,T;L^2(\Omega)), \\ &\chi \nabla \boldsymbol{\eta} \in L^\infty(0,T;V_K), \end{aligned} \tag{6.10}$$

namely, we obtained an additional fluid velocity regularity up to the fluid-structure interface (but excluding mesh) as well as an additional shell displacement regularity up to the mesh provided that

$$\mathbf{u}_0 \in H^1(\Omega),\ \boldsymbol{\eta}_0 \in H^2(\omega),\ \mathbf{v}_0 \in V_K.$$

Step 3: Mesh interior regularity

In this step we calculate the mesh interior regularity. To do so, we need to construct special test functions which "do not see" the mesh vertices and which satisfy the coupling conditions on ω which is an interface between the fluid and shell, and on $\prod_{i=1}^{n_E}(0,l_i)$ which is an interface between the shell and mesh. For each $i = 1,\dots,n_E$, fix any open set $I_i \subset\subset (0,l_i)$ and choose an open set J_i such that $I_i \subset\subset J_i \subset\subset (0,l_i)$. Then select a smooth cut-off function χ_i which satisfies $0 \le \chi_i \le 1$ and

$$\begin{cases} \chi_i = 1 \text{ on } I_i, \\ \chi_i = 0 \text{ on } \mathbb{R}\backslash J_i. \end{cases}$$

Set

$$\chi := \prod_{i=1}^{n_E} \chi_i$$

and let $\bar{\chi}$ be the extension of $\chi \circ \boldsymbol{\pi}^{-1}$ in the interior of the shell domain ω such that $\bar{\chi} = 0$ on $\partial\omega$ and let $\tilde{\chi}$ be the extension of $\bar{\chi} \circ \boldsymbol{\varphi}^{-1}$ in the interior of the fluid domain Ω such that $\tilde{\chi} = 0$ on $\Gamma_{in/out}$.

As we already mentioned, we have a mismatch between the variables, the mesh displacement and infinitesimal rotation are functions in variable s, the shell displacement is a function of (z,θ) while the fluid velocity is a function of (z,r,θ). For that reason, we will have to adapt the test functions belonging to the fluid and shell part.

First, for the mesh test function, we take the pair

$$(\boldsymbol{\xi}, \boldsymbol{\zeta}) = -(D_s^{-h}(\chi^2 D_s^h \partial_t \mathbf{d}), D_s^{-h}(\chi^2 D_s^h \partial_t \mathbf{w})).$$

It is then clear that for the shell test function we should take

$$\boldsymbol{\psi} = -D_s^{-h}(\bar{\chi}^2 D_s^h \partial_t \boldsymbol{\eta})$$

and for the fluid test function

$$\boldsymbol{\upsilon} = -D_s^{-h}(\tilde{\chi}^2 D_s^h \mathbf{u}).$$

Finally, take $(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ as a test function in the weak formulation of the full, coupled problem. In the previous two steps, we have shown how to estimate the terms coming from the fluid and shell equations. The only difference here is that we now obtain the same results as before but in s-direction, $s \in (0, l_i), i = 1, \dots, n_E$. What is left is to see what we get from the mesh part:

$$\begin{aligned}
\rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_{tt}\mathbf{d}_i \cdot \boldsymbol{\xi}_i &= -\rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_{tt}\mathbf{d}_i \cdot D_s^{-h}(\chi_i^2 D_s^h \partial_t \mathbf{d}_i) \\
&= \rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} D_s^h \partial_{tt}\mathbf{d}_i \cdot \chi_i^2 D_s^h \partial_t \mathbf{d}_i \\
&= \frac{\rho_S}{2} \int_0^T \frac{d}{dt} \|\chi D_s^h \partial_t \mathbf{d}\|^2_{L^2(\mathcal{N})},
\end{aligned}$$

$$\begin{aligned}
\rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_{tt}\mathbf{w}_i \cdot \boldsymbol{\zeta}_i &= -\rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_{tt}\mathbf{w}_i \cdot D_s^{-h}(\chi_i^2 D_s^h \partial_t \mathbf{w}_i) \\
&= \rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i D_s^h \partial_{tt}\mathbf{w}_i \cdot \chi_i^2 D_s^h \partial_t \mathbf{w}_i \\
&= \frac{\rho_S}{2} \frac{d}{dt} \int_0^T \|\chi D_s^h \partial_t \mathbf{w}\|^2_m,
\end{aligned}$$

$$\begin{aligned}
\int_0^T a_S(\mathbf{w}, \boldsymbol{\zeta}) &= \int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s \boldsymbol{\zeta}_i \\
&= -\int_0^T \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T \partial_s \mathbf{w}_i \cdot \partial_s (D_s^{-h}(\chi_i^2 D_s^h \partial_t \mathbf{w}_i)) \\
&= \int_0^T \frac{d}{dt} \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T D_s^h \partial_s \mathbf{w}_i \cdot \partial_s (\chi_i^2 D_s^h \mathbf{w}_i) \\
&= \int_0^T \frac{d}{dt} \sum_{i=1}^{n_E} \int_0^{l_i} Q_i H_i Q_i^T D_s^h \partial_s \mathbf{w}_i \cdot 2\chi_i \chi_i' D_s^h \mathbf{w}_i
\end{aligned}$$

$$+\int_0^T \frac{d}{dt}\sum_{i=1}^{n_E}\int_0^{l_i} Q_i H_i Q_i^T D_s^h \partial_s \mathbf{w}_i \cdot \chi_i^2 D_s^h \partial_s \mathbf{w}_i.$$

The first term from the right-hand side is easily estimated, and since again we do not have any boundary data due to the choice of a cut-off function $\tilde{\chi}$, we see that:

$$\chi \partial_{ss}\mathbf{w} \in L^\infty(0,T;L^2(\mathcal{N})).$$

Notice that we did not get the corresponding bound on the mesh displacement so we use the condition of inextensibility and unshearability:

$$\|\chi\partial_{ss}\mathbf{d}\|_{L^2(\mathcal{N})} = \|\chi\partial_s(-\mathbf{t}\times\mathbf{w})\|_{L^2(\mathcal{N})} \le C\left(\|\chi\mathbf{w}\|_{L^2(\mathcal{N})} + \|\chi\partial_s\mathbf{w}\|_{L^2(\mathcal{N})}\right),$$

to see that

$$\chi\partial_{ss}\mathbf{d} \in L^\infty(0,T;L^2(\mathcal{N})).$$

For the fluid and the shell we obtained:

$$\tilde{\chi}\partial_{ss}\mathbf{u} \in L^2(0,T;L^2(\Omega)), \qquad \bar{\chi}\partial_s\boldsymbol{\eta} \in L^\infty(0,T;V_K),$$

where

$$\partial_{ss}\mathbf{u} = \Delta\mathbf{u}\cdot((\boldsymbol{\varphi}(\boldsymbol{\pi}_i))'\circ\boldsymbol{\pi}_i^{-1}(\boldsymbol{\varphi}^{-1}))^2 + \nabla\mathbf{u}\cdot((\boldsymbol{\varphi}(\boldsymbol{\pi}_i))''\circ\boldsymbol{\pi}_i^{-1}(\boldsymbol{\varphi}^{-1})) \tag{6.11}$$

and

$$\partial_s\boldsymbol{\eta} = \nabla\boldsymbol{\eta}\cdot(\boldsymbol{\pi}_i'\circ\boldsymbol{\pi}_i^{-1}). \tag{6.12}$$

This is all true provided that the initial data satisfy:

$$\mathbf{u}_0 \in H^2(\Omega),\ \boldsymbol{\eta}_0 \in H^2(\omega),\ \mathbf{v}_0 \in V_K,\ (\mathbf{d}_0,\mathbf{w}_0) \in H^2(\mathcal{N};\mathbb{R}^6),\ (\mathbf{k}_0,\mathbf{z}_0) \in V_S.$$

The proof of Theorem 4.1 is now completed.

7. Conclusion

In this manuscript, we proved that a weak solution of the fluid-composite structure interaction problem, introduced in [12], enjoys an additional regularity property for initial and boundary data satisfying the appropriate regularity as well as compatibility conditions imposed on the interface. The regularity result is valid up to the boundary, i.e. up to mesh vertices. Even though the techniques we used for proving regularity are standard tool in analysis of partial differential equations, due to multi-physics background of the considered problem, the undertaken procedure is quite challenging. Namely, the fluid equations are of parabolic type, while the shell and mesh equations, which constitute the composite structure, are of hyperbolic type. For that reason, we have a mismatch between the parabolic and hyperbolic regularity on the fluid-composite structure interface. One of the aims of our future work is to address the question of regularity of a weak solution in the case of nonlinear, moving boundary fluid-composite structure interaction, introduced in [13], locally in time, and in case of small initial data. Moreover, future research in the direction of pressure reconstruction (which was extricated from the weak formulation by using the divergence-free test functions) is underway.

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