https://doi.org/10.31896/k.25.2
Original scientific paper
Accepted 22. 2. 2021.

Hiroshi Okumura

Semicircles in the Arbelos with Overhang and Division by Zero

Semicircles in the Arbelos with Overhang and Division by Zero

ABSTRACT

We consider special semicircles, whose endpoints lie on a circle, for a generalized arbelos called the arbelos with overhang considered in [4] with division by zero.

Key words: arbelos, arbelos with overhang, Aida arbelos, semicircle touching at the endpoints, insemicircle, Archimedean semicircle, division by zero

MSC2010: 01A27 51M04

Polukružnice u arbelosima s produžecima i dijeljenje s nulom

SAŽETAK

U radu proučavamo posebne polukružnice, one čije krajnje točke leže na jednoj kružnici, u poopćenim arbelosima s produžecima kao u [4] uz korištenje dijeljenja s nulom.

Ključne riječi: arbelosi, arbelosi s produžecima, Aida arbelosi, polukružnice s diranjem u krajnjim točkama, unutarnje polukružnice, Arhimedove polukružnice, dijeljenje s nulom

1 Introduction

For a point O on the segment AB such that $|AO| = 2a$, $|BO| = 2b$, let A_h (resp. B_h) be a point on the half line OA (resp. OB) with initial point O such that $|OA_h| = 2(a+h)$ (resp. $|OB_h| = 2(b+h)$) for a real number h satisfying $-\min(a,b) < h$. In [4] we have considered a generalized arbelos consisting of the three semicircles α, β and γ of diameters A_hO, B_hO and AB, respectively, constructed on the same side of AB. The figure is denoted by $(\alpha,\beta,\gamma)_h$ and is called the arbelos with overhang h (see Figure 1). The ordinary arbelos is obtained from $(\alpha,\beta,\gamma)_h$ if $h = 0$, which is denoted by $(\alpha,\beta,\gamma)_0$.

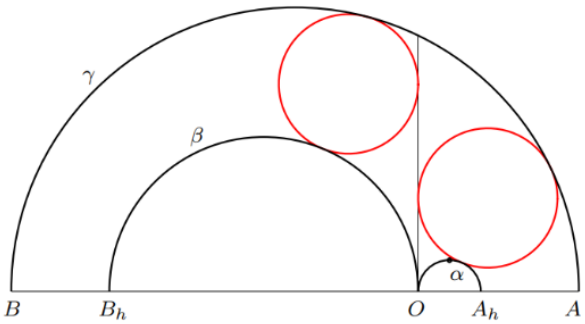

Figure 1: $(\alpha,\beta,\gamma)_h$, $-\min(a,b) < h < 0$.

Let $c = a+b$. The circle touching α (resp. β) externally, γ internally, and the axis from the side opposite to B (resp. A) has radius

$$r_{\mathrm{A}} = \frac{ab}{c+h}.$$

The two circles are called the twin circles of Archimedes of $(\alpha,\beta,\gamma)_h$. Circles of radius r_{A} are called Archimedean circles of $(\alpha,\beta,\gamma)_h$ or said to be Archimedean with respect to $(\alpha,\beta,\gamma)_h$.

In this article we consider special semicircles, which are counterpart to the incircle and Archimedean circles of $(\alpha,\beta,\gamma)_h$ using division by zero. At the last part of this paper we consider special case of $(\alpha,\beta,\gamma)_h$ considered by Aida [1]. We consider using a rectangular coordinate system with origin O such that the farthest point on α have coordinates $(a+h, a+h)$ (see Figure 1). The radical axis of α and β is called the axis.

2 Incircle and insemicircle

In this section we consider the incircle of $(\alpha,\beta,\gamma)_h$ and an inscribed semicircle in $(\alpha,\beta,\gamma)_h$. If a circle touches α and

β externally and γ internally, we call the circle the incircle of $(\alpha,\beta,\gamma)_h$ (see Figure 2). If the endpoints of a semicircles lie on a circle, we say that the semicircle touches the circle at the endpoints. If a semicircle touches α and β, and γ at the endpoints, we say that the semicircle is inscribed in $(\alpha,\beta,\gamma)_h$. We have considered such a semicircle in [2] for $(\alpha,\beta,\gamma)_0$. We use the next proposition.

Proposition 1 *A semicircle of radius s touches a circle of radius r at the endpoints if and only if $d^2+s^2=r^2$, where d is the distance between the centers of the semicircle and the circle.*

Let $v=\sqrt{(c+h)^2-2ab+h^2}$.

Theorem 1 *The following statements hold.*
(i) *The incircle of $(\alpha,\beta,\gamma)_h$ has radius*

$$i_c=\frac{ab(c+2h)}{(c+h)^2-ab}. \tag{1}$$

(ii) *If a semicircle is inscribed in $(\alpha,\beta,\gamma)_h$, then it has radius*

$$i_s=\frac{-v^2+\sqrt{8ab(c+2h)^2+v^4}}{2(c+2h)}. \tag{2}$$

Proof. We prove (ii). Let (x,y) and i_s be the coordinates of the center and the radius of the semicircle inscribed in $(\alpha,\beta,\gamma)_h$. Then we get $(x-(a+h))^2+y^2=((a+h)+i_s)^2$, $(x+(b+h))^2+y^2=((b+h)+i_s)^2$ and $(x-(a-b))^2+y^2+i_s^2=c^2$ by Proposition 1. Eliminating x and y from the three equations and solving the resulting equation for i_s, we get (2). The part (i) is proved similarly. □

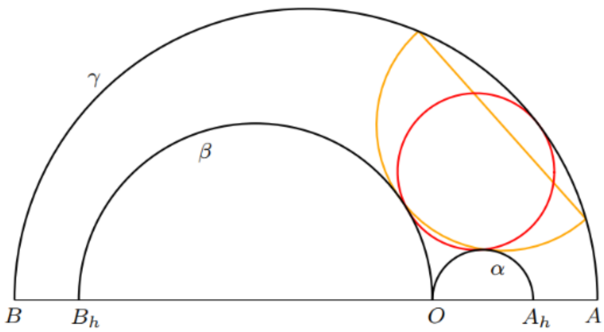

Figure 2

The theorem shows that an inscribed semicircle in $(\alpha,\beta,\gamma)_h$ is determined uniquely. Hence we can call it the insemicircle of $(\alpha,\beta,\gamma)_h$.

We consider a condition where a semicircle of radius i_s touches γ. If one of the endpoints of a semicircle S_1 lies on a semicircle S_2 and the other endpoints of S_1 lies on the reflection of S_2 in its diameter, we still say that S_1 touches S_2 at the endpoints. The circle of center of coordinates $((a+h)m,0)$ (resp. $(-(b+h)n,0)$ and passing through O is denoted by α_m (resp. β_n) for a real number m (resp. n) (see Figure 3). For points P and Q on a semicircle δ, we say that P, Q and the endpoints of δ lie counterclockwise if P, Q and one of the endpoints of δ lie counterclockwise. If a circle touches α_m, β_n and γ internally so that the points of tangency of this circle and each of β_m, α_n and γ lie counterclockwise, we say that the circle touches α_m, β_n and γ appropriately. Also if a semicircle touches α_m and β_n, and γ at the endpoints so that the points of tangency of the semicircle and each of β_n, α_m, and the endpoints lie counterclockwise, then we say that the semicircle touches α_m, β_n and γ appropriately.

Theorem 2 *If $m\neq 0$ and $n\neq 0$, the following three statements are equivalent.*
(i) *A circle of radius i_c touches α_m, β_n and γ appropriately.*
(ii) *A semicircle of radius i_s touches α_m, β_n and γ appropriately.*
(iii) $c+2h=\dfrac{a+h}{m}+\dfrac{b+h}{n}$.

Proof. Assume that (i) and (x,y) are the coordinates of the center of the circle in (i). Then we have $(x-m(a+h))^2+y^2=(m(a+h)+i_c)^2$, $(x+n(b+h))^2+y^2=(n(b+h)+i_c)^2$ and $(x-(a-b))^2+y^2=(c-i_c)^2$. Eliminating x and y from the three equations with (1), we get (iii). Conversely we assume (iii), and a circle of radius i_c touches α_m, $\beta_{n'}$ and γ appropriately for a real number n'. Then we have $a+b+2h=(a+h)/m+(b+h)/n'$ just as we have shown, i.e., $n=n'$. Hence $\beta_n=\beta_{n'}$, i.e., (iii) implies (i). Therefore (i) and (iii) are equivalent. The equivalence of (ii) and (iii) is proved similarly. □

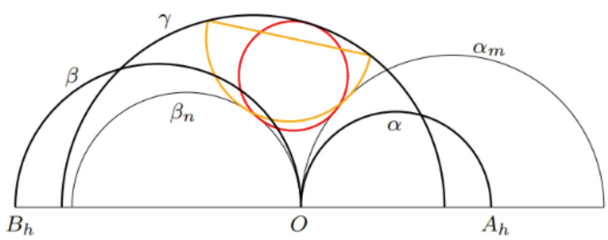

Figure 3: $1<m$ and $0<n<1$.

Theorem 2 does not consider the case in which α_m or β_n coincides with the axis. We consider the case in the next theorem (see Figure 4).

Theorem 3 *The following statements hold.*
(i) *A circle of radius i_c touches α_m $(m>0)$ externally, γ*

internally and the axis if and only if

$$m = m_0 = \frac{a+h}{c+2h}. \tag{3}$$

(ii) *A semicircle of radius i_s touches α_m $(m > 0)$ and the axis, and γ at the endpoints if and only if* (3) *holds.*
(iii) *A circle of radius i_c touches β_n $(n > 0)$ externally, γ internally and the axis if and only if*

$$n = n_0 = \frac{b+h}{c+2h}. \tag{4}$$

(iv) *A semicircle of radius i_s touches β_n $(n > 0)$ and the axis, and γ at the endpoints if and only if* (4) *holds.*

Proof. We prove (i). Let (x,y) be the coordinates of the center of the circle of radius i_c in (i). Then we have $x = i_c$, $(x - m(a+h))^2 + y^2 = (m(a+h)+i_c)^2$ and $(x-(a-b))^2 + y^2 = (a+b-i_c)^2$. Eliminating x and y from the three equations with (1), and solving the resulting equation for m, we get (3). Conversely, we assume that (3) and a circle of radius i_c touches $\alpha_{m'}$ $(m' > 0)$ externally, γ internally and the axis for a real number m'. Then we have $m' = m_0 = m$ as just we have proved. Therefore $\alpha_{m'} = \alpha_m$ and the converse is true. The rest of the theorem is proved similarly. □

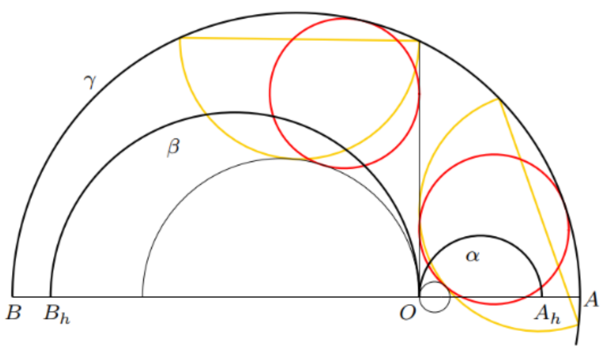

Figure 4

If $m = m_0$, then $(a+h)/m = c+2h$. Therefore if $(b+h)/n_x = 0$, and β_{n_x} coincides with the axis, then we can consider that Theorem 2 is true in the case $(m,n) = (m_0, n_x)$. Similarly if $n = n_0$ and $(a+h)/m_x = 0$ and α_{m_x} coincides with the axis, we can also consider that Theorem 2 holds in the case $(m,n) = (m_x, n_0)$. Therefore Theorems 2 and 3 can be unified in this case. We consider about this in section 4.

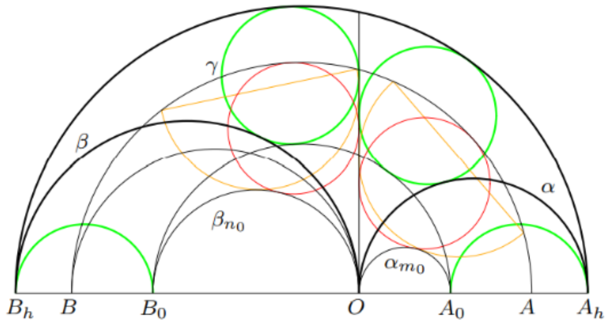

Figure 5

Theorem 4 *If A_0O and B_0O are the diameters of the circles α_{m_0} and β_{n_0}, respectively, then the circles of diameters A_0A_h and B_0B_h are Archimedean circles of the arbelos made by α, β and the semicircle of diameter A_hB_h constructed on the same side of AB as γ. Therefore the circle of diameter A_0B_0 is concentric to γ and touches the twin circles of Archimedes of the arbelos.*

Proof. Since the radius of the circle α_{m_0} equals $(a+h)m_0 = (a+h)^2/(c+2h)$ by (3), the circle of diameter A_0A_h has radius

$$(a+h) - \frac{(a+h)^2}{c+2h} = \frac{(a+h)(b+h)}{c+2h},$$

which equals the radius of Archimedean circles of the arbelos made by α, β and the semicircle of diameter A_hB_h (see Figure 5). Since the radius of the circle is symmetric in a and b, the other circle also has the same radius. □

3 Archimedean semicircles

In this section we consider another kind of semicircles touching γ at the endpoints.

Theorem 5 *The semicircle touching α and the axis and γ at the endpoints is congruent to the semicircle touching β and the axis and γ at the endpoints. The common radius equals*

$$s_{\mathrm{A}} = \frac{1}{2}\left(\sqrt{(c+2h)^2 + 8ab} - c - 2h\right). \tag{5}$$

Proof. Let (s,y) be the coordinates of the center of the semicircle touching α and the axis, and γ at the endpoints. Then s equals the radius of the semicircle, and we have $(s-(a-b))^2 + y^2 + s^2 = c^2$ by Proposition 1 and $(s-(a+h))^2 + y^2 = ((a+h)+s)^2$. Eliminating y from the two equations and solving the resulting equation for s, we have $s = s_{\mathrm{A}}$. Since s is symmetric in a and b, the other semicircle also has the same radius. □

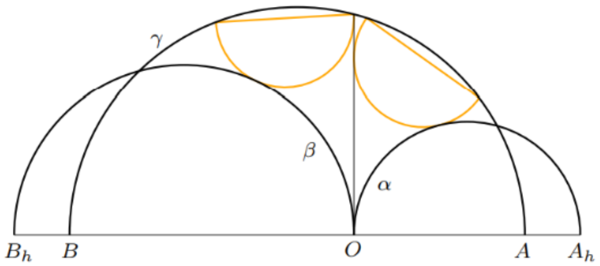

Figure 6

The two congruent semicircles in Theorem 5 may be called *the twin semicircles of Archimedes* (see Figure 6). A semicircle of radius s_A is called an *Archimedean semicircle* of $(\alpha,\beta,\gamma)_h$ or said to be *Archimedean* with respect to $(\alpha,\beta,\gamma)_h$. Let $w_k = \sqrt{a^2+kab+b^2}$. Theorem 5 shows that $(\alpha,\beta,\gamma)_0$ has Archimedean semicircles of radius $(w_{10}-c)/2$.

Theorem 6 *Assume that $(m,n) \neq (1,0),(0,1)$ and a semicircle touches α_m, β_n and γ appropriately. Then the semicircle is Archimedean with respect to $(\alpha,\beta,\gamma)_h$ if and only if*

$$\frac{1}{m}+\frac{1}{n}=1. \tag{6}$$

Proof. Assume that a semicircle of radius s_A touches α_m, β_n and γ appropriately and (x,y) are the coordinates of its center. Then we get $(x-m(a+h))^2+y^2=(m(a+h)+s_A)^2$, $(x+n(b+h))^2+y^2=(n(b+h)+s_A)^2$, and $(x-(a-b))^2+y^2+{s_A}^2=c^2$. Eliminating x and y from the three equations, we have (6). Conversely we assume (6) and assume that a semicircle of radius s_A touches α_m, $\beta_{n'}$ and γ appropriately. Then we have $1/m+1/n'=1$. Hence we get $n=n'$, i.e., $\beta_n=\beta_{n'}$. Hence the converse holds. □

While we have obtained the next theorem in [4].

Theorem 7 *If $(m,n) \neq (1,0),(0,1)$ and a circle touches α_m, β_n and γ appropriately, then the circle is Archimedean with respect to $(\alpha,\beta,\gamma)_h$ if and only if (6) holds.*

By Theorems 6 and 7 we have the next theorem.

Theorem 8 *If $(m,n) \neq (1,0),(0,1)$, the following statements are equivalent.*
(i) *The circle touching α_m, β_n, and γ appropriately is Archimedean with respect to $(\alpha,\beta,\gamma)_h$.*
(ii) *The semicircle touching α_m, β_n, and γ appropriately is Archimedean with respect to $(\alpha,\beta,\gamma)_h$.*
(iii) (6) *holds.*

It is commonly considered that the circles α_0 and β_0 are point circles and coincide with the origin O. This implies that Theorem 8 is not true in the cases $(m,n)=(1,0),(0,1)$. Therefore Theorems 8 does not consider the case of the twin circles of Archimedes and the case of the twin semicircles of Archimedes. We consider the case in the next section.

4 Division by zero

In this section we show that we can consider that the circles α_0 and β_0 coincide with the axis using recently made definition of division by zero [5].
For a field F we consider the following bijection $\psi : F \to F$:

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

It is a custom to denote $z\psi(a)$ by z/a if $a \neq 0$, i.e., $z\psi(a) = a/z$ for $a \neq 0$. Following to this, we write

$$z \cdot \psi(0) = \frac{z}{0} \quad for\ \forall z \in F. \tag{7}$$

Then we have

$$z \cdot \psi(a) = \frac{z}{a} \quad for\ \forall a,z \in F. \tag{8}$$

Especially we have

$$\frac{z}{0} = z \cdot 0 = 0 \quad for\ \forall z \in F. \tag{9}$$

Notice that the concept of the reduction to common denominator can not be used for $z/0$, i.e., we have the following relation in general in the case $b=0$ or $d=0$:

$$\frac{a}{b}+\frac{c}{d} \neq \frac{ad+bc}{bd}.$$

We consider the circle α_m in the case $m=0$. The circle α_m has an equation $(x-m(a+h))^2+y^2=m^2(a+h)^2$, or

$$-2m(a+h)x+(x^2+y^2)=0. \tag{10}$$

This implies $x^2+y^2=0$ if $m=0$. Hence α_0 coincides with the origin in this case. On the other hand, (10) can be written as

$$-2(a+h)x+\frac{x^2+y^2}{m}=0. \tag{11}$$

Therefore we get $-2(a+h)x=0$, i.e., $x=0$ if $m=0$ by (9), i.e., α_0 coincides with the axis in this case. Now we can consider that α_0 is the origin or the axis, or the axis as the union of them. Similarly β_0 can be considered as the origin or the axis.
We can now consider that α_0 and β_0 coincide with the axis. Then Theorem 2 holds in the case $(m,n)=(m_0,0),(0,n_0)$ by (9). Also Theorem 8 holds in the case $(m,n)=$

$(1,0),(0,1)$. Our current mathematics avoids to consider (9). But our above observation shows that (9) is useful. Division by zero was founded by Saburou Saitoh in 2014. He has been making a list of successful example applying division by zero and its generalization called division by zero calculus, and there are more than 1200 evidences. It shows that a new world of mathematics can be opened if we admit them. For an extensive reference of division by zero and division by zero calculus including those evidences, see [5].

5 Aida arbelos

Aida (1747-1817) considered a figure consisting of two touching semicircles at their midpoints and the circle passing through the endpoints of the semicircles [1] (see Figure 7). He gave several notable properties of this figure, which are summarized in [3]. We conclude this paper by considering special circles and special semicircles for this figure.

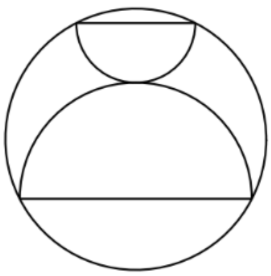

Figure 7: *Aida's figure.*

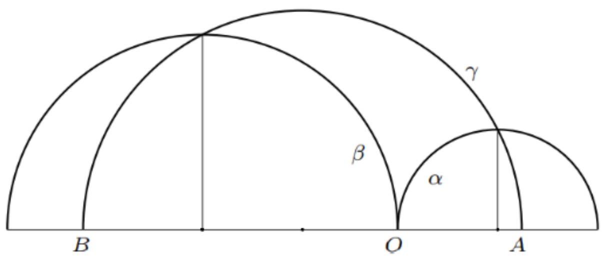

Figure 8: *Aida arbelos.*

Aida's figure is obtained from $(\alpha,\beta,\gamma)_h$, when $h = r_A$ [3], or

$$h = \frac{ab}{c+h}. \qquad (12)$$

Because (12) is equivalent to

$$r_A = h = \frac{1}{2}(w_6 - c), \qquad (13)$$

and (13) implies that the farthest points on α and β from AB lie on γ, where recall $w_k = \sqrt{a^2+kab+b^2}$. In this case we call $(\alpha,\beta,\gamma)_h$ an Aida arbelos (see Figure 8). Replacing h in the denominator of the right side of (12) by the right side of (12) repeatedly, we get a continued fraction expansion of r_A for the Aida arbelos:

$$r_A = \frac{ab}{c+h} = \frac{ab}{c+\dfrac{ab}{c+h}} = \frac{ab}{c+\dfrac{ab}{c+\dfrac{ab}{c+\ddots}}}.$$

We assume $h \geq 0$. Let $\overline{\alpha}$ and $\overline{\beta}$ be the semicircles of diameters AO and BO, respectively, constructed on the same side of AB as γ, i.e., $\overline{\alpha}$, $\overline{\beta}$ and γ form $(\alpha,\beta,\gamma)_0$. The incircle of the curvilinear triangle made by α, $\overline{\alpha}$ (resp. β, $\overline{\beta}$) and the radical axis of α (resp. β) and γ has radius $(1/r_A + 1/h)^{-1}$ for $(\alpha,\beta,\gamma)_h$ [4]. Therefore the radius equals $r_A/2$ for the Aida arbelos. The circles are denoted by green in Figure 9. The circle touching α or β externally, γ externally and the axis has radius ab/h for $(\alpha,\beta,\gamma)_h$ [4]. Hence the radius equals $ab/r_A = c + r_A$ for the Aida arbelos by (12). The circles are denoted by magenta in Figure 9.

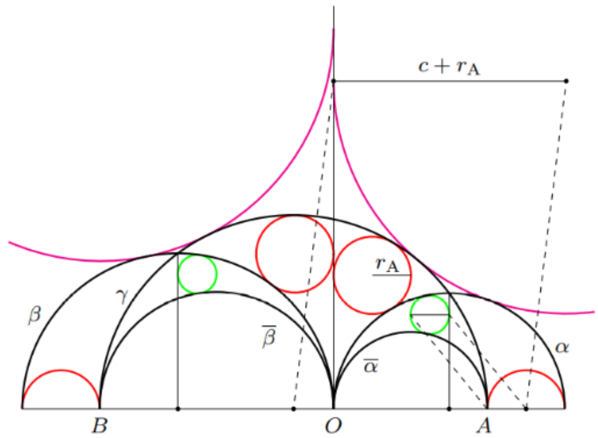

Figure 9: *The green circles have radius $r_A/2$.*

Substituting (13) in (5), we get that the radius of Archimedean semicircles of the Aida arbelos equals

$$s_A = \frac{1}{2}(w_{14} - w_6).$$

Since $i_c = w_6 h/c$ for the Aida arbelos [3], we get that the inradius of the Aida arbelos equals

$$i_c = \frac{w_6(w_6 - c)}{2c}$$

by (13). Therefore we have

$$i_c + r_A = \frac{2ab}{c}.$$

Hence the sum of i_c and r_A for the Aida arbelos equals the diameter of the Archimedean circle of $(\alpha,\beta,\gamma)_0$. Let $u=(w_6^4+16a^2b^2)^{1/4}$.

Theorem 9 *If the insemicircle of the Aida arbelos has center of coordinates (x_s,y_s), we have*

$$i_s=\frac{u^2-c^2}{2w_6}, \tag{14}$$

$$(x_s,y_s)=\left(\frac{(b-a)i_s}{w_6},\frac{4ab\sqrt{4ab+u^2}}{w_6^2}\right). \tag{15}$$

Proof. By (2) and (13), we get (14). Solving the equations $(x_s-(a+h))^2+y_s^2=((a+h)+i_s)^2$ and $(x_s+(b+h))^2+y_s^2=((b+h)+i_s)^2$ with (14), we get (15). □

The next theorem shows that the result for the insemicircle of $(\alpha,\beta,\gamma)_0$ obtained in [2] also holds for the Aida arbelos (see Figure 10).

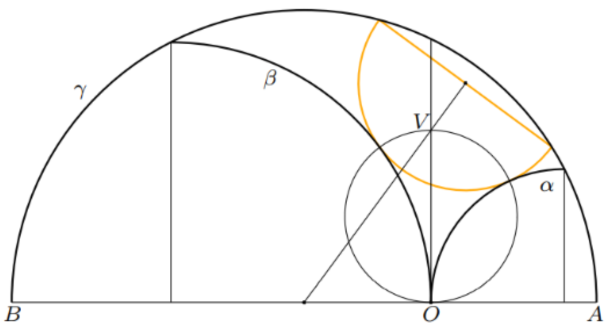

Figure 10

Theorem 10 *If the line joining the centers of γ and the insemicircle of the Aida arbelos meets the axis in a point V, then the circle of diameter OV is orthogonal to the insemicircle. Hence the circle passes through the points of tangency of two of α, β and the insemicircle.*

Proof. From (13) and (15), the circle of diameter OV has radius

$$r_v=\frac{4ab\sqrt{4ab+u^2}}{w_{10}^2+u^2}$$

and the center of coordinates $(0,y_v)=(0,r_v)$. Then we have $(x_s-0)^2+(y_s-y_v)^2=r_v^2+i_s^2$. □

References

[1] Aida ed., Sampō Chikusaku Jutsu, Volume 2, Tohoku University Digital Collection.

[2] H. Okumura, The arbelos in Wasan geometry, the inscribed semicircle in the arbelos, *Sangaku J. Math.* **5** (2021), 1–8.

[3] H. Okumura, Wasan geometry. In: B. Sriraman ed., *Handbook of the Mathematics of the Arts and Sciences*, Springer, 2020, 1–52, doi: 10.1007/978-3-319-70658-0_122-1

[4] H. Okumura, The arbelos with overhang, *KoG* **18** (2014), 19–27.

[5] S. Saitoh, Introduction to the division by zero calculus, *Scientific Research Publishing*, 2021.

Hiroshi Okumura
orcid.org/0000-0003-4332-0189
e-mail: hokmr@yandex.com

Takahanadai Maebashi Gunma 371-0123, Japan