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Polyhedrons the Faces of which are Special Quadric Patches

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ABSTRACT

We seize an idea of Oswald Giering (see [1] and [2]), who replaced pairs of faces of a polyhedron by patches of hyperbolic paraboloids and link up edge-quadrilaterals of a polyhedron with the pencil of quadrics determined by that quadrilateral. Obviously only ruled quadrics can occur. There is a simple criterion for the existence of a ruled hyperboloid of revolution through an arbitrarily given quadrilateral. Especially, if a (not planar) quadrilateral allows one symmetry, there exist two such hyperboloids of revolution through it, and if the quadrilateral happens to be equilateral, the pencil of quadrics through it contains even three hyperboloids of revolution with pairwise orthogonal axes. To mention an example, for right double pyramids, as for example the octahedron, the axes of the hyperboloids of revolution are, on one hand, located in the plane of the regular guiding polygon, and on the other, they are parallel to the symmetry axis of the double pyramid.

Not only for platonic solids, but for all polyhedrons, where one can define edge-quadrilaterals, their pairs of face-triangles can be replaced by quadric patches, and by this one could generate roofing of architectural relevance. Especially patches of hyperbolic paraboloids or, as we present here, patches of hyperboloids of revolution deliver versions of such roofing, which are also practically simple to realize.

Key words: polyhedron, quadric, hyperboloid of revolution, Bézier patch

MSC2010: 51Mxx, (51M20, 51M30), 51N05, 51N20, 15Axx

Poliedri čije su strane dijelovi posebnih kvadrika

SAŽETAK

Preuzimamo ideju Oswalda Gieringa (vidi [1] i [2]), koji je par strana poliedra zamijenio dijelom hiperboličnog paraboloida i povezao bridni četverostran poliedra s pramenom kvadrika određenim tim četverostranom. Očito se samo pravčaste kvadrike mogu pojaviti. Postoji jednostavan nužan uvjet postojanja pravčastog rotacijskog hiperboloida kroz dani četverostran. Posebno, ako (prostorni) četverostran ima jednu ravninu simetrije, onda postoje dva rotacijska hiperboloida kroz njega, a ako je četverostran jednakostraničan, onda pramen kvadrika kroz njega sadrži čak tri rotacijska hiperboloida s međusobno okomitim osima. Na primjer, kod pravilne dvostruke piramide, kao što je oktaedar, osi rotacijskih hiperboloida su, s jedne strane, u ravnini pravilnog mnogokuta (osnovke), a s druge strane, su usporedne s osi simetrije dvostruke piramide.

Parove strana (trokute) ne samo Platonovih tijela, već svih poliedara kod kojih se mogu definirati bridni četverostrani, moguće je zamijeniti dijelovima kvadrika, i na taj način proizvesti krovišta od arhitektonskog značaja. Posebno zanimljiva krovišta mogu nastati primjenom dijelova hiperboličnih paraboloida, ili kao što je ovdje prikazano, rotacijskih hiperboloida koje je jednostavno i realizirati u praksi.

Ključne riječi: poliedar, kvadrika, rotacijski hiperboloid, Bézierova zakrpa

Excerpt of what we aim to present in the following chapters

Chapter 1 deals with the regular octahedron $\mathfrak{p}$ as a standard example and replace pairs of triangles by quadric patches. Here we can already show the principle of how to proceed. Among the pencil of quadrics through an edge quadrilateral of $\mathfrak{p}$ we look for the hyperbolic paraboloid ("HP-surface") and for hyperboloids of revolution ("R-hyperboloids"). It turns out that descriptive geometric methods highly support an analytic treatment of the problem.

In Chapter 2 we deal with a criterion for quadrilaterals, which are generators of an R-hyperboloid. For a quadrilateral fulfilling the criterion we give a construction of the axis and the skirt circle of an R-hyperboloid through it as well as analytic descriptions of the R-hyperboloid by its equation and as a tensor-product patch ("TP-patch"). Additionally, we also ask for the set of R-hyperboloids through two skew given lines. This set is, to some extent, a 3D-generalisation of a (planar) elliptic pencil of circles.

The third chapter concerns polyhedrons $\mathfrak{p}$, the faces of which are n-gons ($n > 3$). By adding pyramids of a certain height h to these faces one can interpret the original polyhedron $\mathfrak{p}$ as the limit of the set of polyhedrons $\mathfrak{p}(h)$ for $h \to 0$. This gives a more "natural" set of edge-quadrilaterals than that proposed by Giering [1] and [2] for the cube. We apply this way of splitting an n-gon-face into triangles for e.g. a box shaped polyhedron. Finally we show images of some Johnson polyhedrons with R-hyperboloid patches as faces.

Concluding we note that Giering's idea to replace pairs of planar faces by HP-surfaces works for any polyhedron, while R-hyperboloids exist only for edge-quadrilaterals fulfilling the criterion mentioned in Chapter 2. Anyway, by choosing a certain quadric out of the pencil of quadrics through an edge-quadrilateral and describe it as a TP-patch one wins an additional design parameter, what works for all polyhedrons independent from the criterion. This could be of relevance for architectural design, too.

1 The regular octahedron and its R-hyperboloid faces

We connect a Cartesian frame with the regular octahedron $\mathfrak{p} = \{A,B,C,D,E,F\}$ such that its midpoint becomes the origin O and one of its diagonals becomes the z-axis. The x- and y-axes are parallel to edges BC and AB (Figure 1). We consider the (equilateral) edge-quadrilateral $\mathcal{H} = \{A,B,E,F\}$ and the pencil $\mathfrak{Q}$ of quadrics $\Phi(t)$ through it. Setting the edge length $\overline{AB} = \sqrt{2}$ we obtain the vertex coordinates $A = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$, $B = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)$, $E = (0,0,1)$, $F = (0,0,-1)$.

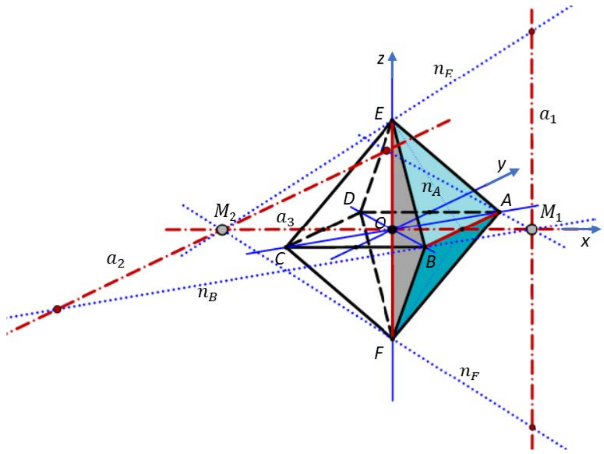

Figure 1: *The octahedron $\mathfrak{p}$, its edge-quadrilateral $\mathcal{H} = \{A,B,E,F\}$, and the normals $n\ldots$, which are common for all quadrics of the pencil $\mathfrak{Q}$ through $\mathcal{H}$. The lines a_1, a_2, a_3 (dashed red) represent the axes of three* R*-hyperboloids through $\mathcal{H}$.*

The pencil $\mathfrak{Q}$ is spanned by the pairs of face planes $\Phi_1 = (AEF) \cup (BEF)$ and $\Phi_2 = (ABE) \cup (ABF)$, such that a general ruled quadric $\Phi(t)$ can be written as

$$\Phi(t) = (1-t)\Phi_1 + t\Phi_2. \tag{1}$$

By the equations of Φ_1, Φ_2

$$\Phi_1 \ldots (x+y)(x-y) = 0,$$
$$\Phi_2 \ldots (z + (\sqrt{2}.x - 1))(z - (\sqrt{2}.x - 1)) = 0, \tag{2}$$

follows

$$\Phi(t) \ldots (1-t)(x^2 - y^2) + t(z^2 - 2x^2 - 2\sqrt{2}.x - 1) = 0. \tag{3}$$

We see immediately that for $t = \frac{1}{2}$ one gets the R-hyperboloid Φ_{R1}

$$\Phi_{R1} \ldots (x - \sqrt{2})^2 + y^2 - z^2 = 1, \tag{4}$$

and for $t = \frac{1}{4}$ the R-hyperboloid Φ_{R2}

$$\Phi_{R2} \ldots (x + \sqrt{2})^2 + z^2 - 3y^2 = 3. \tag{5}$$

For $t = \frac{1}{3}$ we obtain the hyperbolic paraboloid Φ_P

$$\Phi_P \ldots 2y^2 - z^2 - 2\sqrt{2}.x + 1 = 0. \tag{6}$$

These results (4), (5), and (6) verify what one already knows because of geometric properties of the pencil $\mathfrak{Q}$:

(a) The quadrics $\Phi(t)$ have the same symmetries as the quadrilateral $\mathcal{H}$. In our special case of $\mathcal{H}$ being equilateral, the planes xy and xz are symmetry planes. Therefore, the x-axis is a common axis of $\Phi(t)$. If $\Phi(t)$ is a hyperboloid with three axes, a second axis is parallel to EF, while the third one is parallel to AB.

(b) The diagonals of an arbitrarily given quadrilateral $\mathcal{H}$ are reciprocal polar lines for all quadrics $\Phi(t)$.

(c) The quadrics $\Phi(t)$ through $\mathcal{H}$ have the surface normals n_A, n_B, n_E, n_F at the vertices A, B, E, F in common. For an R-hyperboloid Φ_{Ri} all surface normals meet the rotation axis a_i. Therefore, a_i must of course intersect these special normals n_A, n_B, n_E, n_F. In the general case, when $\mathcal{H}$ has no symmetries, the normals n_A, n_B, n_E, n_F are pairwise skew, and we expect (in algebraic sense) two lines l_i, which meet these four lines. Such a line l_i is an axis of an R-hyperboloid, if and only if it includes the same angle with each of the four edges of $\mathcal{H}$.

Finally, we visualise the octahedron $\mathfrak{p}$ with its edge-quadrilateral $\mathcal{H}$ and the three R-hyperboloids Φ_{R1}, Φ_{R2}, Φ_{R3} though $\mathcal{H}$ in Figure 2, 3 and 4:

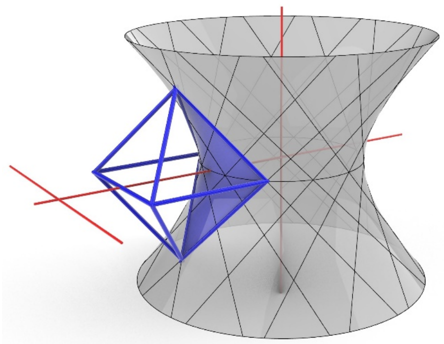

Figure 2: *R-hyperboloid Φ_{R1} through an edge-quadrilateral of an octahedron*

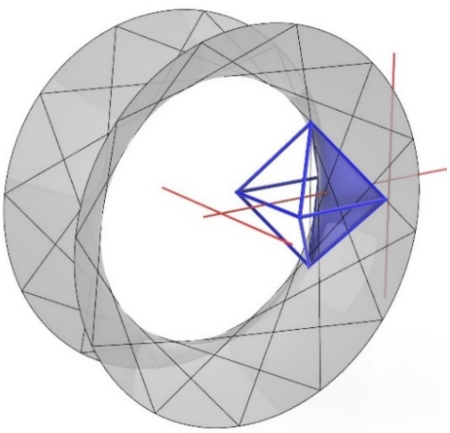

Figure 3: *R-hyperboloid Φ_{R2} through an edge-quadrilateral of an octahedron*

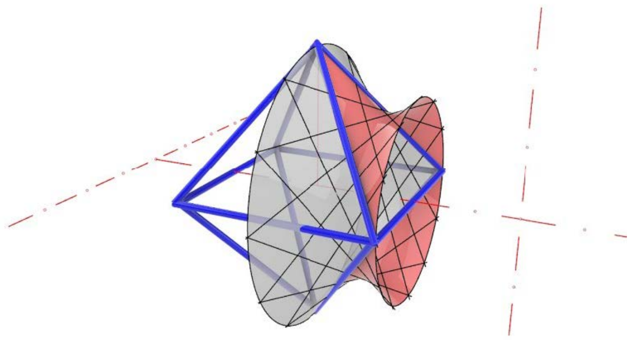

Figure 4: *R-hyperboloid Φ_{R3} through an edge-quadrilateral of an octahedron*

2 A criterion for quadrilaterals, which are generators of an R-hyperboloid

An arbitrarily given quadrilateral $\mathcal{H}$ consists of two pairs of skew generators (e_1, e_2), (f_1, f_2) of different reguli of the quadrics through $\mathcal{H}$. We look for properties of $\mathcal{H}$, such that there exists an R-hyperboloid Φ_R among the pencil of quadrics through $\mathcal{H}$, (we continue the numbering of properties of Chapter 1):

(d) Generators of an R-hyperboloid Φ_R include a fixed angle with its axis a and they are equidistant from a.

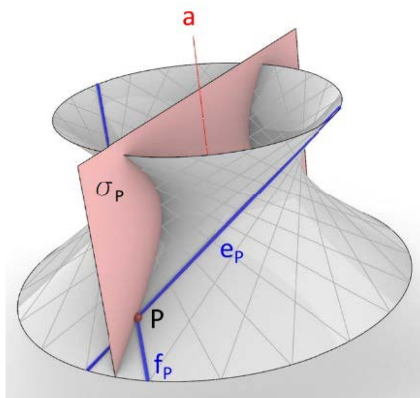

Figure 5: *One symmetry plane of two intersecting generators of an R-hyperboloid Φ_R contains the axis a of Φ_R.*

If we had a quadrilateral of generators on an R-hyperboloid Φ_R, then its normal projection in direction of the axis a of Φ_R yields a planar quadrilateral subscribed to the image of the circle of the gorge g. Because of property (d) yields, the lengths of the quadrilateral's edges are distorted by the same factor such that relations deduced for the lengths of edge images also hold for the situation in space.

There can occur different cases of such a normal projection, see Figures 6 and 7.

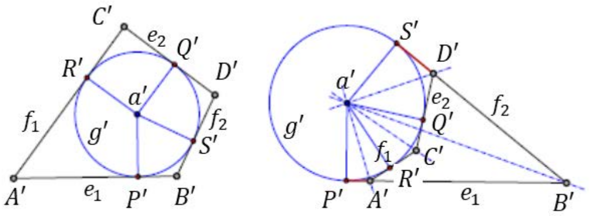

Figure 6: *Normal projection of a quadrilateral $\mathcal{H} = (ABCD)$ contained on an R-hyperboloid Φ_R; direction of projection parallel to the axis a of Φ_R*

For example, for the case shown in Figure 6, left, by adding segment lengths we obtain (see also [4])

$$\overline{A'B'} + \overline{C'D'} = \overline{A'C'} + \overline{B'D'} \iff |e_1| + |e_2| = |f_1| + |f_2|. \tag{7.1}$$

For the case shown in Figure 6, right, because of
$\overline{P'A'} = \overline{R'A'}$, $\overline{S'D'} = \overline{Q'D'}$ and
$\overline{A'B'} + \overline{P'A'} - \overline{S'D'} - \overline{D'B'} = 0$ and
$\overline{C'D'} - \overline{Q'D'} - \overline{C'A'} + \overline{R'A'} = 0$,
one derives

$$|e_1| - |e_2| = |f_1| - |f_2|. \tag{7.2}$$

In the left case in Figure 6 the R-hyperboloid does fill the interior of the quadrilateral, and therefore it is not suited for a TP-representation, because a TP-patch is contained in the interior of the convex hull of $\mathcal{H}$. (An f-generator passing to an inner point of segment e_1 cannot meet segment e_2 in an inner point, see Figure 6, left.)
A similar calculation shows that the cases shown in Figure 7 both lead to

$$|e_1| - |e_2| = |f_2| - |f_1|. \tag{7.3}$$

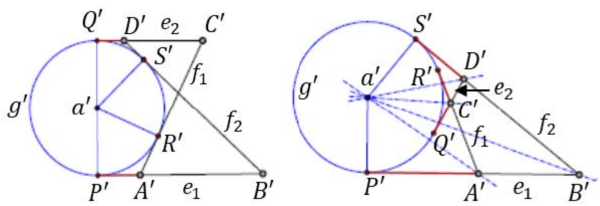

Figure 7: *Additional cases of images of $\mathcal{H}$*

Therewith we can formulate a criterion for the existence of an R-hyperboloid Φ_R through a given quadrilateral $(ABCD)$, (c.f. [4]):

Criterion 1 *The pencil of quadrics through a quadrilateral $\mathcal{H} = (ABCD)$ contains an R-hyperboloid Φ_R, if and only if at least one of the three conditions (7.1), (7.2), (7.3) holds.*

We complete this section by the following

Theorem 1 *If $\mathcal{H}$ is symmetric with respect to a symmetry plane through CB, then (7.1) and (7.2) are automatically fulfilled and there are two R-hyperboloids Φ_{R1}, Φ_{R2} through $\mathcal{H}$. If $\mathcal{H}$ is equilateral, all three conditions (7.1), (7.2), (7.3) are fulfilled and there are three R-hyperboloids Φ_{R1}, Φ_{R2}, Φ_{R3} through $\mathcal{H}$, and the R-hyperboloids have pairwise orthogonal axes.*

The case with three R-hyperboloids occurs as shown with the example in Chapter 1.
In the following we identify the points of the quadrilateral $\mathcal{H} = (A, B, C, D)$ with their coordinate vectors, such that $\vec{e}_1 = B - A$, $\vec{e}_2 = D - C$, $\vec{f}_1 = A - C$, $\vec{f}_2 = D - B$. Therewith the edge vectors are oriented such that the following closure condition (8) is fulfilled

$$\vec{e}_1 + \vec{f}_2 - \vec{e}_2 - \vec{f}_1 = 0. \tag{8}$$

We will also omit vector arrows, but keep in mind the orientation of the edges of $\mathcal{H}$. As (7.1) does not suit for a TP-patch representation of the R-hyperboloid, we can focus on the conditions (7.2) and (7.3), where we assume that at least one of them is fulfilled.

3 Further conditions for R-hyperboloids through a given quadrilateral

Two generators e and f of an R-hyperboloid Φ intersecting in $P \in \Phi$ are symmetric with respect to the plane spanned by the axis a of Φ and by P (see Figure 5). This property can be used for finding a condition, that the pencil $\mathfrak{Q}$ of hyperboloids through a given quadrilateral $\mathcal{H} = (e_1, e_2, f_1, f_2)$ contains an R-hyperboloid: Four of the symmetry planes of (e_i, f_j) must belong to a pencil of planes. If so, then they will intersect in the axis a of an R-hyperboloid. In each vertex of $\mathcal{H}$ there exist two symmetry planes σ_X^i spanned by the normal $e_i \times f_j$ and the symmetry lines s_X^i in the planes $e_i \vee f_j$, see Figure 8.
From Figure 8 we read off that of all possible combinations of symmetry planes there are only $\frac{1}{2}\binom{4}{2} = 3$, which make sense: *a*) $\{\sigma_A^1, \sigma_B^1, \sigma_C^1, \sigma_D^1\}$, *b*) $\{\sigma_A^2, \sigma_B^2, \sigma_C^2, \sigma_D^2\}$, and *c*) $\{\sigma_A^2, \sigma_B^1, \sigma_C^1, \sigma_D^2\}$. This suits again to the maximally three R-hyperboloids in the pencil $\mathfrak{Q}$. (Here and in the following we use the labelling in Figure 8.)
The normal vector of σ_A^1 resp. σ_A^2 is

$$s_A^2 = \frac{e_1}{\|e_1\|} + \frac{f_1}{\|f_1\|} \quad \text{resp.} \quad s_A^1 = \frac{e_1}{\|e_1\|} - \frac{f_1}{\|f_1\|}, \tag{9}$$

and, similarly, also for the other symmetry planes, σ_X^1 has normal vector s_X^2, while s_X^1 is normal to σ_X^2.

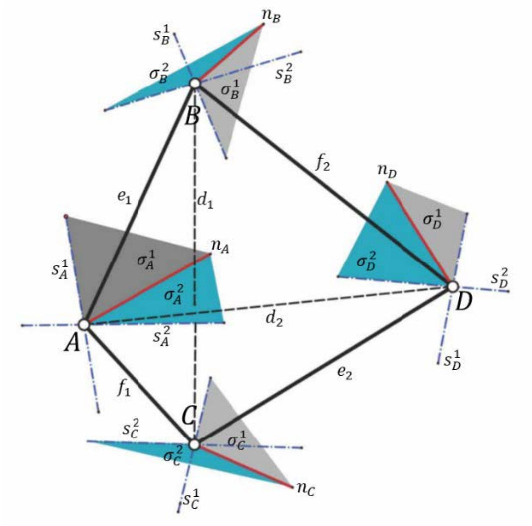

Figure 8: *A quadrilateral $\mathcal{H}$ and the symmetry planes of its pairs of consecutive edges.*

In case of *a*) we demand that $\{s_A^2, s_B^2, s_C^2, s_D^2\}$ necessarily are parallel to a plane. This means that

$$det(s_A^2, s_B^2, s_C^2) = 0 \quad \wedge \quad det(s_A^2, s_B^2, s_D^2) = 0. \tag{10}$$

By replacing s_X^2 by $\frac{e_i}{\|e_i\|} \pm \frac{f_j}{\|f_j\|}$ in (10) we obtain the same condition (11) for both equations:

$$\begin{aligned}&\|e_1\|.det(e_2, f_1, f_2) - \|e_2\|.det(e_1, f_1, f_2) =\\ &= \|f_1\|.det(e_1, e_2, f_2) - \|f_2\|.det(e_1, e_2, f_1).\end{aligned} \tag{11}$$

This means that, if one of the necessary conditions (10) is fulfilled, then the other is fulfilled, too. When we substitute the closure condition (8) $e_2 = e_1 + f_2 - f_1$ into (11) we get $det(e_1, f_1, f_2)(\|e_1\| - \|f_2\| - \|e_2\| + \|f_1\|) = 0$, which is equivalent to (7.3).
In case of *b*), if we proceed in the same manner for the two conditions $(s_A^1, s_B^1, s_C^1) = 0$, $(s_A^1, s_B^1, s_D^1) = 0$, and we obtain the equation

$$\begin{aligned}&\|e_1\|.det(e_2, f_1, f_2) + \|e_2\|.det(e_1, f_1, f_2) =\\ &= \|f_1\|.det(e_1, e_2, f_2) - \|f_2\|.det(e_1, e_2, f_1),\end{aligned} \tag{12}$$

which turns out to be equivalent to (7.1).
For case *c*) the conditions read as $(s_A^1, s_B^2, s_C^2) = 0$ and $(s_A^1, s_B^2, s_D^1) = 0$. The resulting single condition now becomes

$$\begin{aligned}&\|e_1\|.det(e_2, f_1, f_2) + \|e_2\|.det(e_1, f_1, f_2) =\\ &= -\|f_1\|.det(e_1, e_2, f_2) - \|f_2\|.det(e_1, e_2, f_1),\end{aligned} \tag{13}$$

which is equivalent to (7.2). We collect these statements as

Theorem 2 *Four symmetry planes of consecutive edges of a quadrilateral $\mathcal{H}$ intersect in a common line a, if and only if at least one of the conditions (11), (12), (13) is fulfilled. These conditions are equivalent to the conditions (7.3), (7.1) and (7.2) respectively. Therefore, such a common line a is the axis of an R-hyperboloid Φ through $\mathcal{H}$.*

4 Bézier representation of quadrics through a given quadrilateral

We consider the quadrangle $\mathcal{H}$ again and want to calculate the generators of a hyperboloid $\Phi(p)$ through it aiming at a Bézier-patch representation of $\Phi(p)$, see Figure 9. We use the fact that the f-generators intersect two e-generators of a ruled quadric "with equal cross-ratios". This means that

$$CR(U, E, A, B) = CR(U', E', C, D). \tag{14}$$

The generator $e_1 = AB$ is parameterised by $A \widehat{=} 0$, $B \widehat{=} 1$ and the midpoint $E \widehat{=} \frac{1}{2}$ of segment $[AB]$ and similarly for generator $e_2 = CD$. A third "f-generator" passing through $E \in e_1$ intersects e_2 in a point $E' \widehat{=} (\frac{1}{2})' =: p + \frac{1}{2}$. Obviously, for $p = 0$ one gets the paraboloid $\Phi(0) \in \mathfrak{Q}$.

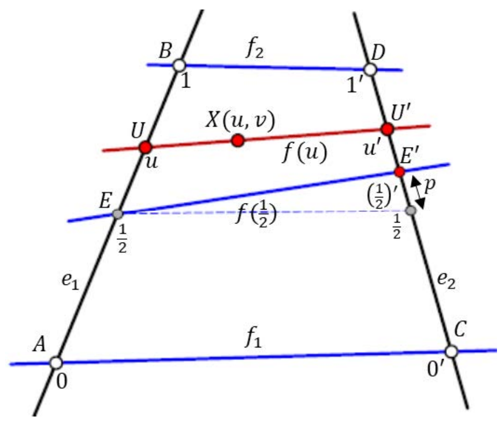

Figure 9: *The fixed f-generators f_1, f_2 of $\mathcal{H}$ together with a third f-generator define a hyperboloid $\Phi(p) \in \mathfrak{Q}$.*

Putting $u' = \frac{u+s}{qu+r}$ according to (14), then with $u = 0 \mapsto u' = 0$, $u = 1 \mapsto u' = 1$, $u = \frac{1}{2} \mapsto u' = \frac{1}{2} + p$ we obtain $s = 0$, $r = 1 - q$ and finally

$$u' = \frac{u}{qu + r} \quad \text{with} \quad q(p) = \frac{4p}{1+2p},\ r(p) = \frac{1-2p}{1+2p}. \tag{15}$$

Another convenient representation of condition (14) then is

$$t' := \frac{u'}{1-u'} = \frac{u(1-2p)}{(1-u)(1+2p)} =: t\frac{1}{r}. \tag{16}$$

Therewith follows for a Bézier-patch representation for $\Phi(p)$

$$X(u,v) = (1-v)((1-u)A + uB) + v((1-u')C + u'D),$$
$$(u,u',v \in [0,1]), \tag{17}$$

with v the parameter on generator $f(u) = vU + (1-v)U'$. (As before, we use the same symbols for points and their coordinate vectors.)

The form parameter $p = 0$ in (15) describes the unique paraboloid $\Phi(0) \in \mathfrak{Q}$. The parameter values $p = \pm\frac{1}{2}$ describe the singular quadrics, namely the pairs of planes in the pencil $\mathfrak{Q}$. We are now interested in the parameter value p for an R-hyperboloid in the quadric pencil $\mathfrak{Q}$ through $\mathcal{H}$, which is assumed to fulfil one of the conditions (7.2), (7.3). Because of the cross-ratio condition (14) it is enough to demand that one further generator, say $f(\frac{1}{2})$, together with $\frac{1}{2}e_1$, $(\frac{1}{2}+p)e_2$ and f_1 fulfils (7.2) or (7.3). For the vector $f(\frac{1}{2})$ follows

$$f(\tfrac{1}{2}) = f_1 + (\tfrac{1}{2}+p)e_2 - \tfrac{1}{2}e_1, \tag{18}$$

its squared norm is therefore

$$f^2(\tfrac{1}{2}) = f_1^2 + (\tfrac{1}{2}+p)^2 e_2^2 + \tfrac{1}{4}e_1^2 + 2(\tfrac{1}{2}+p)(e_2 f_1)$$
$$-(\tfrac{1}{2}+p)(e_1 e_2) - (e_1 f_1). \tag{19}$$

The R-hyperboloid conditions (7.2), (7.3) for $f(\frac{1}{2})$ are

$$\mp\|f(\tfrac{1}{2})\| = \tfrac{1}{2}\|e_1\| - (\tfrac{1}{2}+p)\|e_2\| \mp \|e_1\|\|f_1\|. \tag{20}$$

and we square (20) receiving

$$f^2(\tfrac{1}{2}) = \tfrac{1}{4}e_1^2 + (\tfrac{1}{2}+p)^2 e_2^2 + f_1^2 \pm 2(\tfrac{1}{2}+p)\|e_2\|\|f_1\|$$
$$-(\tfrac{1}{2}+p)\|e_1\|\|e_2\| \mp \|e_1\|\|f_1\|. \tag{21}$$

Now we compare (19) and (21) and get a linear equation in p. (In fact, there occur two such equations because of the different signs.)

$$(e_1 f_1) \pm \|e_1\|\|f_1\| = (\tfrac{1}{2}+p)[(-\|e_1\|\|e_2\| + (e_1 e_2))$$
$$+2(\pm\|e_2\|\|f_1\| - (e_2 f_1))]. \tag{22}$$

Here we see that (22) involves the angles between consecutive edges of $\mathcal{H}$, too:

$$(\frac{1}{2}+p) =$$
$$\frac{\|e_1\|\|f_1\|(\cos\sphericalangle e_1 f_1 \pm 1)}{\|e_1\|\|e_2\|(\cos\sphericalangle e_1 e_2 - 1) + 2\|e_2\|\|f_1\|(\pm 1 - \cos\sphericalangle e_2 f_1)}. \tag{23}$$

We put $\sphericalangle e_1 f_1 =: \alpha$, $\sphericalangle f_1 e_2 =: \gamma$, $\sphericalangle e_1 e_2 =: \varepsilon$; then, because of $1-\cos\xi = 2\sin^2\xi/2$ and $1+\cos\xi = 2\cos^2\xi/2$ equation (23) can be written as

$$p_1 = \frac{\|e_1\|\|f_1\|\cos^2\alpha/2}{2\|e_2\|\|f_1\|\sin^2\gamma/2 - \|e_1\|\|e_2\|\sin^2\varepsilon/2} - \frac{1}{2} \tag{24.1}$$

$$p_2 = \frac{\|e_1\|\|f_1\|\sin^2\alpha/2}{2\|e_2\|\|f_1\|\cos^2\gamma/2 + \|e_1\|\|e_2\|\sin^2\varepsilon/2} - \frac{1}{2} \tag{24.2}$$

Now we can state

Theorem 3 *An R-hyperboloid $\Phi(p)$ through a quadrilateral $\mathcal{H}$, which fulfils the conditions (7.2) resp. (7.3) allows the tensor-product representation (17), whereby the form parameter p takes the value p_1 (24.1) resp. p_2 (24.2).*

In the following chapter we will apply these results to some polyhedrons. As the chosen starting polyhedrons have regular faces, edge quadrilaterals are symmetric. This facilitates the calculation of the parameters p_1 and p_2.

5 Examples of polyhedrons with patches of R-hyperboloids as faces

If the start polyhedron has n-gons as faces ($n > 3$), see Figure 10 and 11, we split such a face into triangles. It is also possible to add pyramids to such a face to obtain an additional form parameter by the pyramid's height.

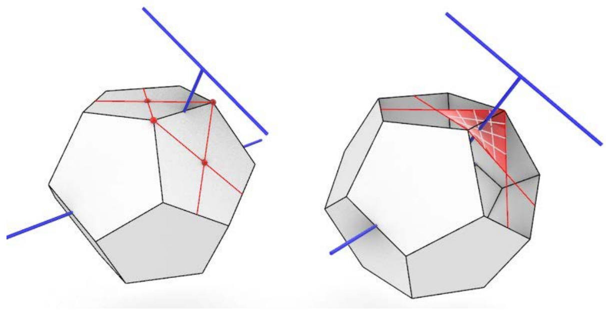

Figure 10: *The principle, how one can proceed in case of non-triangular faces of a polyhedron, shown at a regular dodecahedron*

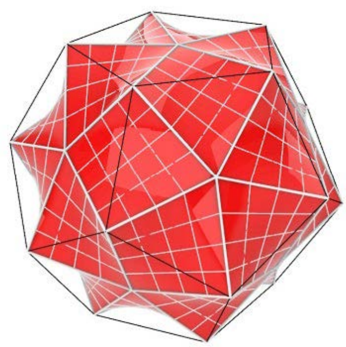

Figure 11: *The dodecahedron's faces are completely replaced by paraboloid patches.*

Because the pentagonal faces are tangential to the five patches connected at the midpoint of the face, the 12 midpoints must be interpreted as additional vertices, such that the object has got 32 vertices and 30 quadric patches. Almost the same object emerges by adding pyramids to the pentagonal faces of a dodecahedron, such that it gets 60 isosceles triangles as faces, see Figure 12. This object is a Catalan polyhedron and is called pentakis-dodecahedron or kisdodecahedron. Again pairs of triangles are replaced by quadric patches.

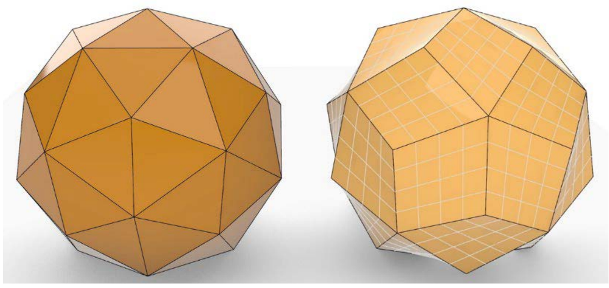

Figure 12: *The dodecahedron's faces are completely replaced by paraboloid patches.*

Choosing the height of the pyramids added to the faces of a dodecahedron suitably one can get a Kepler star. We show the principle of replacing two adjacent triangles by R-hyperboloid patches through equilateral edge quadrilateral in Figure 13.

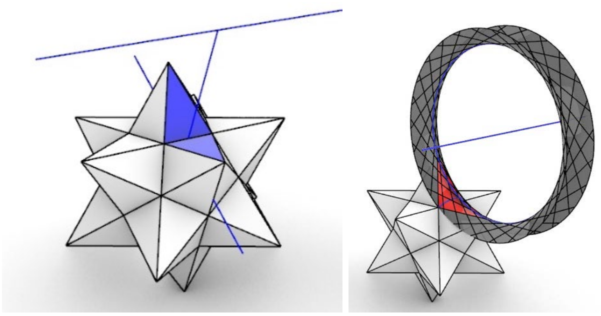

Figure 13: *A Kepler star with an R-hyperboloid patch through an equilateral edge quadrilateral*

The next object, an elongated pentagonal cupola, might have at least some architectural relevance by its "windows" formed by R-hyperboloids, Figure 15. The used edge quadrilaterals are equilateral. In this case we refrained from the patch representation according Theorem 3 and applied condition (7.1) as well as geometric properties derived from the octahedron in Chapter 1.

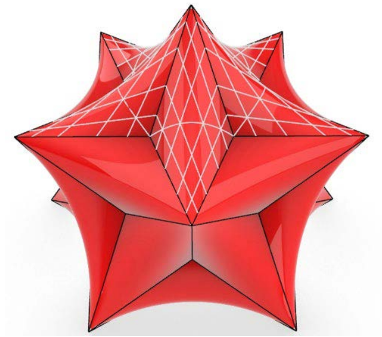

Figure 14: *A Kepler star completely covered with R-hyperboloid patches*

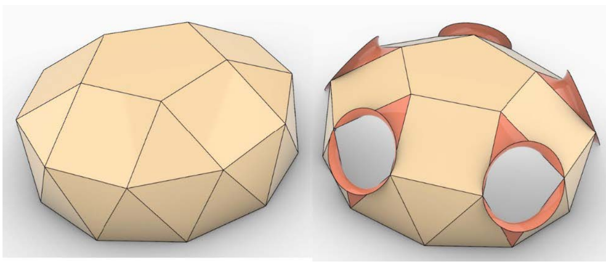

Figure 15: *R-hyperboloids through equilateral edge quadrangles forming "windows" into an elongated pentagonal cupola*

6 Pencils of R-hyperboloids and final remarks

The previous chapters were concerned with R-hyperboloids through a given quadrilateral of generators $\mathcal{H} = (e_1e_2f_1f_2)$ and we derived conditions for the existence of an R-hyperboloid through $\mathcal{H}$. Another approach could be to consider the pencil of R-hyperboloids through the skew generators e_1, e_2 and the second pencil through f_1 and f_2. The axes of such a pencil of R-hyperboloids are generators of the symmetry paraboloid $\Psi(e)$ of e_1 and e_2 resp. $\Psi(f)$ of f_1 and f_2, c.f. [3]. The two pencils have an R-hyperboloid in common, if and only if $\Psi(e)$ and $\Psi(f)$ have a common generator a, which acts as axis of the common R-hyperboloid. Obviously the conditions for that must be again (7.1), (7.2) and (7.3).

In [3] the symmetry paraboloid of two skew lines e_1 and e_2 is considered as the set of points, which are equidistant from these lines. When interpreting it as set of axes of R-hyperboloids through these lines one takes a line geometric viewpoint. (For line geometry c.f. e.g. [5]). The place of action is the projectively enclosed Euclidean 3-space. Indeed, it seems worthwhile to look at pencils of R-hyperboloids that way. They can be seen as 3D-generalisations of pencils of circles. The skew (and real) proper lines e_1 and e_2 span a hyperbolic linear congruence of lines meeting both, e_1 and e_2. If e_1 and e_2 coincide in the way that the line congruence becomes parabolic, we might ask again for the then parabolic pencil of R-hyperboloids in this congruence of lines. If e_1 and e_2 are skew and imaginary, they are axes of an elliptic linear congruence. Here pops up a case, where all R-hyperboloids are coaxial, such that the symmetry paraboloid $\Psi(e)$ degenerates into a single line.

There are many other ways to replace the planar faces of a polyhedron by patches of curved surfaces. One could e.g. blow up a balloon in the materialised edge frame of a closed polyhedron. Such structures are almost omnipresent in our environment. Pairs of faces replaced by minimal surfaces, a topic of differential geometry, will, in the most cases differ not essentially from quadric patches. This might justify the use of patches of paraboloids or R-hyperboloids instead for architectural purposes.

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