Polyhedrons the Faces of which are Special Quadric Patches

ABSTRACT

We seize an idea of Oswald Giering (see [1] and [2]), who replaced pairs of faces of a polyhedron by patches of hyperbolic paraboloids and link up edge-quadrilaterals of a polyhedron with the pencil of quadrics determined by that quadrilateral. Obviously only ruled quadrics can occur. There is a simple criterion for the existence of a ruled hyperboloid of revolution through an arbitrarily given quadrilateral. Especially, if a (not planar) quadrilateral allows one symmetry, there exist two such hyperboloids of revolution through it, and if the quadrilateral happens to be equilateral, the pencil of quadrics through it contains even three hyperboloids of revolution with pairwise orthogonal axes. To mention an example, for right double pyramids, as for example the octahedron, the axes of the hyperboloids of revolution are, on one hand, located in the plane of the regular guiding polygon, and on the other, they are parallel to the symmetry axis of the double pyramid.

Not only for platonic solids, but for all polyhedrons, where one can define edge-quadrilaterals, their pairs of face-triangles can be replaced by quadric patches, and by this one could generate roofing of architectural relevance. Especially patches of hyperbolic paraboloids or, as we present here, patches of hyperboloids of revolution deliver versions of such roofing, which are also practically simple to realize.

Key words: polyhedron, quadric, hyperboloid of revolution, Bézier patch

MSC2010: 51Mxx, (51M20, 51M30), 51N05, 51N20, 15Axx

Excerpt of what we aim to present in the following chapters

Chapter 1 deals with the regular octahedron $p$ as a standard example and replace pairs of triangles by quadric patches. Here we can already show the principle of how to proceed. Among the pencil of quadrics through an edge quadrilateral of $p$ we look for the hyperbolic paraboloid (“HP-surface”) and for hyperboloids of revolution (“R-
hyperboloids”). It turns out that descriptive geometric methods highly support an analytic treatment of the problem.

In Chapter 2 we deal with a criterion for quadrilaterals, which are generators of an R-hyperboloid. For a quadrilateral fulfilling the criterion we give a construction of the axis and the skirt circle of an R-hyperboloid through it as well as analytic descriptions of the R-hyperboloid by its equation and as a tensor-product patch (“TP-patch”). Additionally, we also ask for the set of R-hyperboloids through two skew given lines. This set is, to some extent, a 3D-generalisation of a (planar) elliptic pencil of circles.

The third chapter concerns polyhedrons p, the faces of which are n-gons \( n > 3 \). By adding pyramids of a certain height \( h \) to these faces one can interpret the original polyhedron \( p \) as the limit of the set of polyhedrons \( p(h) \) for \( h \to 0 \). This gives a more “natural” set of edge-quadrilaterals than that proposed by Giering [1] and [2] for the cube. We apply this way of splitting an n-gon-face into triangles for e.g. a box shaped polyhedron. Finally we show images of some Johnson polyhedrons with R-hyperboloid patches as faces.

Concluding we note that Giering’s idea to replace pairs of planar faces by HP-surfaces works for any polyhedron, while R-hyperboloids exist only for edge-quadrilaterals fulfilling the criterion mentioned in Chapter 2. Anyway, by choosing a certain quadric out of the pencil of quadrics through an edge-quadrilateral and describe it as a TP-patch one wins an additional design parameter, what works for all polyhedrons independent from the criterion. This could be of relevance for architectural design, too.

1 The regular octahedron and its R-hyperboloid faces

We connect a Cartesian frame with the regular octahedron \( p = \{A,B,C,D,E,F\} \) such that its midpoint becomes the origin \( O \) and one of its diagonals becomes the z-axis. The x- and y-axes are parallel to edges \( BC \) and \( AB \) (Figure 1). We consider the (equilateral) edge-quadrilateral \( H = \{A,B,E,F\} \) and the pencil \( \Omega \) of quadrics \( \Phi(t) \) through it. Setting the edge length \( AB = \sqrt{2} \) we obtain the vertex coordinates \( A = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), B = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0), E = (0,0,1), F = (0,0,-1) \).

![Figure 1: The octahedron p, its edge-quadrilateral \( H = \{A,B,E,F\} \), and the normals \( \Phi(t) \), which are common for all quadrics of the pencil \( \Phi(t) \) through \( H \). The lines \( \alpha_1, \alpha_2, \alpha_3 \) (dashed red) represent the axes of three R-hyperboloids through \( H \).](image-url)
(b) The diagonals of an arbitrarily given quadrilateral $H$ are reciprocal polar lines for all quadrics $\Phi(t)$.

(c) The quadrics $\Phi(t)$ through $H$ have the surface normals $n_A, n_B, n_E, n_F$ at the vertices $A, B, E, F$ in common. For an R-hyperboloid $\Phi_R$ all surface normals meet the rotation axis $a_i$. Therefore, $a_i$ must of course intersect these special normals $n_A, n_B, n_E, n_F$. In the general case, when $H$ has no symmetries, the normals $n_A, n_B, n_E, n_F$ are pairwise skew, and we expect (in algebraic sense) two lines $l_i$, which meet these four lines. Such a line $l_i$ is an axis of an R-hyperboloid, if and only if it includes the same angle with each of the four edges of $H$.

Finally, we visualise the octahedron $p$ with its edge-quadrilateral $H$ and the three R-hyperboloids $\Phi_{R1}, \Phi_{R2}, \Phi_{R3}$ though $H$ in Figure 2, 3 and 4:

![Figure 2: R-hyperboloid $\Phi_{R1}$ through an edge-quadrilateral of an octahedron](image)

![Figure 3: R-hyperboloid $\Phi_{R2}$ through an edge-quadrilateral of an octahedron](image)

Figure 4: R-hyperboloid $\Phi_{R3}$ through an edge-quadrilateral of an octahedron

2 A criterion for quadrilaterals, which are generators of an R-hyperboloid

An arbitrarily given quadrilateral $H$ consists of two pairs of skew generators $(e_1, e_2), (f_1, f_2)$ of different reguli of the quadrics through $H$. We look for properties of $H$, such that there exists an R-hyperboloid $\Phi_R$ among the pencil of quadrics through $H$, (we continue the numbering of properties of Chapter 1):

(d) Generators of an R-hyperboloid $\Phi_R$ include a fixed angle with its axis $a$ and they are equidistant from $a$.

![Figure 5: One symmetry plane of two intersecting generators of an R-hyperboloid $\Phi_R$ contains the axis $a$ of $\Phi_R$.](image)

If we had a quadrilateral of generators on an R-hyperboloid $\Phi_R$, then its normal projection in direction of the axis $a$ of $\Phi_R$ yields a planar quadrilateral subscribed to the image of the circle of the gorge $g$. Because of property (d) yields, the lengths of the quadrilateral’s edges are distorted by the same factor such that relations deduced for the lengths of edge images also hold for the situation in space.
There can occur different cases of such a normal projection, see Figures 6 and 7.

For example, for the case shown in Figure 6, left, by adding segment lengths we obtain (see also [4])
\[
\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AC} + \overrightarrow{BD} \iff |e_1| + |e_2| = |f_1| + |f_2|.
\] (7.1)

For the case shown in Figure 6, right, because of
\[
\overrightarrow{PA} = \overrightarrow{RA}, \quad \overrightarrow{SD} = \overrightarrow{QD} \quad \text{and} \quad \overrightarrow{AB} + \overrightarrow{PA} - \overrightarrow{QD} - \overrightarrow{DB} = 0 \quad \text{and} \quad \overrightarrow{CD} - \overrightarrow{QD} - \overrightarrow{CA} + \overrightarrow{RA} = 0,
\]
one derives
\[
|e_1| - |e_2| = |f_1| - |f_2|.
\] (7.2)

In the left case in Figure 6 the R-hyperboloid does fill the interior of the quadrilateral, and therefore it is not suited for a TP-representation, because a TP-patch is contained in the interior of the convex hull of \( \mathcal{H} \). (An \( e \)-generator passing to an inner point of segment \( e_1 \) cannot meet segment \( e_2 \) in an inner point, see Figure 6, left.)

A similar calculation shows that the cases shown in Figure 7 both lead to
\[
|e_1| - |e_2| = |f_2| - |f_1|.
\] (7.3)

We complete this section by the following

**Theorem 1** If \( \mathcal{H} \) is symmetric with respect to a symmetry plane through \( CB \), then (7.1) and (7.2) are automatically fulfilled and there are two R-hyperboloids \( \Phi_R^1, \Phi_R^2 \) through \( \mathcal{H} \). If \( \mathcal{H} \) is equilateral, all three conditions (7.1), (7.2), (7.3) are fulfilled and there are three R-hyperboloids \( \Phi_R^1, \Phi_R^2, \Phi_R^3 \) through \( \mathcal{H} \), and the R-hyperboloids have pairwise orthogonal axes.

The case with three R-hyperboloids occurs as shown with the example in Chapter 1.

In the following we identify the points of the quadrilateral \( \mathcal{H} = (A,B,C,D) \) with their coordinate vectors, such that \( e_1 = B - A, e_2 = D - C, f_1 = A - C, f_2 = D - B \). Thereewith we can formulate a criterion for the existence of an R-hyperboloid \( \Phi_R \) through \( \mathcal{H} \).

Two generators \( e \) and \( f \) of an R-hyperboloid \( \Phi \) intersecting in \( P \in \Phi \) are symmetric with respect to the plane spanned by the axis \( a \) of \( \Phi \) and by \( P \) (see Figure 5). This property can be used for finding a condition, that the pencil \( \Omega \) of hyperboloids through a given quadrilateral \( \mathcal{H} = (e_1, e_2, f_1, f_2) \) contains an R-hyperboloid: Four of the symmetry planes of \( (e_i, f_j) \) must belong to a pencil of planes. If so, then they will intersect in the axis \( a \) of an R-hyperboloid. In each vertex of \( \mathcal{H} \) there exist two symmetry planes \( \sigma_i^a \) spanned by the normal \( e_i \times f_j \) and the symmetry lines \( s_X^a \) in the planes \( e_i \vee f_j \), see Figure 8.

From Figure 8 we read off that of all possible combinations of symmetry planes there are only \( \frac{1}{2} \binom{4}{2} = 3 \) which make sense: \( a) \{ \sigma_1^a, \sigma_2^a, \sigma_3^a, \sigma_4^a \}, b) \{ \sigma_1^a, \sigma_2^a, \sigma_3^a, \sigma_4^a \}, \) and \( c) \{ \sigma_1^a, \sigma_2^a, \sigma_3^a, \sigma_4^a \} \). This suits again to the maximally three R-hyperboloids in the pencil \( \Omega \). (Here and in the following we use the labelling in Figure 8.)

The normal vector of \( \sigma_i^a \) resp. \( \sigma_i^b \) is
\[
s_{iX}^a = \frac{e_i}{||e_i||} + \frac{f_i}{||f_i||} \quad \text{resp.} \quad s_{iX}^b = \frac{e_i}{||e_i||} - \frac{f_i}{||f_i||},
\] (9)
and, similarly, also for the other symmetry planes, \( \sigma_i^a \) has normal vector \( s_{iX}^a \), while \( s_{iX}^b \) is normal to \( \sigma_i^b \).

3 Further conditions for R-hyperboloids through a given quadrilateral

Further conditions for R-hyperboloids through a given quadrilateral
A quadrilateral $K$ condition (11) for both equations:

This means that, if one of the necessary conditions (10) is fulfilled, then the other is fulfilled, too. When we substitute the condition (8) $e_2 = e_1 + f_2 - f_1$ into (11) we get $\det(e_1, f_1, f_2)(|e_1| - |f_2| - |e_2| + |f_1|) = 0$, which is equivalent to (7.3).

In case of (a) we demand that $(s_{A_1}^1, s_{B_2}^2, s_{C_2}^2)$ necessarily are parallel to a plane. This means that

$$\det(s_{A_1}^1, s_{B_2}^2, s_{C_2}^2) = 0 \quad \land \quad \det(s_{A_1}^2, s_{B_2}^2, s_{D_2}^2) = 0.$$ (10)

By replacing $s_{A_1}^1$ by $c_{\parallel} \pm f_{\parallel}$ in (10) we obtain the same condition (11) for both equations:

$$\|e_1\| \det(e_2, f_1, f_2) - |e_2| \det(e_1, f_1, f_2) = \|f_1\| \det(e_1, e_2, f_2) - |f_2| \det(e_1, e_2, f_1).$$ (11)

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For case (c) the conditions read as $(s_{A_1}^1, s_{B_2}^2, s_{C_2}^2) = 0$ and $(s_{A_1}^2, s_{B_2}^2, s_{D_2}^2) = 0$. The resulting single condition now becomes

$$\|e_1\| \det(e_2, f_1, f_2) + |e_2| \det(e_1, f_1, f_2) = -\|f_1\| \det(e_1, e_2, f_2) - |f_2| \det(e_1, e_2, f_1),$$ (13)

which is equivalent to (7.2). We collect these statements as

**Theorem 2** Four symmetry planes of consecutive edges of a quadrilateral $H$ intersect in a common line $a$, if and only if at least one of the conditions (11), (12), (13) is fulfilled. These conditions are equivalent to the conditions (7.3), (7.1) and (7.2) respectively. Therefore, such a common line $a$ is the axis of an $R$-hyperboloid $\Phi$ through $H$.

### 4 Bézier representation of quadrics through a given quadrilateral

We consider the quadrangle $H$ again and want to calculate the generators of a hyperboloid $\Phi(p)$ through it aiming at a Bézier-patch representation of $\Phi(p)$, see Figure 9. We use the fact that the $f$-generators intersect two $e$-generators of a ruled quadric “with equal cross-ratios”. This means that

$$CR(U, E, A, B) = CR(U', E', C, D).$$ (14)

The generator $e_1 = AB$ is parameterised by $A \equiv 0, B \equiv 1$ and the midpoint $E \equiv \frac{1}{2}$ of segment $[AB]$ and similarly for generator $e_2 = CD$. A third “$f$-generator” passing through $E \in e_1$ intersects $e_2$ in a point $E' \equiv \frac{1}{2} + \frac{p}{q}$. Obviously, for $p = 0$ one gets the paraboloid $\Phi(0) \in \Omega$.

$$u' = \frac{u}{qu + r} \quad \text{with} \quad q(p) = \frac{4p}{1 + 2p}, \quad r(p) = \frac{1 - 2p}{1 + 2p}.$$ (15)

Another convenient representation of condition (14) then is

$$u' = \frac{u}{1 - u} = \frac{u(1 - 2p)}{(1 - u)(1 + 2p)} = \frac{t}{r}.$$ (16)
Therewith follows for a Bézier-patch representation for $\Phi(p)$

$$X(u,v) = (1-v)((1-u)A + uB) + v((1-u')C + u'D),$$

$$(u,u',v \in [0,1]),$$

(17)

with $v$ the parameter on generator $f(u) = vU + (1-v)U'$. (As before, we use the same symbols for points and their coordinate vectors.)

The form parameter $p = 0$ in (15) describes the unique paraboloid $\Phi(0) \in \mathcal{Q}$. The parameter values $p = \pm \frac{1}{2}$ describe the singular quadrics, namely the pairs of planes in the pencil $\mathcal{Q}$. We are now interested in the parameter value $p$ for an R-hyperboloid in the quadric pencil $\mathcal{Q}$ through $\mathcal{H}$, which is assumed to fulfil one of the conditions (7.2), (7.3).

Because of the cross-ratio condition (14) it is enough to demand that one further generator, say $f(\frac{1}{2})$, together with $\frac{1}{2}e_1$, $(\frac{1}{2} + p)e_2$ and $f_1$ fulfils (7.2) or (7.3). For the vector $f(\frac{1}{2})$ follows

$$f(\frac{1}{2}) = f_1 + (\frac{1}{2} + p)e_2 - \frac{1}{2}e_1,$$

(18)

its squared norm is therefore

$$f^2(\frac{1}{2}) = f_1^2 + (\frac{1}{2} + p)^2 e_2^2 + \frac{1}{4} e_1^2 + 2(\frac{1}{2} + p)(e_2f_1) - (\frac{1}{2} + p)(e_1e_2) - (e_1f_1).$$

(19)

The R-hyperboloid conditions (7.2), (7.3) for $f(\frac{1}{2})$ are

$$\mp \|f(\frac{1}{2})\| = \frac{1}{2}\|e_1\| - (\frac{1}{2} + p)\|e_2\| \mp \|e_1\|\|f_1\|,$$

(20)

and we square (20) receiving

$$f^2(\frac{1}{2}) = \frac{1}{4} e_1^2 + (\frac{1}{2} + p)^2 e_2^2 + f_1^2 + 2(\frac{1}{2} + p)\|e_2\|\|f_1\| - (\frac{1}{2} + p)\|e_1\|\|e_2\| \mp \|e_1\|\|f_1\|.$$  

(21)

Now we compare (19) and (21) and get a linear equation in $p$. (In fact, there occur two such equations because of the different signs.)

$$(e_1f_1) \pm \|e_1\|\|f_1\| = (\frac{1}{2} + p)[(-\|e_1\||e_2\| + (e_1e_2)) + 2(\mp \|e_2\|\|f_1\| - (e_2f_1))].$$  

(22)

Here we see that (22) involves the angles between consecutive edges of $\mathcal{H}$, too:

$$\frac{1}{2} + p = \frac{\|e_1\|\|f_1\|\cos \angle e_1f_1 \pm 1}{\|e_1\||e_2\|\cos \angle e_1e_2 - 1 + 2\|e_2\||f_1\|\|\pm 1 - \cos \angle e_2f_1\|.}$$  

(23)

We put $\angle e_1f_1 =: \alpha$, $\angle f_1e_1 =: \gamma$, $\angle e_1e_2 =: \varepsilon$; then, because of $1 - \cos \varepsilon = 2 \sin^2 \frac{\varepsilon}{2}$ and $1 + \cos \varepsilon = 2 \cos^2 \frac{\varepsilon}{2}$ equation (23) can be written as

$$p_1 = \frac{\|e_1\|\|f_1\|\cos \alpha / 2}{2\|e_2\||f_1\|\sin^2 \gamma / 2 - \|e_1\||e_2\|\sin^2 \varepsilon / 2 - \frac{1}{2}}$$  

(24.1)

$$p_2 = \frac{\|e_1\||f_1||\sin^2 \alpha / 2}{2\|e_2\||f_1||\cos^2 \gamma / 2 + \|e_1\||e_2||\sin^2 \varepsilon / 2 - \frac{1}{2}}$$  

(24.2)

Now we can state

**Theorem 3** An R-hyperboloid $\Phi(p)$ through a quadrilateral $\mathcal{H}$, which fulfils the conditions (7.2) resp. (7.3) allows the tensor-product representation (17), whereby the form parameter $p$ takes the value $p_1$ (24.1) resp. $p_2$ (24.2).

In the following chapter we will apply these results to some polyhedrons. As the chosen starting polyhedrons have regular faces, edge quadrilaterals are symmetric. This facilitates the calculation of the parameters $p_1$ and $p_2$.

5 Examples of polyhedrons with patches of R-hyperboloids as faces

If the start polyhedron has $n$-gons as faces ($n > 3$), see Figure 10 and 11, we split such a face into triangles. It is also possible to add pyramids to such a face to obtain an additional form parameter by the pyramid’s height.

![Figure 10: The principle, how one can proceed in case of non-triangular faces of a polyhedron, shown at a regular dodecahedron](image)

![Figure 11: The dodecahedron’s faces are completely replaced by paraboloid patches.](image)
Because the pentagonal faces are tangential to the five patches connected at the midpoint of the face, the 12 mid-points must be interpreted as additional vertices, such that the object has got 32 vertices and 30 quadric patches. Almost the same object emerges by adding pyramids to the pentagonal faces of a dodecahedron, such that it gets 60 isosceles triangles as faces, see Figure 12. This object is a Catalan polyhedron and is called pentakis-dodecahedron or kisdodecahedron. Again pairs of triangles are replaced by quadric patches.

Figure 12: The dodecahedron's faces are completely replaced by paraboloid patches.

Choosing the height of the pyramids added to the faces of a dodecahedron suitably one can get a Kepler star. We show the principle of replacing two adjacent triangles by R-hyperboloid patches through equilateral edge quadrilateral in Figure 13.

Figure 13: A Kepler star with an R-hyperboloid patch through an equilateral edge quadrilateral

The next object, an elongated pentagonal cupola, might have at least some architectural relevance by its “windows” formed by R-hyperboloids, Figure 15. The used edge quadrilaterals are equilateral. In this case we refrained from the patch representation according Theorem 3 and applied condition (7.1) as well as geometric properties derived from the octahedron in Chapter 1.

Figure 14: A Kepler star completely covered with R-hyperboloid patches

Figure 15: R-hyperboloids through equilateral edge quadrangles forming “windows” into an elongated pentagonal cupola

6 Pencils of R-hyperboloids and final remarks

The previous chapters were concerned with R-hyperboloids through a given quadrilateral of generators \( H = (e_1, e_2, f_1, f_2) \) and we derived conditions for the existence of an R-hyperboloid through \( H \). Another approach could be to consider the pencil of R-hyperboloids through the skew generators \( e_1 \), \( e_2 \) and the second pencil through \( f_1 \) and \( f_2 \). The axes of such a pencil of R-hyperboloids are generators of the symmetry paraboloid \( \Psi(e) \) of \( e_1 \) and \( e_2 \) resp. \( \Psi(f) \) of \( f_1 \) and \( f_2 \), cf. [3]. The two pencils have an R-hyperboloid in common, if and only if \( \Psi(e) \) and \( \Psi(f) \) have a common generator \( a \), which acts as axis of the common R-hyperboloid. Obviously the conditions for that must be again (7.1), (7.2) and (7.3).

In [3] the symmetry paraboloid of two skew lines \( e_1 \) and \( e_2 \) is considered as the set of points, which are equidistant from these lines. When interpreting it as set of axes of R-hyperboloids through these lines one takes a line geometric viewpoint. (For line geometry c.f. e.g. [5]). The place of action is the projectively enclosed Euclidean 3-space. Indeed, it seems worthwhile to look at pencils
of R-hyperboloids that way. They can be seen as 3D-generalisations of pencils of circles. The skew (and real) proper lines $e_1$ and $e_2$ span a hyperbolic linear congruence of lines meeting both, $e_1$ and $e_2$. If $e_1$ and $e_2$ coincide in the way that the line congruence becomes parabolic, we might ask again for the then parabolic pencil of R-hyperboloids in this congruence of lines. If $e_1$ and $e_2$ are skew and imaginary, they are axes of an elliptic linear congruence. Here pops up a case, where all R-hyperboloids are coaxial, such that the symmetry paraboloid $\Psi(e)$ degenerates into a single line.

There are many other ways to replace the planar faces of a polyhedron by patches of curved surfaces. One could e.g. blow up a balloon in the materialised edge frame of a closed polyhedron. Such structures are almost omnipresent in our environment. Pairs of faces replaced by minimal surfaces, a topic of differential geometry, will, in the most cases differ not essentially from quadric patches. This might justify the use of patches of paraboloids or R-hyperboloids instead for architectural purposes.

References


