Generalized Regularity and the Symmetry of Branches of “Botanological” Networks

To the weighted regularity of Euclid.

1 Introduction

Let $A_1, A_2, \ldots, A_n$ be the vertices of a polygon $A_1A_2A_3\ldots A_n$ in a cyclic order.
If we set one point (node) weighted Steiner problem, by setting $B_i$ outwardly (or inwardly) on the sides of any affine regular triangles are affine regular and parallelograms are affine regular quadrilaterals in $\mathbb{R}^2$.

Gerber connected the affine regularity with the Euclidean regularity of $n-gons$ in [34], (see also [2] and [3]) and proved the result: If you construct regular $n-gons$ outwardly (or inwardly) on the sides of any affine regular $n-gon$, then their centers form the vertices of a regular $n-gon$. The case $n=4$ was proved by Thebault, who gave the first generalization of Napoleon’s regularity for the case $n=3$ (Napoleon’s theorem) ([2] p. 185).

We start by giving the definitions of a weighted Fermat-Torricelli tree and weighted Steiner tree for a boundary quadrilateral, in order to derive a new regularity of quadrilaterals which is different from Coxeter’s, Gerber’s and Thebault’s approach. The new regularity of quadrilaterals is achieved by the construction of isosceles triangles outwardly on the parallel sides of a rectangle or a trapezoid. Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$. We denote by $A_i(x_i,y_i)$ the vertices of $A_1A_2A_3A_4$, by $B_i$ a positive real number (weight) which corresponds to $A_i$, by $O_{12}(x_{012},y_{012})$, by $O_{34}(x_{034},y_{034})$ two points in $\mathbb{R}^2$ with given weights $B_{12}$ in $O_{12}$ and $B_{34}$ in $O_{34}$, by $d(X,Y)$ the Euclidean distance $\|XY\|$, for $X,Y \in \mathbb{R}^2$.

The weighted Steiner problem for $A_1A_2A_3A_4$ in $\mathbb{R}^2$ states that:

**Problem 1** Find $O_i(x_{0i},y_{0i})$, for $i = \{12,34\}$, such that

$$f(O_{12},O_{34}) = B_1d(O_{12},A_1) + B_2d(O_{12},A_2) + B_3d(O_{34},A_3) + B_4d(O_{34},A_4) + \frac{B_{12} + B_{34}}{2}d(O_{12},O_{34}) \rightarrow \min.$$  \hspace{1cm} (1)

For $B_1 = B_2 = B_3 = B_4$, the solution of the (unweighted) Steiner problem is called a Steiner tree. Gilbert and Pollack introduce the Steiner tree topologies for $A_1A_2A_3A_4$, in their classical study ([35]). They mention three topologies of solutions w.r.t. the boundary $A_1A_2A_3A_4$:

1. If we set one point (node) $F$ (Fermat-Torricelli point) different from $A_i$, the solution is called a Fermat-Torricelli tree. The Fermat-Torricelli point $F$ has four connections \{FA_1,FA_2,FA_3,FA_4\}. This is a special case of the unweighted Steiner problem, by setting $B_{12} = 0$ or $B_{34} = 0$.

2. If we set two points (nodes) $O_{12}$ and $O_{34}$ (Steiner points) and $B_{12} + B_{34} = 2$, such that the objective function \[40\] is minimized, then we derive a solution which is called a full Steiner tree. The Steiner points $O_{12}$ and $O_{34}$ have three connections \{A_1O_{12},A_2O_{12},O_{12}O_{34}\} and \{A_3O_{34},A_4O_{34},O_{12}O_{34}\}, respectively.

3. If we set one point (node) Steiner point $O_{12}$ and $O_{34} \equiv A_{x0rA_4}$, such that the objective function \[40\] is minimized, then we derive a degenerate Steiner tree.

It is well known that the Steiner point with three connections possesses the equiangular property $\frac{360}{3}$. The angle formed by the Steiner point as a vertex and two connections is $120^\circ$, for the unweighted case and by assuming that $B_{12} + B_{34} = 2$ ([3]). The same property holds for the Fermat-Torricelli point for a boundary triangle, which coincides with the Steiner point. The Fermat-Torricelli tree of a convex quadrilateral consists of the two diagonals $A_1A_3$ and $A_2A_4$, which meet at the intersection point $F$ (Fermat-Torricelli point) for the unweighted case.

Rubinstein, Thomas and Weng studied in [8] the unweighted Steiner problem for tetrahedra in $\mathbb{R}^3$. They succeeded in locating the Simpson line, which passes through the two Steiner points $O_{12}$ and $O_{34}$ in $\mathbb{R}^3$. The vertex $A_{12}$ of the equilateral $\triangle A_1A_2A_3$, which lies on the opposite side of $A_1A_2$ to $O_{12}$ is referred to as the $e$-point of $A_1A_2$. The vertex $A_{34}$ of the equilateral $\triangle A_3A_4A_2$, which lies on the opposite side of $A_3A_4$ to $O_{34}$ is referred to as the $e$-point of $A_3A_4$. The Simpson line passes through the $e$-points of $A_1A_2$ and $A_3A_4$, respectively, and

$$d(A_{12},A_{34}) = d(O_{12},A_1) + d(O_{12},A_2) + d(O_{34},A_3) + d(O_{34},A_4) = L.$$  

The Melzak Circle is a circle $C(O_1,r_{12})$, which passes through $A_1$, $A_2$, $A_{12}$ and intersects the Simpson line at $O_{12}$. Similarly, the Melzak Circle $C(O_2,r_{34})$ passes through $A_3$, $A_4$, $A_{34}$ and intersects the Simpson line at $O_{34}$. The Melzak construction via the method of $e$-points is established in [7]. Furthermore, Rubinstein, Thomas and Weng gave explicit formulas for computing Steiner trees for four points in $\mathbb{R}^2$, for all possible cases, in which the lines defined by $A_1A_2$ and $A_3A_4$ either intersect or are parallel ([8] Chapter 3, Cases (1), (2))). We set $\varphi \equiv \angle (A_1A_2,A_3A_4)$. For $\varphi = 0$, $A_2A_4$ and $A_3A_4$ are parallel, we refer to this solution as the Steiner zero solution. The Steiner zero solution depends on the distance $h$ between the two parallel lines, the midpoints of $A_1A_2$ and $A_3A_4$, respectively and the radius of Melzak circles $r_{12}$ and $r_{34}$ ([8] Chapter 3, Explicit formulas Case (2), page 65)).

Ivanov and Tuzhilin introduced the concept of the weighted Simpson line and they found the relation of the length of the weighted network with the length of a Simpson line ([6] Theorem 1) which gives

$$\frac{B_{12} + B_{34}}{2}L = B_1d(O_{12},A_1) + B_2d(O_{12},A_2) + B_3d(O_{34},A_3) + B_4d(O_{34},A_4).$$  

We note that $A_{12}$ and $A_{34}$ are not the $e$-points for the weighted case.

In this paper, we introduce the generalized (weighted) regularity of convex quadrilaterals and tetrahedra, which gives a new evolutionary class of convex quadrilaterals and tetrahedra in $\mathbb{R}^3$. 


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The property of generalized regularity states that the Simpson line defined by the two Steiner points $O_{12}$ and $O_{34}$ passes through the corresponding Fermat-Torricelli point of the same convex quadrilateral. The property of weighted regularity for weighted rectangles states that the weighted Simpson line defined by the two weighted Steiner points passes through the corresponding weighted Fermat-Torricelli point of the same rectangle.

The main results are:

1. The property of generalized regularity possess a class of convex quadrilaterals (generalized regular quadrilaterals), which corresponds to the Steiner zero solution and it consists of quadrilaterals having two of their opposite sides parallel (Theorem 1).

2. Let $A_1A_2A_3A_4$ be a rectangle in $\mathbb{R}^2$ and $A_1F$, $A_2F$ be the two roots of the corresponding weighted Fermat-Torricelli tree (thumb), the weighted Fermat-Torricelli point $F$ is located on the ground and $A_1F$, $A_1F$ are two branches of the weighted Fermat-Torricelli tree (thumb). If the weighted Simpson line $A_{12}A_{34}$ is perpendicular to the ground and $A_1A_2A_3A_4$ is a generalized regular quadrilateral, we prove that $B_1^2 + B_2^2 = B_3^2 + B_4^2$ (Theorem 2).

3. Two branches have equal weights and the two roots have equal weights, if the thumb inherits a symmetry w.r.t. the midperpendicular line of the two opposite sides of the rectangle, which is perpendicular to the ground (equal branches and equal roots, Proposition 3).

4. The dynamic Plasticity of weighted network with two roots and two growing branches states that:

Given the weighted Fermat-Torricelli point $A_0$ that has got a subconscious $B_0$ to be an interior point of the tetrahedron $A_1A_2A_3A_4$ with the vertices lie on four prescribed rays that meet at $A_0$, the positive real weights $B_{ij}$ depends on the five given values of $\alpha_{12}$, $\alpha_{13}$, $\alpha_{14}$, $\alpha_{23}$, $\alpha_{24}$ and $B_{0}$ (Theorem 3).

5. We assume that the common perpendicular line of each tetrahedron $A_1A_2A_3A_4$ passes through the common midpoints $m_{12}$ and $m_{34}$ of $A_1A_2$ and $A_3A_4$, respectively and $m_{12}m_{34} \gg A_1A_2$. We prove the following theorem for a botanical thumb (without a main branch) (Theorem 4).

If $A_0$ lies on the common perpendicular segment $m_{12}m_{34}$, then $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$.

6. We prove the following theorem for a “botanical” network (with a main branch) (Theorem 5).

If $A_0$ lies on the common perpendicular segment $m_{12}m_{34}$, then $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$.

The dynamic plasticity (Theorem 6), geometric plasticity (Lemma 2) and rotational plasticity (Proposition 4) of generalized regular tetrahedra (Definition 7) and generalized weighted regular tetrahedra (Definition 8) develops a symmetry for the weights for a “botanical” thumb (Theorem 1 Evolutionary scheme) or a botanical network in $\mathbb{R}^3$ (Theorem 10, Evolutionary scheme).

2 The property of generalized regularity of convex quadrilaterals in $\mathbb{R}^2$

Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$, such that $B_1 = B_2 = B_3 = B_4 = 1$ and $B_{12} + B_{34} = 2$. We recall that a weight $B_i$ corresponds to the vertex $A_i$, for $i = 1, 2, 3, 4$, a weight $B_{12} \equiv 1$ corresponds to the Steiner point $O_{12}$ and $B_{34} \equiv 1$ corresponds to the Steiner point $O_{34}$. The Fermat-Torricelli point $F$ is the intersection of the two diagonals of $A_1A_3$ and $A_2A_4$. We denote by $L$ the Simpson line, which passes through the $e$-points $A_{12}$, $A_{34}$ and $O_{12}$, $O_{34}$ and by $T_{12}$, $T_{34}$ the intersection points of the common angle bisector of the vertical angles $A_1F_2A_2$ and $A_3F_4A_4$ and the line segments $A_1A_2$ and $A_3A_4$, respectively.

Definition 1 (Generalized regularity) A generalized regular quadrilateral is a convex quadrilateral in $\mathbb{R}^2$, such that the Simpson line $L$ passes through the Fermat-Torricelli point $F$.

Definition 2 (Weighted regularity) A weighted regular quadrilateral is a convex quadrilateral in $\mathbb{R}^2$, such that the weighted Simpson line $L$ passes through the weighted Fermat-Torricelli point $F$.

Without loss of generality, we assume that:

$A_i = A_i(x_i, y_i)$, for $i = 1, 2, 3, 4$, $F = (x_F, y_F)$, $A_{34} = A_{34}(x_{34}, y_{34})$ and $A_{12} = A_{12}(x_{12}, y_{12})$, such that:

$y_4 > y_3 > y_2 > y_1$, $x_1 < x_4 < x_3 < x_2$.

Theorem 1 The property of generalized regularity possess a class of convex quadrilaterals (generalized regular quadrilaterals), which corresponds to the Steiner zero solution and it consists of quadrilaterals having two of their opposite sides parallel.

Proof. The intersection of the two diagonals $A_1A_3$, $A_2A_4$ is the unweighted Fermat-Torricelli point $F = (x_F, y_F)$, where

\[
x_F = \frac{x_1(x_3 - y_1) - x_2(x_4 - y_2)}{x_3 - x_1} - \frac{x_1(x_4 - y_2) - y_1 + y_2}{x_4 - x_2} = \frac{x_1(x_3 - y_1) - x_2(x_4 - y_2) - y_1 + y_2}{x_3 - x_1} \quad (2)
\]

and

\[
y_F = \frac{x_1(x_3 - y_1)(x_3 - x_1) - x_2(x_4 - y_2)(x_4 - x_2) + y_1 y_2}{x_3 - x_1} + y_1. \quad (3)
\]

We shall express the coordinates of the $e$-point $A_{34} = A_{34}(x_{34}, y_{34})$ and $y_{34}$ w.r. to $x_3, y_3, x_4, y_4$ (see Fig 1).
By working similarly, we derive a second order degree polynomial w.r. to $x_{12}$ and taking into account $x_{12} < \frac{34 + \sqrt{21}}{2}$, we obtain:

$$x_{12} = \frac{x_{12}^3 + x_{12} y_{12}^2 - 2 x_{12} y_{12} - \sqrt{3} M + x_{12} y_{12}^2}{2 (x_{12} - y_{12})^2 + (y_{12} - y_{12})^2} + \frac{x_{12} y_{12}^2 - 2 x_{12} y_{12} + x_{12}^2 - x_{12} y_{12}^2 - 3 x_{12} y_{12} + x_{12}^3}{2 (x_{12} - y_{12})^2 + (y_{12} - y_{12})^2}$$

(7)

where

$N \equiv (x_{12} - y_{12})^2 + (y_{12} - y_{12})^2 |y_{12} - y_{12}|$.

The area of $\triangle A_{12} A_{34} F$ is given by:

$$A(\triangle A_{12} A_{34} F) = |\det \begin{pmatrix} x_F & y_F & 1 \\ x_{12} & y_{12} & 1 \\ x_{34} & y_{34} & 1 \end{pmatrix}|.$$  

(8)

By substituting $y_1 = y_3 + \frac{y_{12} - y_{12}}{x_{12} - x_{12}} (x_4 - x_3)$ in (8) and by getting as a common factor $d(A_1 A_2)^2 (y_{12} - y_{12})^2$, we derive that

$$A(\triangle A_{12} A_{34} F) = f(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) g(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$$

where

$$g(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = x_3 - x_4 + (x_1 - x_3) \frac{|x_3 - x_4|}{|x_1 - x_2|}.$$  

(9)

Without loss of generality, we assume that $x_2 > x_1$ and $x_3 > x_4$.

Hence, by calculating (9), we deduce that $A(\triangle A_{12} A_{34} F) = 0$ and $A_{12}, A_{34}$ and $F$ are collinear only when $A_1 A_2$ is parallel to $A_3 A_4$.

We denote by $H$ the distance between $A_1 A_2$ and $A_3 A_4$. Suppose that $H > d(A_1 A_2) + d(A_3 A_4)$ and $\varphi \geq \angle A_2 A_1 A_3 \leq 120^\circ$, $\varphi \geq \angle A_1 A_2 A_4 \leq 120^\circ$, where $\varphi = \arctan\left(\frac{d(A_1 A_2) + H}{d(A_3 A_4)}\right)$.

**Proposition 1** If $A_1 A_2 \parallel A_3 A_4$, the intersection point of the common angle bisector of $\angle A_1 F A_2$ and $\angle A_3 F A_4$ and the Simpson line defined by $A_{12} A_{34}$ is the Fermat-Torricelli point $F$.

**Proof.** By applying Theorem 1, $F$ lies on the Simpson line. Therefore, the common angle bisector of $\angle A_1 F A_2$ and $\angle A_3 F A_4$ and the Simpson line defined by $A_{12} A_{34}$ passes through the Fermat-Torricelli point $F$. \qed

**Remark 1** If $x_{12} < \frac{x_{12} + x_{12}}{2}$ and $x_{12} < \frac{x_{12} + x_{12}}{2}$, we derive:

$$x_{34} = \frac{x_{34} y_{12}^2 - 2 x_{34} y_{12} + x_{34}^2 - x_{34} y_{12}^2 - 3 x_{34} y_{12} + x_{34}^3}{2 (x_{12} - y_{12})^2 + (y_{12} - y_{12})^2} + \frac{x_{34} y_{12}^2 - 2 x_{34} y_{12} + x_{34}^2 - x_{34} y_{12}^2 - 3 x_{34} y_{12} + x_{34}^3}{2 (x_{12} - y_{12})^2 + (y_{12} - y_{12})^2}$$

(10)
and taking into account

\[ x_{12} = \frac{x_1 y_2^2 + y_1 y_2^2 - 2x_1 y_1 y_2 - \sqrt{3} N + x_2 y_1^2}{2 (x_1 - x_2)^2 + (y_1 - y_2)^2} + \frac{x_2 y_1^2 - 2x_2 y_1 y_2 + x_1^2 - x_2 x_1^2 - x_1 y_1 + x_2^3}{2 (x_1 - x_2)^2 + (y_1 - y_2)^2}, \]

the corresponding determinant of the area \( A(\triangle A_{12}A_{34}F) \) is non-zero.

Examples of generalized regular quadrilaterals are the square, rectangle and the isosceles trapezoid.

The following results are a direct consequence of Theorem 1:

**Proposition 2** A square is a generalized regular quadrilateral, which corresponds to two Steiner zero solutions, having their Simpson lines perpendicular and meet at the Fermat-Torricelli point \( F \).

**Corollary 1** A square is a generalized regular quadrilateral, such that the two Simpson lines and the two corresponding angle bisectors w.r.t the vertical angles coincide (two minimum Steiner trees).

**Corollary 2** A rectangle is a generalized regular quadrilateral, such that the two Simpson lines and the two corresponding angle bisectors w.r.t the vertical angles coincide and the Simpson line which is midperpendicular w.r.t. to the parallel sides with greater length does not give a minimum Steiner tree (a unique minimum Steiner tree).

**Corollary 3** An isosceles trapezoid is a generalized regular quadrilateral, such that the Simpson line (midperpendicular) which passes through the Fermat-Torricelli point \( F \) and the corresponding angle bisector w.r.t the vertical angles coincide.

3 Creation of a “botanological” thumb for a boundary rectangle in \( \mathbb{R}^2 \)

A “botanological” network for four non-collinear points in \( \mathbb{R}^2 \) is introduced and studied in [13] for open systems (Botany).

**Definition 3 (“Botanical” network, [13])** A “botanical” network for four non-collinear points is a two-way communication network, which has the topology of a weighted minimal Steiner tree in \( \mathbb{R}^2 \), having two weighted Fermat-Torricelli nodes (Steiner nodes), two weighted roots, two weighted branches and one main branch.

Let \( A_1A_2A_3A_4 \) be a weighted rectangle in \( \mathbb{R}^2 \), \( B_i \) be a weight which corresponds to each vertex \( A_i \), for \( i = 1, 2, 3, 4 \), \( A_1F, A_2F \) are the two roots of the corresponding weighted Fermat-Torricelli tree (thumb). We assume that the weighted Fermat-Torricelli point \( F \) is located on the ground and \( A_3F, A_4F \) are two branches of the weighted Fermat-Torricelli tree (thumb) and \( A_1A_2 \gg A_1A_2 \).

The weighted Simpson line is a line defined by \( A_{12}F \), where \( A_{12} \) is a vertex of \( \triangle A_{12}A_1A_2 \), which lies on the opposite side of \( A_1A_2 \) to \( O_{12} \) and \( A_{34} \) is a vertex of \( \triangle A_{34}A_3A_4 \), which lies on the opposite side of \( A_3A_4 \) to \( O_{34} \). The weighted Steiner points \( O_{12} \) and \( O_{34} \) are the two nodes of the weighted Steiner tree and they both lie on \( A_{12}A_{34} \), with equal weights \( \frac{B_{12} + B_{34}}{2} \).

**Definition 4** A “botanical” thumb for a boundary rectangle is a two-way communication network, which has the topology of a weighted Fermat-Torricelli tree in \( \mathbb{R}^2 \), having one weighted Fermat-Torricelli node, two weighted roots and two weighted branches, which is enriched by the property of generalized regularity of quadrilaterals, such that \( A_{12}A_{34} \) is perpendicular to \( A_1A_2 \).

We assume that the weighted Fermat-Torricelli point \( F \) of \( A_1A_2A_3A_4 \) (\( B_{12} = B_{34} = 0 \)) lies on the ground and \( A_1A_2 \) is parallel to the ground. Our main result is the following theorem, which gives a weighted condition for the four weights of a thumb whose weighted Simpson line is perpendicular to the ground and \( A_1A_2 \) and passes through the corresponding weighted Fermat-Torricelli point \( F \).

**Theorem 2** If \( A_{12}A_{34} \) is perpendicular to \( A_1A_2 \),

\[ B_1^2 = B_2^2 + B_3^2 - B_4^2. \]  

**Proof.** We consider the weighted Steiner tree for the boundary \( A_1A_2A_3A_4 \). We recall that the objective function is given by:

\[ f(O_{12}, O_{34}) = B_1 d(O_{12}, A_1) + B_2 d(O_{12}, A_2) + B_3 d(O_{34}, A_3) + B_4 d(O_{34}, A_4) + \frac{B_{12} + B_{34}}{2} d(O_{12}, O_{34}) \rightarrow \min, \]

where \( O_{12} \) is the weighted Fermat-Torricelli point (Steiner node) of \( \triangle A_1A_2O_{34} \) with corresponding weights \( B_1, B_2 \) and \( B_{12} + B_{34} \), respectively, and \( O_{34} \) is the weighted Fermat-Torricelli point (Steiner node) of \( \triangle A_3A_4O_{34} \) with corresponding weights \( B_3, B_4 \) and \( B_{12} + B_{34} \), respectively. Hence, the construction of the weighted Simpson line yields the following relations:

\[ B_1 \sin \angle A_1A_2A_{12} = B_2 \sin \angle A_2A_1A_{12} \]  

and

\[ B_3 \sin \angle A_3A_4A_{34} = B_4 \sin \angle A_4A_3A_{34}. \]
Weighted Fermat-Torricelli trees and weighted Steiner trees

We need the following lemma, in order to prove that the weights w.r. to the two symmetrical roots and a pair we obtain that:

\[ B_1 \cos \angle A_1A_2A_{12} = B_4 \cos \angle A_4A_3A_{34} \]

and

\[ B_2 \cos \angle A_2A_1A_{12} = B_3 \cos \angle A_3A_4A_{34}. \]

By squaring both sides of (13), (14), (15) and (16) and by adding the first and third derived relation and the second and fourth derived relation, we deduce (17).

We need the following lemma, in order to prove that the symmetry of a thumb is determined by a pair of equal weights w.r. to the two symmetrical roots and a pair of equal weights w.r. to the two symmetrical branches. Let \( O = O(0,0) \), be the intersection of the diagonals of \( A_1A_2A_3A_4 \).

\textbf{Lemma 1}

\[ d(A_1, F)^2 + d(A_3, F)^2 = d(A_2, F)^2 + d(A_4, F)^2. \]

\textbf{Proposition 3} If the thumb inherits a symmetry w.r to the midperpendicular line of the two opposite sides of the rectangle, which is perpendicular to the ground (equal branches and equal roots), then \( B_1 = B_2 \) and \( B_3 = B_4 \).

\textbf{Proof.} By replacing \( d(A_1, F) = d(A_2, F) \) in (17), we get

\[ d(A_3, F) = d(A_4, F). \]

The weighted Simpson line \( A_{12}A_{34} \) is the midperpendicular line of \( A_1A_2 \) and \( A_3A_4 \) and passes through the weighted Fermat-Torricelli point \( F \). Therefore, \( A_1A_2A_3A_4 \) is a generalized weighted regular rectangle. Thus, we get:

\[ B_1 \sin \angle A_1A_2A_{12} = B_2 \sin \angle A_2A_1A_{12} \]

and

\[ B_3 \sin \angle A_3A_4A_{34} = B_4 \sin \angle A_4A_3A_{34}. \]

By replacing \( \angle A_1A_2A_{12} = \angle A_2A_1A_{12} \) in (18) and \( \angle A_3A_4A_{34} = \angle A_4A_3A_{34} \) in (19), we get: \( B_1 = B_2 \) and \( B_3 = B_4 \).

\section{Creation of a “botanological” thumb with symmetrical branches in the three dimensional Euclidean Space}

Let \( A_1A_2A_3A_4 \) be \( n \) tetrahedra in \( \mathbb{R}^3 \) and \( B_{jk} \) be the weight (positive real number) which corresponds to the vertex \( A_{jk} \), for \( i = 1, 2, ..., n \) and \( j = 1, 2, 3, 4 \). We denote by \( \bar{u}(A_{ik}, A_{jk}) \) the unit vector from \( A_{ik} \) to \( A_{jk} \). We assume that \( \| \sum_{i=1}^{n} B_{jk} \bar{u}(A_{ik}, A_{jk}) \| > B_{jk} \) hold, in order to locate weighted Fermat-Torricelli trees with four branches \( \{A_{0k}A_{1k}, A_{0k}A_{2k}, A_{0k}A_{3k}, A_{0k}A_{4k}\} \) that got a subconscious node.

\textbf{Lemma 2 (Geometric plasticity of weighted Fermat-Torricelli trees that have got a subconscious node)}

If we select a point \( P_{0k} \) with a non-negative weight \( B_{0k} \) on the ray that is defined by the line segment \( A_{0k}A_{4k} \), such that:

\[ \| \sum_{i=1}^{4} B_{jk} \bar{u}(P_{0k}, P_{jk}) \| > B_{jk}. \]

Then the corresponding weighted Fermat-Torricelli node \( P_{0k} \) that has got a subconscious of \( \{P_{0k}P_{1k}, P_{0k}P_{2k}, P_{0k}P_{3k}, P_{0k}P_{4k}\} \) remains the same with \( A_{0k} \), for \( k = 1, 2, 3, ..., n \).

The modified weighted Fermat-Torricelli problem for tetrahedra states that:

\textbf{Problem 2 (Modified weighted Fermat-Torricelli problem)}

Let \( A_{1k}A_{2k}A_{3k}A_{4k} \) be a tetrahedron in \( \mathbb{R}^3 \), \( B_{jk} \) be a non-negative number (weight) which corresponds to each line segment \( A_{0k}A_{4k} \), respectively. Find a point \( A_{0k} \) which minimizes the sum of the lengths of the line segments \( a_{0k} \) that connect every vertex \( A_{jk} \) with \( A_{0k} \) multiplied by the positive weight \( B_{jk} \):

\[ \sum_{i=1}^{4} B_{jk} a_{0k} = \text{minimum}. \]
We denote by \( \alpha \) the dihedral angle which is formed by the planes defined by \( \Delta A_1 A_0 A_2 \) and \( \Delta A_1 A_2 A_3 \), and by \( \alpha_{kl} \), the dihedral angle formed by the planes defined by \( \Delta A_1 A_4 A_2 \) and \( \Delta A_1 A_2 A_3 \), for \( l = 1, 2, \ldots, n \).

**Lemma 4** [\( \{10, \text{Formula (27), p. 997}\} \)]

The variable length \( d_{ij} \) is given by

\[
\begin{align*}
d_{ij} &= a_{ij}^2 + a_{2ij}^2 - 2a_{ij}a_{2ij} \left[ \sqrt{a_{ij}^2 - h_{012}^2} \cos \alpha_{124i} + 
\right. \\
&
\left. + h_{012} \sin \alpha_{124i} \left( \cos \gamma_{ij} \left( \frac{a_{ij}^2 - a_{2ij}^2}{a_{ij}} \right) - \sqrt{a_{ij}^2 - h_{012}^2} \cos \alpha_{123i} \right) \right] + \\
&
\left. + \sin \alpha_{ij} \sin \arccos \left( \frac{a_{ij}^2 - a_{2ij}^2}{a_{ij}} \right) \right) \\
&\quad \left( 2a_{01402i} \right)^2, \\
\end{align*}
\]

and

\[
\begin{align*}
h_{012} &= \frac{a_{01}a_{02}}{a_{12}} \sqrt{1 - \left( \frac{a_{01}^2 + a_{02}^2 - a_{12}^2}{2a_{01402i}} \right)^2}. \\
\end{align*}
\]

**Theorem 3** [Dynamic Plasticity of weighted network with two roots and two growing branches]

Given the weighted Fermat-Torricelli point \( A_0 \) that has got a subconsecutive \( B_{ij} \) to be an interior point of the tetrahedron \( A_1 A_2 A_3 A_4 \) with the vertices lie on four prescribed rays that meet at \( A_0 \) and from the five given values of \( \alpha_{102}, \alpha_{103}, \alpha_{104}, \alpha_{203}, \alpha_{204}, \alpha_{204} \), the positive real weights \( B_{ji} \) are given by:

\[
\begin{align*}
\hat{B}_{11} &= \left( \frac{\sin \alpha_{4,103i}}{\sin \alpha_{3,203i}} \right) \frac{c - B_{0i}}{2}, \\
\hat{B}_{21} &= \left( \frac{\sin \alpha_{4,103i}}{\sin \alpha_{3,203i}} \right) \frac{c - B_{0i}}{2}, \\
\hat{B}_{31} &= \left( \frac{\sin \alpha_{4,102i}}{\sin \alpha_{3,202i}} \right) \frac{c - B_{0i}}{2}, \\
\hat{B}_{41} &= \frac{c - B_{0i}}{2}, \\
\end{align*}
\]

under the weighted conditions

\[
\begin{align*}
\hat{B}_{11} + \hat{B}_{21} + \hat{B}_{31} + \hat{B}_{41} &= c, \\
\end{align*}
\]

and

\[
\begin{align*}
\hat{B}_{11} + \hat{B}_{21} &= \hat{B}_{31} + \hat{B}_{41} + \hat{B}_{0i}, \\
\end{align*}
\]
Proof. By considering a two-way communication network and by assuming mass flow continuity the weights $B_{ij}$, for $i = 1, 2, 3, 4$, are determined by the weighted outward and inward flow conditions (21), (22), which yield the weighted conditions (34) and (35).

Thus, we obtain that:

$$
\sum_{k=1}^{4} B_{ik} a_{0kj} + \sum_{k=1}^{4} B_{kj} a_{0ik} \rightarrow \min,
$$

which gives

$$
\sum_{k=1}^{4} B_{ik} a_{0kj} \rightarrow \min.
$$

By differentiating (37) w.r. to $a_{01i}, a_{02i}, a_{03i}$, respectively, taking into account the derivative of $a_{02i}$ w.r. to $a_{01i}, a_{02i}, a_{03i}$, by lemma [4] we obtain (30), (31), (32) and (33).

Remark 2 We note that the dynamic plasticity equations of Theorem 3 have been derived in [10] for weighted Fermat-Torricelli trees, which consist of two roots one branch and one growing branch that have inherited a subconic (weighted Fermat-Torricelli node) under different weighted (inflow - outflow conditions):

$$
B_{i1} + B_{i2} + B_{i3} = B_{0i} + B_{4i}, \text{ for } i = 1, 2, ..., n.
$$

We assume that the common perpendicular line of $A_1A_2A_3A_4$ passes through the common midpoint $m_{12}$ and $m_{34}$ of $A_1A_2$ and $A_3A_4$, respectively and $m_{12}m_{34} > > A_1A_2$. We denote by $\varphi_i$ the angle formed by $A_1A_2$ and $A_3A_4$ and by $B_{ij}$ the weight (positive real number) which corresponds to the vertex $A_{ji}$, for $j = 1, 2, 3, 4$, $i = 1, 2, ..., n$. Hence, by rotating $A_1A_2A_3A_4$ by $\varphi_i$ with respect to $m_{12}m_{34}$, we obtain $n$ weighted isosceles trapezoids $A_{1i}A_{2i}A_{3i}A_{4i}$ and $B_{j} \rightarrow B_{ji}$. We denote by $O_{i}$ the intersection point of the equal diagonals $A_{1i}A_{3i}$ and $A_{2i}A_{4i}$, by $A_{0i}$ the corresponding weighted Fermat-Torricelli node with remaining weight $B_{0i}$ (one node that has got a subconic) and by $O_{12}$ and $O_{34}$ the two corresponding weighted Steiner nodes with remaining weights $B_{12}$ and $B_{34}$ (two nodes that got a subconic).

Theorem 4 If $A_{0i}$ lies on the common perpendicular segment $m_{12}m_{34}$, then

$$
B_{i1} = B_{i2},
$$

and

$$
B_{i3} = B_{i4}.
$$

Proof. By substituting $\alpha_{i,102} = \alpha_{3,102}$ in (32) and (33), we obtain (39). By working cyclically with the indices and by exchanging the indices $3 \rightarrow 2, 4 \rightarrow 1$ and $1 \rightarrow 4, 2 \rightarrow 3$, we derive (38).

We may consider that $\{A_{1i},A_{2i}\}$ lie on a circular cone $C_{012}$, having $m_{12}m_{34}$ as axis of rotation with vertex the weighted Fermat-Torricelli point $A_{0i}$ and $\{A_{3i},A_{4i}\}$ lie on a circular cone $C_{034}$, having $m_{12}m_{34}$ as axis of rotation with vertex the weighted Fermat-Torricelli point $A_{0i}$. We note that $C_{012}$ and $C_{034}$ intersect only at $A_{0i}$.

Proposition 4 (Rotational plasticity of tetrahedra) If we select $\{R_{1i},R_{2i}\}$ two points with weights $B_{1i}, B_{2i}$, respectively, on $C_{012}$, such that their midpoint $m_{12i}$ lies on the line defined by $m_{12}m_{34}$ and $\{R_{3i},R_{4i}\}$ two points with weights $B_{3i}$ and $B_{4i}$, respectively, on $C_{034}$, such that their midpoint $m_{34i}$ lies on the line defined by $m_{12}m_{34}$, then the corresponding weighted Fermat-Torricelli point $R_{0i}$ of $R_{1i}R_{2i}R_{3i}R_{4i}$ remains the same with $A_{0i}$ for $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$, for $i = 1, 2, ..., n$.

Proof. It is a direct consequence of Theorem 4 and taking into account that $R_{1i}R_{2i}R_{3i}R_{4i}$ are derived by rotating the two isosceles triangles $\triangle R_{1i}A_{0i}R_{2i}$ and $\triangle R_{3i}A_{0i}R_{4i}$ along $m_{12}m_{34}$. By rotating properly $R_{1i}R_{2i}R_{3i}R_{4i}$, we may derive a weighted isosceles trapezoid or a weighted rectangle ($R_{1i}R_{2i} \rightarrow R_{3i}R_{4i}$) for $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$. Thus, the weighted balancing condition $\sum_{i=1}^{n} B_{ij}(A_{0i}, A_{ji}) = 0$, yields $R_{0i} = A_{0i}$.

Definition 5 A “botanological” thumb for a boundary symmetric tetrahedron $A_1A_2A_3A_4$ whose common perpendicular passes through the common midpoints $m_{12}$ and $m_{34}$ of $A_1A_2$ and $A_3A_4$, respectively and $m_{12}m_{34} > > A_1A_2$ is a “botanological” network, which is transformed to a botanological “thumb” for a boundary rectangle or a boundary isosceles trapezoid, by rotating properly $A_1A_2$ w.r. $m_{12}m_{34}$.

Definition 6 A “botanological” thumb is a collection of “botanological” thumbs for a finite number of boundary symmetric tetrahedra in $\mathbb{R}^3$.

We will describe an evolutionary scheme for the creation of a “botanological” thumb in $\mathbb{R}^3$.

1. Evolutionary Phase 1 At time $t = 0$, we consider a point “seed” $A_{0i}$ on the ground.

2. Evolutionary Phase 2 After time $t$, by assuming mass flow continuity two equal roots start to grow underground and two equal branches start to grow overground, such that their endpoints form a boundary rectangle $A_{1i}A_{2i}A_{3i}A_{4i}$. Taking into account Proposition 3 we derive that $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$.

3. Evolutionary Phase 3 We consider two cases: (i) If $A_{0i}$ is the intersection of the diagonals $A_{1i}A_{3i}$ and $A_{2i}A_{4i}$ the weighted Fermat-Torricelli node $A_{0i}$ has acquired a subconic $B_{0i}$. (ii) If $A_{0i}$ lies
on the midperpendicular line segment $m_1m_{34}$ the weighted Fermat-Torricelli node $A_0$ has acquired a subconscious $B_0$.

4. Evolutionary Phase 4

The subconscious $B_0$ may cause a geometric plasticity and/or a rotational plasticity of the weighted Fermat-Toricelli tree $\{A_1A_0, A_2A_0, A_3A_0, A_4A_0\}$.

(i) The geometric plasticity (Theorem 2) yields a weighted Fermat-Toricelli tree $\{R_{1i}A_0, R_{2i}A_0, R_{3i}A_0, R_{4i}A_0\}$, such that their endpoints form an isosceles trapezoid $R_{1i}R_{2i}R_{3i}R_{4i}$, $A_{0i}'' \equiv A_0$ and $B_{ji}$ corresponds to $R_{ji}$, for $j = 1, 2, 3, 4$ and $i = 1, 2, ..., n$.

(ii) The rotational plasticity (Proposition 3), the dynamic plasticity (Theorem 4) and the symmetry of boundary tetrahedra taken from Theorem 2 creates a “botanical” thumb for $i = 1, 2, ..., n$, having the corresponding weighted Fermat-Torricelli node $A_0$ constant on the ground (point “seed”), but with different subconscious quantities $B_0$, for $i = 1, 2, ..., n$.

5 Generalized regularity for tetrahedra in the three dimensional Euclidean Space

The weighted Steiner problem for a boundary weighted tetrahedron $A_1A_2A_3A_4$ in $\mathbb{R}^3$ having two subconscious nodes (weighted Fermat-Torricelli or weighted Steiner points) has been studied recently in [11].

We denote by $A_1A_2A_3A_4$ a tetrahedron in $\mathbb{R}^3$, with $A_i(x_i, y_i, z_i)$ $(i = 1, 2, 3, 4)$, by $b_i$ a positive real number (weight) which corresponds to the vertex $A_i$, $O_{12}$, $O_{34}$, two interior points (nodes) of $A_1A_2A_3A_4$ in $\mathbb{R}^3$, by $b_{12}$ the weight which corresponds to $O_{12}$, $b_{34}$ the weight which corresponds to $O_{34}$, by $H$ the length of the common perpendicular (Euclidean distance) between the two lines defined by $A_1A_2$, $A_3A_4$, by $A_1O_{12}$ the Euclidean distance from $A_1$ to $O_{12}$ and by $A_3O_{34}$ the Euclidean distance from $A_3$ to $O_{34}$, by $T_{12}$ the intersection point of the line defined by $O_{12}O_{34}$ and the line defined by $A_1A_2$, $M_{12}$ the midpoint of $A_1A_2$ and $M_{34}$ the midpoint of $A_3A_4$, for $i = 1, 2, 3, 4$.

We denote by $\beta_i'$ the intersection point of the line defined by $A_1A_2$ and $A_3A_4$ by $\beta_i''$ the intersection point of the line defined by $A_1A_2$ and the line defined by the common perpendicular of $A_3A_4$.

We set $\delta_{ij} \equiv A_iA_j$, for $i, j = 1, 2, 3, 4$, $i \neq j \neq k$, $\alpha_{12} \equiv \angle A_1O_{12}A_2$, $\alpha_{34} \equiv \angle A_3O_{34}A_4$, $\alpha_1 \equiv \angle A_2O_{12}A_3$, $\alpha_2 \equiv \angle A_1O_{12}A_3$, $\alpha_3 \equiv \angle A_3O_{34}A_1$, $\alpha_4 \equiv \angle A_4O_{34}A_2$, $\varphi \equiv \arccos(\frac{b_{12}b_{34}}{b_{12}b_{34}})$ and $b_{ST} = \frac{b_{12}b_{34}}{2}$.

Furthermore, we denote by $A_{12}$ the vertex of $\triangle A_1A_2A_3$, such that: $\angle A_1A_2A_3 = \pi - \alpha_{12}$, $\angle A_1A_2A_3 = \pi - \alpha_1$ and $\angle A_1A_2A_3 = \pi - \alpha_2$, by $A_{34}$ the vertex of $\triangle A_4A_{34}A_3$, such that: $\angle A_4A_{34}A_3 = \pi - \alpha_{34}$, $\angle A_4A_{34}A_3 = \pi - \alpha_3$ and $\angle A_4A_{34}A_3 = \pi - \alpha_4$, by $H_{12}$ the trace of the height of $\triangle A_1A_2A_3$ w.r.t to the base $A_1A_2$ and by $H_{34}$ the vertex of $\triangle A_4A_{34}A_3$ such that: $\angle A_4A_{34}A_3 = \pi - \alpha_{34}$, $\angle A_4A_{34}A_3 = \pi - \alpha_3$ and $H_{34}$ the trace of the height of $\triangle A_4A_{34}A_3$ w.r.t to the base $A_4A_3$.

We set $H \equiv A_4''T_34 \equiv A_4''T_34 \equiv A_4''T_{12} k_1 \equiv A_4''A_1$ and $k_2 \equiv A_4''A_4, m_{12} \equiv A_4''M_{12}$ and $m_{34} \equiv A_4''M_{34}, h_{12} \equiv A_4''H_{12}$ and $h_{34} \equiv A_4''H_{34}$.

We assume that: $A_1A_4 + A_2A_3 > A_1A_2 + A_3A_4$.

The weighted Steiner problem for $A_1A_2A_3A_4$ in $\mathbb{R}^3$ states that:

**Problem 3** (Problem 5) Find $O_{12}(x_0, y_0, z_0)$ and $O_{34}(x_0'\gamma_0, y_0', z_0')$ with given weights $b_{12}$ in $O_{12}$ and $b_{34}$ in $O_{34}$, such that:

$$f(O_{12}, O_{34}) = b_1A_1O_{12} + b_2A_2O_{34} + b_3A_3O_{34} + b_4A_4O_{34} + \frac{b_{12} + b_{34}}{2}O_{12}O_{34} \rightarrow \min.$$ (40)

**Theorem 5** (Theorem 3) The solution of the weighted Steiner problem is a weighted Steiner tree in $\mathbb{R}^3$ whose nodes $O_{12}$ and $O_{34}$ (weighted Fermat-Torricelli points) are seen by the angles:

$$\cos \alpha_{12} = \frac{b_{12}^2 - b_1^2 - b_2^2}{2b_1b_2},$$

$$\cos \alpha_{1} = \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_1b_{ST}},$$

$$\cos \alpha_{34} = \frac{b_{34}^2 - b_3^2 - b_4^2}{2b_3b_{4}},$$

$$\cos \alpha_{4} = \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_3b_{ST}}.$$ (41)

The inradius $r_{12}$ is the radius of the inscribed circle of triangle $\triangle A_1A_2A_3$ with sides $A_1A_2 = \lambda b_{12} + b_{34}, A_1A_2 = \lambda b_1, A_2A_3 = \lambda b_2 + b_{34}$ and $A_1A_2 = \lambda b_1$. The inradius $r_{34}$ is the radius of the inscribed circle of triangle $\triangle A_3A_4A_3$ with sides $A_3A_4 = \lambda b_{12} + b_{34}, A_3A_4 = \lambda b_1, A_3A_4 = \lambda b_4$ and $A_3A_4 = \lambda b_4$.

We use the substitutions for $r_{12}$ and $r_{34}$, ([11] Section 2, p. 6):

$$r_{12} = \frac{A_1A_2}{(b_1 + b_2 + b_{12} + b_{34})} \left((b_1 + b_2 - b_{12}) - (b_1 + b_{12} + b_{34} - b_2)\right),$$

$$r_{34} = \frac{A_3A_4}{(b_3 + b_4 + b_{34})} \left((b_3 + b_4 - b_{34}) - (b_3 + b_{34} - b_4)\right).$$

$$\beta_{12} = \arccos(\frac{A_1A_2}{2r_{12}}),$$

$$\beta_{34} = \arccos(\frac{A_3A_4}{2r_{34}}).$$
Theorem 6 (Theorem 4) The following system of equations w.r. to \( t_{34} \) and \( t_{12} \) allows the computation of the position of the weighted Simpson line \( O_{12}O_{34} \) of the weighted full Steiner tree for \( A_1A_2A_3A_4 \):

\[
\frac{t_{34} - t_{12} \cos \phi}{\sqrt{H^2 + t_{12}^2 \sin^2 \phi}} = \frac{h'_{34} - t_{34}}{t_{34}}
\]

and

\[
\frac{t_{12} - t_{34} \cos \phi}{\sqrt{H^2 + t_{34}^2 \sin^2 \phi}} = \frac{h'_{12} - t_{12}}{t_{12}}
\]

Proposition 5 (Proposition 1) \( t_{34} - t_{12} \cos \phi = \frac{m_{34} - t_{34}}{a_{34} \frac{\sqrt{3}}{2}} \)

\[
\frac{t_{12} - t_{34} \cos \phi}{\sqrt{H^2 + t_{34}^2 \sin^2 \phi}} = \frac{m_{12} - t_{12}}{a_{12} \frac{\sqrt{3}}{2}}
\]

Theorem 7 (Theorem 5) The following system of equations w.r. to \( t_{34}, t_{12} \) and \( \angle A_3F_1A_4 \) allows the computation of the position of the line defined by \( T_{12}T_{34} \) of the (unweighted) Fermat-Torricelli tree of \( A_1A_2A_3A_4 \):

\[
\cos \angle A_3F_1A_4 = \frac{\frac{m_{34} - t_{34}}{\sqrt{2} \sin \angle A_3F_1A_4}}{2}
\]

\[
\cos \angle A_1T_1O_1 = \frac{\frac{m_{12} - t_{12}}{\sqrt{2} \sin \angle A_1T_1O_1}}{2}
\]

Remark 3 We correct two typographical errors that occur in [11] by replacing \( \sqrt{H^2 + t_{34}^2 \sin^2 \phi} \) by \( \sqrt{H^2 + t_{34}^2 \sin^2 \phi} \) and the angle \( \phi_{34} \) by \( \sin \phi_{34} \) in [11] Formula (3.1).

Definition 7 A generalized regular tetrahedron is a tetrahedron, which determines a generalized (weighted) regular quadrilateral, formed by rotating \( A_1A_2 \) or \( A_3A_4 \) by the twist angle \( \omega \), w.r. to the (weighted) Simpson line \( A_1A_2A_3A_4 \).

We denote by \( \omega_F \) the twist angle formed by the planes defined by \( \triangle A_1F_2A_4 \) and \( \triangle A_3F_4A_4 \) and by \( \omega_S \) the twist angle formed by the planes \( \triangle A_1O_{12}A_2 \) and \( \triangle A_3O_{34}A_4 \).

Theorem 9 (Generalized regularity of tetrahedra) If \( A_1A_2A_3A_4 \) is a generalized regular quadrilateral, then generalized regular tetrahedra are derived by:

(i) rotating the twist angle \( \omega_F \) w.r. to the line defined by \( M_{12}M_{34} \)

\[
\cos \omega_F = \frac{\cos \phi - \cos^2 \angle A_1M_1F_2}{\sin^2 \angle A_1M_1F_2}
\]

or (ii) rotating the twist angle \( \omega_S \) w.r. to the Simpson line defined by \( T_{12}T_{34} \)

\[
\cos \omega_S = \frac{\cos \phi - \cos^2 \angle A_1T_1O_1}{\sin^2 \angle A_1T_1O_1}
\]

Proof. A generalized regular convex quadrilateral is a trapezoid having the property: \( A_1A_2 \parallel A_3A_4 \). Thus, the Fermat-Torricelli point \( F \) is the intersection of diagonals \( A_1A_3 \) and \( A_2A_4 \) and lies on the line defined by \( M_{12}M_{34} \), which yields \( \angle A_1M_1F_2 = \angle A_3M_3F_2 \). By substituting \( \angle A_1M_1F_2 = \angle A_3M_3F_2 \) in (49), we obtain (50). We recall that \( A_1A_2A_1 \) and \( A_3A_4A_3 \) are equilateral triangles outside of \( A_1A_2A_3A_4 \) and the Simpson line intersects \( A_1T_2 \) and \( A_3T_4 \) at \( T_{12} \) and \( T_{34} \), respectively. By substituting \( \angle A_1T_1O_{12} = \angle A_3T_3O_{34} \) in (49), we obtain (51). \( \square \)

Remark 4 The position of \( A_1^0 \) and \( A_4^0 \) may be calculated by Theorem 2.

Definition 8 A weighted regular tetrahedron is a tetrahedron in \( \mathbb{R}^3 \), such that the weighted Simpson line \( L \) passes through the weighted Fermat-Torricelli point \( F \).

We assume that \( A_1A_2A_3A_4 \) is a weighted regular tetrahedron \( A_1A_2A_3A_3 \), such that: \( M_{12}M_{34} > \max A_1A_2, A_3A_4 \).

Theorem 10 (Weighted regularity of tetrahedra) The common perpendicular line of \( A_1A_2 \) and \( A_3A_4 \) passes through the common midpoints \( M_{12} \) and \( M_{34} \), respectively, if and only if \( b_1 = b_2 \) and \( b_3 = b_4 \).
Proof. The weighted Simpson line passes through $A_{12}, A_{34}$, the weighted Steiner nodes $O_{12}, O_{34}$, the weighted Fermat-Torricelli point $F$ and $M_{12}, M_{34}$. Therefore, $\triangle A_{12}A_{12}$ and $A_{1}A_{2}A_{34}$ are isosceles triangles $A_{1}A_{12} = A_{2}A_{12}$ and $A_{3}A_{34} = A_{4}A_{34}$, which yield $b_1 = b_2$ and $b_3 = b_4$. Hence, it is shown one direction.

We assume that the common perpendicular line of $A_{1}A_{2}$ and $A_{3}A_{4}$ does not pass through the common midpoints $M_{12}$ and $M_{34}$, $b_1 = b_2$ and $b_3 = b_4$. By substituting $b_1 = b_2$ and $b_3 = b_4$ and given a subconscious weight $B_3$ in (41), we derive that $\angle A_{1}O_{12}O_{34} = \angle A_{2}O_{12}O_{34}$ and $\angle A_{3}O_{34}O_{12} = \angle A_{4}O_{34}O_{12}$. By substituting $b_1 = b_2$, $b_3 = b_4$ in (42) and (43) we obtain the values of $t_{12}$ and $t_{34}$, in order to calculate the twist angle $\omega_5$. By rotating $A_{1}A_{2}$ w.r. to $A_{12}A_{34}$ by $\omega_5$, $A_{1}A_{2} \parallel A_{3}A_{4}$, and $A_{12}A_{34}$ passes through $M_{12}, M_{34}$, otherwise $O_{12}, O_{34}$ and $F$ are not collinear. It proves another direction and the theorem as well. □

We may follow the same evolutionary scheme for a “botanological” thumb in $\mathbb{R}^3$. Taking into account that the point seed which has got a subconscious $B_{ST}$ is located underground, an evolutionary two way communication network will start to grow having two roots one main branch and two branches. By assuming a constant mass flow continuity that corresponds to the two roots $b_1 = b_2$ ($O_{12}$ is located underground) one main branch with remaining weight $B_{ST}$ and two branches with weights $b_3 = b_4$ ($O_{34}$ is located overground). Therefore, by applying Theorem 10 we obtain a boundary weighted regular tetrahedron formed by the endpoints of two symmetrical roots and two symmetrical branches, such that the main branch is perpendicular to the ground.

References


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