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Generalized Regularity and the Symmetry of Branches of "Botanological" Networks

To the weighted regularity of Euclid.

Generalized Regularity and the Symmetry of Branches of "Botanological" Networks

ABSTRACT

We derive the generalized regularity of convex quadrilaterals in $\mathbb{R}^2$, which gives a new evolutionary class of convex quadrilaterals that we call generalized regular quadrilaterals in $\mathbb{R}^2$. The property of generalized regularity states that the Simpson line defined by the two Steiner points passes through the corresponding Fermat-Torricelli point of the same convex quadrilateral. We prove that a class of generalized regular convex quadrilaterals consists of convex quadrilaterals, such that their two opposite sides are parallel. We solve the problem of vertical evolution of a "botanological" thumb (a two way communication weighted network) w.r to a boundary rectangle in $\mathbb{R}^2$ having two roots,two branches and without having a main branch, by applying the property of generalized regularity of weighted rectangles. We show that the two branches have equal weights and the two roots have equal weights, if the thumb inherits a symmetry w.r to the midperpendicular line of the two opposite sides of the rectangle, which is perpendicular to the ground (equal branches and equal roots). The geometric, rotational and dynamic plasticity of weighted networks for boundary generalized regular tetrahedra and weighted regular tetrahedra lead to the creation of "botanological" thumbs and "botanological" networks (with a main branch) having symmetrical branches.

Key words: Fermat-Torricelli problem, Fermat-Torricelli point, Steiner tree, Steiner points, generalized regular quadrilaterals, generalized regularity, "thumb"

MSC2010: 51N20, 51M20, 51E10, 52A15

Generalizirana regularnost i simetrija "botanologičnih" mreža

SAŽETAK

Izvodimo generaliziranu regularnost konveksnih četverokuta u $\mathbb{R}^2$ koja daje novu evolucijsku klasu konveksnih četverokuta koju mi nazivamo generalizirani regularni četverokuti u $\mathbb{R}^2$. Svojstvo generalizirane regularnosti kaže da Simpsonov pravac definiran s dvije Steinerove točke prolazi odgovarajućom Fermat-Torricellijevom točkom tog istog četverokuta. Dokazujemo da se klasa generaliziranih regularnih konveksnih četverokuta sastoji od konveksnih četverokuta takvih da su njihove dvije nasuprotne stranice paralelne. Rješavamo problem vertikalne evolucije "botanologičnog palca" (težinska mreža, u oba smjera) s obzirom na granični pravokutnik u $\mathbb{R}^2$ koji ima dva korijena, dvije grane, bez da ima glavnu granu, primjenjujući svojstvo generalizirane regularnosti težinskih pravokutnika. Pokazujemo da dvije grane imaju jednake težine kao i dva korijena ako "palac" nasljeđuje simetriju s obzirom na poluokomit pravac dvaju nasuprotnih stranica pravokutnika koji je okomit na tlo (jednake grane i jednaki korijeni). Geometrijski, rotacijski i dinamični plasticitet težinskih mreža za granični generalizirani regularni tetraedar i težinski regularni tetraedar vodi ka stvaranju "botanologičnih palčeva" i "botanologičnih" mreža (s glavnom granom) koja ima simetrične grane.

Ključne riječi: Fermat-Torricellijev problem, Fermat-Torricellijeva točka, Steinerovo stablo, Steinerove točke, generalizirani regularni četverokuti, generalizirana regularnost, "palac"

1 Introduction

Let $A_1, A_2, \ldots, A_n$ be the vertices of a polygon $A_1A_2A_3\ldots A_n$ in a cyclic order.

An affinely regular polygon in $\mathbb{R}^2$ is derived by applying an affine transformation to a regular polygon ([1]). Coxeter introduced the affine regularity of polygons and proved the following result ([2], [3]):

$A_1A_2A_3...A_n$ is affinely regular if and only if there is $m \geq 0$, such that
$\overrightarrow{A_{i-1}A_{i+2}} = m\overrightarrow{A_iA_{i+1}}$, for $i = 1,2,...,n$.
Triangles are affine regular and parallelograms are affine regular quadrilaterals in $\mathbb{R}^2$.
Gerber connected the affine regularity with the Euclidean regularity of $n-gons$ in [4], (see also [2] and [3])) and proved the result: If you construct regular $n-gons$ outwardly (or inwardly) on the sides of any affine regular $n-gon$, then their centers form the vertices of a regular $n-gon$. The case $n=4$ was proved by Thebault, who gave the first generalization of Napoleon's regularity for the case $n=3$ (Napoleon's theorem) ([2, p. 185]).
We start by giving the definitions of a weighted Fermat-Torricelli tree and weighted Steiner tree for a boundary quadrilateral, in order to derive a new regularity of quadrilaterals which is different from Coxeter's, Gerber's and Thebault's approach. The new regularity of quadrilaterals is achieved by the construction of isosceles triangles outwardly on the parallel sides of a rectangle or a trapezoid.
Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$. We denote by $A_i(x_i,y_i)$ the vertices of $A_1A_2A_3A_4$, by B_i a positive real number (weight) which corresponds to A_i, by $O_{12}(x_{012},y_{012})$, by $O_{34}(x_{034},y_{034})$ two points in $\mathbb{R}^2$ with given weights B_{12} in O_{12} and B_{34} in O_{34}, by $d(X,Y)$ the Euclidean distance $\|XY\|$, for $X,Y \in \mathbb{R}^2$.
The weighted Steiner problem for $A_1A_2A_3A_4$ in $\mathbb{R}^2$ states that:

Problem 1 *Find $O_i(x_{0i},y_{0i})$, for $i = \{12,34\}$, such that*

$$\begin{aligned} f(O_{12},O_{34}) &= B_1d(O_{12},A_1)+B_2d(O_{12},A_2)+ \\ &+B_3d(O_{34},A_3)+B_4d(O_{34},A_4)+ \\ &+\frac{B_{12}+B_{34}}{2}d(O_{12},O_{34}) \to min. \quad (1) \end{aligned}$$

For $B_1 = B_2 = B_3 = B_4$, the solution of the (unweighted) Steiner problem is called a Steiner tree. Gilbert and Pollack introduce the Steiner tree topologies for $A_1A_2A_3A_4$, in their classical study ([5]). They mention three topologies of solutions w.r to the boundary $A_1A_2A_3A_4$:
1. If we set one point (node) F (Fermat-Torricelli point) different from A_i, the solution is called a Fermat-Torricelli tree. The Fermat-Torricelli point F has four connections $\{FA_1,FA_2,FA_3,FA_4\}$. This is a special case of the unweighted Steiner problem, by setting $B_{12} = 0$ or $B_{34} = 0$.
2. If we set two points (nodes) O_{12} and O_{34} (Steiner points) and $B_{12}+B_{34} = 2$, such that the objective function (40) is minimized, then we derive a solution which is called a full Steiner tree. The Steiner points O_{12} and O_{34} have three connections $\{A_1O_{12},A_2O_{12},O_{12}O_{34}\}$ and $\{A_3O_{34},A_4O_{34},O_{12}O_{34}\}$, respectively.
3. If we set one point (node) Steiner point O_{12} and $O_{34} \equiv A_3 or A_4$, such that the objective function (40) is minimized, then we derive a degenerate Steiner tree.

It is well known that the Steiner point with three connections possesses the equiangular property $\frac{360^o}{3}$. The angle formed by the Steiner point as a vertex and two connections is 120^o, for the unweighted case and by assuming that $B_{12}+B_{34} = 2$ ([5]). The same property holds for the Fermat-Torricelli point for a boundary triangle, which coincides with the Steiner point. The Fermat-Torricelli tree of a convex quadrilateral consists of the two diagonals A_1A_3 and A_2A_4, which meet at the intersection point F (Fermat-Torricelli point) for the unweighted case.
Rubinstein, Thomas and Weng studied in [8] the unweighted Steiner problem for tetrahedra in $\mathbb{R}^3$. They succeeded in locating the Simpson line, which passes through the two Steiner points O_{12} and O_{34} in $\mathbb{R}^3$. The vertex A_{12} of the equilateral $\triangle A_{12}A_1A_2$, which lies on the opposite side of A_1A_2 to O_{12} is referred to as the e-point of A_1A_2. The vertex A_{34} of the equilateral $\triangle A_{34}A_4A_3$, which lies on the opposite side of A_3A_4 to O_{34} is referred to as the e-point of A_3A_4. The Simpson line passes through the e-points of A_1A_2 and A_3A_4, respectively, and

$$\begin{aligned} &d(A_{12},A_{34}) = \\ &d(O_{12},A_1)+d(O_{12},A_2)+d(O_{34},A_3)+d(O_{34},A_4) = L. \end{aligned}$$

The Melzak Circle is a circle $C(O_1,r_{12})$, which passes through A_1, A_2, A_{12} and intersects the Simpson line at O_{12}. Similarly, the Melzak Circle $C(O_2,r_{34})$ passes through A_3, A_4, A_{34} and intersects the Simpson line at O_{34}. The Melzak construction via the method of e-points is established in [7]. Furthermore, Rubinstein, Thomas and Weng gave explicit formulas for computing Steiner trees for four points in $\mathbb{R}^2$, for all possible cases, in which the lines defined by A_1A_2 and A_3A_4 either intersect or are parallel ([8, Chapter 3, Cases (1), (2)]). We set $\varphi \equiv \angle(\vec{A_1A_2},\vec{A_3A_4})$. For $\varphi = 0$, (A_1A_2 and A_3A_4 are parallel), we refer to this solution as the Steiner zero solution. The Steiner zero solution depends on the distance h between the two parallel lines, the midpoints of A_1A_2 and A_3A_4, respectively and the radius of Melzak circles r_{12} and r_{34} ([8, Chapter 3, Expicit formulas Case (2), page 65]).
Ivanov and Tuzhilin introduced the concept of the weighted Simpson line and they found the relation of the length of the weighted network with the length of a Simpson line ([6, Theorem 1]) which gives

$$\begin{aligned} &\frac{B_{12}+B_{34}}{2}L = \\ &B_1d(O_{12},A_1)+B_2d(O_{12},A_2)+B_3d(O_{34},A_3)+B_4d(O_{34},A_4). \end{aligned}$$

We note that A_{12} and A_{34} are not the e-points for the weighted case.
In this paper, we introduce the generalized (weighted) regularity of convex quadrilaterals and tetrahedra, which gives a new evolutionary class of convex quadrilaterals and tetrahedra in $\mathbb{R}^3$.

The property of generalized regularity states that the Simpson line defined by the two Steiner points O_{12} and O_{34} passes through the corresponding Fermat-Torricelli point of the same convex quadrilateral. The property of weighted regularity for weighted rectangles states that the weighted Simpson line defined by the two weighted Steiner points passes through the corresponding weighted Fermat-Torricelli point of the same rectangle.
The main results are:
1. The property of generalized regularity possess a class of convex quadrilaterals (generalized regular quadrilaterals), which corresponds to the Steiner zero solution and it consists of quadrilaterals having two of their opposite sides parallel (Theorem 1).
2. Let $A_1A_2A_3A_4$ be a rectangle in $\mathbb{R}^2$ and A_1F, A_2F be the two roots of the corresponding weighted Fermat-Torricelli tree (thumb), the weighted Fermat-Torricelli point F is located on the ground and A_3F, A_4F are two branches of the weighted Fermat-Torricelli tree (thumb).
If the weighted Simpson line $A_{12}A_{34}$ is perpendicular to the ground and $A_1A_2A_3A_4$ is a generalized regular quadrilateral, we prove that $B_1^2+B_3^2=B_2^2+B_4^2$ (Theorem 2).
3. Two branches have equal weights and the two roots have equal weights, if the thumb inherits a symmetry w.r to the midperpendicular line of the two opposite sides of the rectangle, which is perpendicular to the ground (equal branches and equal roots, Proposition 3).
4. The dynamic Plasticity of weighted network with two roots and two growing branches states that:
Given the weighted Fermat-Torricelli point A_{0i} that has got a subconscious $\bar{B}_{0i}$ to be an interior point of the tetrahedron $A_{1i}A_{2i}A_{3i}A_{4i}$ with the vertices lie on four prescribed rays that meet at A_{0i} the positive real weights $\bar{B}_{ji}$ depends on the five given values of α_{102i}, α_{103i}, α_{104i}, α_{203i}, α_{204i} and $\bar{B}_{0i}$ (Theorem 3).
5. We assume that the common perpendicular line of each tetrahedron $A_{1i}A_{2i}A_{3i}A_{4i}$ passes through the common midpoints m_{12} and m_{34} of $A_{1i}A_{2i}$ and $A_{4i}A_{3i}$, respectively and $m_{12}m_{34} >> A_{1i}A_{2i}$. We prove the following theorem for a botanological thumb (without a main branch) (Theorem 4):
If A_{0i} lies on the common perpendicular segment $m_{12}m_{34}$, then $\bar{B_1}_i=\bar{B_2}_i$ and $\bar{B_3}_i=\bar{B_4}_i$.
6. We prove the following theorem for a "botanological" network (with a main branch) (Theorem 4):
If A_{0i} lies on the common perpendicular segment $m_{12}m_{34}$, then $\bar{B_1}_i=\bar{B_2}_i$ and $\bar{B_3}_i=\bar{B_4}_i$.
The dynamic plasticity (Theorem 3), geometric plasticity (Lemma 2) and rotational plasticity (Proposition 4) of generalized regular tetrahedra (Definition 7) and generalized weighted regular tetrahedra (Definition 8) develops a symmetry for the weights for a "botanological" thumb (Theorem 4, Evolutionary scheme) or a botanological network in $\mathbb{R}^3$ (Theorem 10, Evolutionary scheme).

2 The property of generalized regularity of convex quadrilaterals in $\mathbb{R}^2$

Let $A_1A_2A_3A_4$ be a convex quadrilateral in $\mathbb{R}^2$, such that $B_1=B_2=B_3=B_4=1$ and $B_{12}+B_{34}=2$. We recall that a weight B_i corresponds to the vertex A_i, for $i=1,2,3,4$, a weight $B_{12}\equiv 1$ corresponds to the Steiner point O_{12} and $B_{34}\equiv 1$ corresponds to the Steiner point O_{34}. The Fermat-Torricelli point F is the intersection of the two diagonals of A_1A_3 and A_2A_4. We denote by L the Simpson line, which passes through the e-points A_{12}, A_{34} and O_{12}, O_{34} and by T_{12}, T_{34} the intersection points of the common angle bisector of the vertical angles A_1FA_2 and A_3FA_4 and the line segments A_1A_2 and A_3A_4, respectively.

Definition 1 (Generalized regularity) *A generalized regular quadrilateral is a convex quadrilateral in $\mathbb{R}^2$, such that the Simpson line L passes through the Fermat-Torricelli point F.*

Definition 2 (Weighted regularity) *A weighted regular quadrilateral is a convex quadrilateral in $\mathbb{R}^2$, such that the weighted Simpson line L passes through the weighted Fermat-Torricelli point F.*

Without loss of generality, we assume that:
$A_i=A_1(x_i,y_i)$, for $i=1,2,3,4$, $F=(x_F,y_F)$, $A_{34}=A_{34}(x_{34},y_{34})$ and $A_{12}=A_{12}(x_{12},y_{12})$, such that:
$y_4>y_3>y_2>y_1$, $x_1<x_4<x_3<x_2$.

Theorem 1 *The property of generalized regularity possess a class of convex quadrilaterals (generalized regular quadrilaterals), which corresponds to the Steiner zero solution and it consists of quadrilaterals having two of their opposite sides parallel.*

Proof. The intersection of the two diagonals A_1A_3, A_2A_4 is the unweighted Fermat-Torricelli point $F=(x_F,y_F)$, where

$$x_F=\frac{\frac{x_1(y_3-y_1)}{x_3-x_1}-\frac{x_2(y_4-y_2)}{x_4-x_2}-y_1+y_2}{\frac{y_3-y_1}{x_3-x_1}-\frac{y_4-y_2}{x_4-x_2}} \tag{2}$$

and

$$y_F=\frac{(y_3-y_1)\left(\frac{\frac{x_1(y_3-y_1)}{x_3-x_1}-\frac{x_2(y_4-y_2)}{x_4-x_2}-y_1+y_2}{\frac{y_3-y_1}{x_3-x_1}-\frac{y_4-y_2}{x_4-x_2}}-x_1\right)}{x_3-x_1}+y_1. \tag{3}$$

We shall express the coordinates of the e-point $A_{34}=A_{34}(x_{34},y_{34}$ x_{34} and y_{34} w.r. to x_3,y_3,x_4,y_4 (see Fig 1).

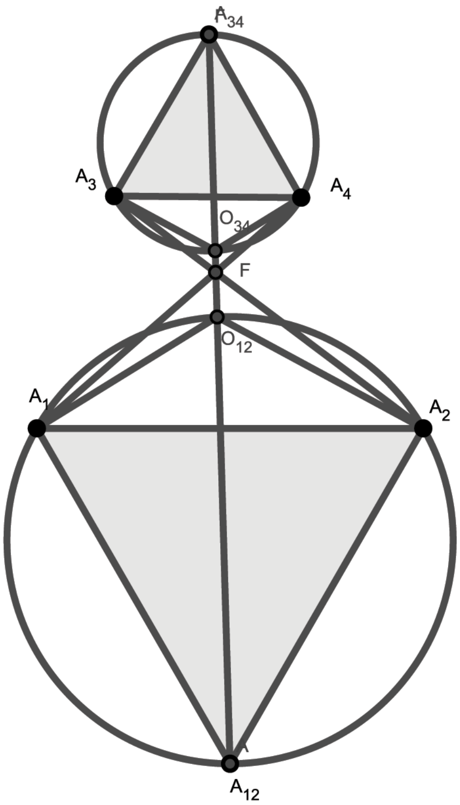

Figure 1: *Generalized regularity of quadrilaterals*

The relation $A_{34}A_3 = A_3A_4$ yields:

$$(x_{34}-x_3)^2+(y_{34}(x_3,y_3,x_4,y_4,x_{34})-y_3)^2 = (x_3-x_4)^2+(y_3-y_4)^2. \tag{4}$$

The midperpendicular line which is defined by $A_{34} = A_{34}(x_{34},y_{34})$ and the midpoint $M_{34} = (\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2})$ yields:

$$(y_{34}(x_3,y_3,x_4,y_4,x_{34}) = \frac{(x_4-x_3)\left(\left(x_{34}-\frac{x_3+x_4}{2}\right)\right)}{y_3-y_4}+\frac{1}{2}(y_3+y_4). \tag{5}$$

By replacing (5) in (4), we derive a second order degree polynomial w.r. to x_{34} and taking into account $x_{34} > \frac{x_3+x_4}{2}$, we obtain:

$$x_{34} = \frac{x_3y_3^2+x_3y_4^2-2x_3y_3y_4+\sqrt{3}M+x_4y_3^2}{2(x_3-x_4)^2+(y_3-y_4)^2} + \frac{x_4y_4^2-2x_4y_3y_4+x_3^3-x_4x_3^2-x_4^2x_3+x_4^3}{2(x_3-x_4)^2+(y_3-y_4)^2} \tag{6}$$

where

$$M \equiv (x_3-x_4)^2+(y_3-y_4)^2|y_3-y_4|.$$ □

By working similarly, we derive a second order degree polynomial w.r. to x_{12} and taking into account $x_{12} < \frac{x_1+x_2}{2}$, we obtain:

$$x_{12} = \frac{x_1y_1^2+x_1y_2^2-2x_1y_1y_2-\sqrt{3}N+x_2y_1^2}{2(x_1-x_2)^2+(y_1-y_2)^2} + \frac{x_2y_2^2-2x_2y_1y_2+x_1^3-x_2x_1^2-x_2^2x_1+x_2^3}{2(x_1-x_2)^2+(y_1-y_2)^2} \tag{7}$$

where

$$N \equiv (x_1-x_2)^2+(y_1-y_2)^2|y_1-y_2|.$$

The area of $\triangle A_{12}A_{34}F$ is given by:

$$A(\triangle A_{12}A_{34}F) = |\det\begin{pmatrix} x_F & y_F & 1\\ x_{12} & y_{12} & 1\\ x_{34} & y_{34} & 1\end{pmatrix}|. \tag{8}$$

By substituting $y_4 = y_3+\frac{y_2-y_1}{x_2-x_1}(x_4-x_3)$ in (8) and by getting as a common factor $\frac{d(A_1,A_2)}{|y_1-y_2|}$, we derive that
$A(\triangle A_{12}A_{34}F) =$
$f(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)g(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4)$
where

$$g(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4) = x_3-x_4+(x_1-x_2)\frac{|x_3-x_4|}{|x_1-x_2|}. \tag{9}$$

Without loss of generality, we assume that $x_2 > x_1$ and $x_3 > x_4$.
Hence, by calculating (9), we deduce that $A(\triangle A_{12}A_{34}F) = 0$ and A_{12}, A_{34} and F are collinear only when A_1A_2 is parallel to A_4A_3.
We denote by H the distance between A_1A_2 and A_3A_4. Suppose that $H > d(A_1,A_2)+d(A_3,A_4)$ and $\varphi \geq \angle A_2A_1A_3 \leq 120^\circ$, $\varphi \geq \angle A_1A_2A_4 \leq 120^\circ$, where $\varphi = \arctan(\frac{H}{d(A_1,A_2)+H\frac{\sqrt{3}}{3}+d(A_3,A_4)})$.

Proposition 1 *If $A_1A_2 \parallel A_4A_3$, the intersection point of the common angle bisector of $\angle A_1FA_2$ and $\angle A_3FA_4$ and the Simpson line defined by $A_{12}A_{34}$ is the Fermat-Torricelli point F.*

Proof. By applying Theorem 1, F lies on the Simpson line. Therefore, the common angle bisector of $\angle A_1FA_2$ and $\angle A_3FA_4$ and the Simpson line defined by $A_{12}A_{34}$ passes through the Fermat-Torricelli point F. □

Remark 1 *If $x_{34} < \frac{x_3+x_4}{2}$ and $x_{12} < \frac{x_1+x_2}{2}$, we derive:*

$$x_{34} = \frac{x_3y_3^2+x_3y_4^2-2x_3y_3y_4-\sqrt{3}M+x_4y_3^2}{2(x_3-x_4)^2+(y_3-y_4)^2} + \frac{x_4y_4^2-2x_4y_3y_4+x_3^3-x_4x_3^2-x_4^2x_3+x_4^3}{2(x_3-x_4)^2+(y_3-y_4)^2} \tag{10}$$

and taking into account

$$x_{12}=\frac{x_1y_1^2+x_1y_2^2-2x_1y_1y_2-\sqrt{3}N+x_2y_1^2}{2(x_1-x_2)^2+(y_1-y_2)^2}+ \\ +\frac{x_2y_2^2-2x_2y_1y_2+x_1^3-x_2x_1^2-x_2^2x_1+x_2^3}{2(x_1-x_2)^2+(y_1-y_2)^2},$$

the corresponding determinant of the area $A(\triangle A_{12}A_{34}F)$ is non-zero.

Examples of generalized regular quadrilaterals are the square, rectangle and the isosceles trapezoid.
The following results are a direct consequence of Theorem 1:

Proposition 2 *A square is a generalized regular quadrilateral, which corresponds to two Steiner zero solutions, having their Simpson lines perpendicular and meet at the Fermat-Torricelli point F.*

Corollary 1 *A square is a generalized regular quadrilateral, such that the two Simpson lines and the two corresponding angle bisectors w.r to the vertical angles coincide (two minimum Steiner trees).*

Corollary 2 *A rectangle is a generalized regular quadrilateral, such that the two Simpson lines and the two corresponding angle bisectors w.r to the vertical angles coincide and the Simpson line which is midperpendicular w.r. to the parallel sides with greater length does not given a minimum Steiner tree (a unique minimum Steiner tree).*

Corollary 3 *An isosceles trapezoid is a generalized regular quadrilateral, such that the Simpson line (midperpendicular) which passes through the Fermat-Torricelli point F and the corresponding angle bisector w.r to the vertical angles coincide.*

3 Creation of a "botanological" thumb for a boundary rectangle in $\mathbb{R}^2$

A "botanological" network for four non-collinear points in $\mathbb{R}^2$ is introduced and studied in [13] for open systems (Botany).

Definition 3 ("Botanological" network, [13]) *A "botanological" network for four non-collinear points is a two-way communication network, which has the topology of a weighted minimal Steiner tree in $\mathbb{R}^2$, having two weighted Fermat-Torricelli nodes (Steiner nodes), two weighted roots, two weighted branches and one main branch.*

Let $A_1A_2A_3A_4$ be a weighted rectangle in $\mathbb{R}^2$, B_i be a weight which corresponds to each vertex A_i, for $i=1,2,3,4$, A_1F, A_2F are the two roots of the corresponding weighted Fermat-Torricelli tree (thumb). We assume that the weighted Fermat-Torricelli point F is located on the ground and A_3F, A_4F are two branches of the weighted Fermat-Torricelli tree (thumb) and $A_1A_4 >> A_1A_2$.
The weighted Simpson line is a line defined by $A_{12}A_{34}$, where A_{12} is a vertex of $\triangle A_{12}A_1A_2$, which lies on the opposite side of A_1A_2 to O_{12} and A_{34} is a vertex of $\triangle A_{34}A_4A_3$, which lies on the opposite side of A_3A_4 to O_{34}. The weighted Steiner points O_{12} and O_{34} are the two nodes of the weighted Steiner tree and they both lie on $A_{12}A_{34}$, with equal weights $\frac{B_{12}+B_{34}}{2}$.

Definition 4 *A "botanological" thumb for a boundary rectangle is a two-way communication network, which has the topology of a weighted Fermat-Torricelli tree in $\mathbb{R}^2$, having one weighted Fermat-Torricelli node, two weighted roots and two weighted branches, which is enriched by the property of generalized regularity of quadrilaterals, such that $A_{12}A_{34}$ is perpendicular to A_1A_2.*

We assume that the weighted Fermat-Torricelli point F of $A_1A_2A_3A_4$ ($B_{12}=B_{34}=0$) lies on the ground and A_1A_2 is parallel to the ground.
Our main result is the following theorem, which gives a weighted condition for the four weights of a thumb whose weighted Simpson line is perpendicular to the ground and A_1A_2 and passes through the corresponding weighted Fermat-Torricelli point F.

Theorem 2 *If $A_{12}A_{34}$ is perpendicular to A_1A_2,*

$$B_1^2=B_2^2+B_4^2-B_3^2. \tag{11}$$

Proof. We consider the weighted Steiner tree for the boundary $A_1A_2A_3A_4$. We recall that the objective function is given by:

$$f(O_{12},O_{34})=B_1d(O_{12},A_1)+B_2d(O_{12},A_2)+B_3d(O_{34},A_3) \\ +B_4d(O_{34},A_4)+\frac{B_{12}+B_{34}}{2}d(O_{12},O_{34})\to min, \tag{12}$$

where O_{12} is the weighted Fermat-Torricelli point (Steiner node) of $\triangle A_1A_2O_{34}$ with corresponding weights B_1, B_2 and $\frac{B_{12}+B_{34}}{2}$, respectively, and O_{34} is the weighted Fermat-Torricelli point (Steiner node) of $\triangle A_3A_4O_{34}$ with corresponding weights B_3, B_4 and $\frac{B_{12}+B_{34}}{2}$, respectively.
Hence, the construction of the weighted Simpson line yields the following relations:

$$B_1\sin\angle A_1A_2A_{12}=B_2\sin\angle A_2A_1A_{12} \tag{13}$$

and

$$B_3\sin\angle A_3A_4A_{34}=B_4\sin\angle A_4A_3A_{34}. \tag{14}$$

The weighted balancing condition of the weighted Fermat-Torricelli point F for $A_1A_2A_3A_4$ taking into account that

$\vec{B_{14}} = -\vec{B_{23}}$, $\vec{B_{12}} = -\vec{B_{34}}$ and $\vec{B_{12}}$ is perpendicular to $\vec{B_{23}}$, we obtain that:

$$B_1 \cos\angle A_1A_2A_{12} = B_4 \cos\angle A_4A_3A_{34} \tag{15}$$

and

$$B_2 \cos\angle A_2A_1A_{12} = B_3 \cos\angle A_3A_4A_{34}. \tag{16}$$

By squaring both sides of (13),(14),(15) and (16) and by adding the first and third derived relation and the second and fourth derived relation, we deduce (11). □

We need the following lemma, in order to prove that the symmetry of a thumb is determined by a pair of equal weights w.r. to the two symmetrical roots and a pair of equal weights w.r. to the two symmetrical branches. Let $O = O(0,0)$, be the intersection of the diagonals of $A_1A_2A_3A_4$.

Lemma 1

$$d(A_1,F)^2 + d(A_3,F)^2 = d(A_2,F)^2 + d(A_4,F)^2. \tag{17}$$

Proposition 3 *If the thumb inherits a symmetry w.r to the midperpendicular line of the two opposite sides of the rectangle, which is perpendicular to the ground (equal branches and equal roots), then $B_1 = B_2$ and $B_3 = B_4$.*

Proof. By replacing $d(A_1,F) = d(A_2,F)$ in (17), we get

$d(A_3,F) = d(A_4,F)$.

The weighted Simpson line $A_{12}A_{34}$ is the midperpendicular line of A_1A_2 and A_3A_4 and passes through the weighted Fermat-Torricelli point F. Therefore, $A_1A_2A_3A_4$ is a generalized weighted regular rectangle. Thus, we get:

$$B_1 \sin\angle A_1A_2A_{12} = B_2 \sin\angle A_2A_1A_{12} \tag{18}$$

and

$$B_3 \sin\angle A_3A_4A_{34} = B_4 \sin\angle A_4A_3A_{34}. \tag{19}$$

By replacing $\angle A_1A_2A_{12} = \angle A_2A_1A_{12}$ in (18) and $\angle A_3A_4A_{34} = \angle A_4A_3A_{34}$ in (19), we get: $B_1 = B_2$ and $B_3 = B_4$. □

4 Creation of a "botanological" thumb with symmetrical branches in the three dimensional Euclidean Space

Let $A_{1i}A_{2i}A_{3i}A_{4i}$ be n tetrahedra in $\mathbb{R}^3$ and B_{ji} be the weight (positive real number) which corresponds to the vertex A_{ji}, for $i = 1,2,...,n$ and $j = 1,2,3,4$.

Weighted Fermat-Torricelli trees and weighted Steiner trees that have got a subconscious have been established in [10] and [11].

We denote by $\vec{u}(A_{ik},A_{jk})$ the unit vector from A_{ik} to A_{jk}. We assume that $\|\sum_{j=1}^{4} B_{jk}\vec{u}(A_{ik},A_{jk})\| > B_{ik}$ hold, in order to locate weighted Fermat-Torricelli trees with four branches $\{A_{0k}A_{1k}, A_{0k}A_{2k}, A_{0k}A_{3k}, A_{0k}A_{4k}\}$ that got a subconscious node.

Lemma 2 (Geometric plasticity of weighted Fermat-Torricelli trees that have got a subconscious node[10]) *If we select a point P_{ik} with a non-negative weight B_{ik} on the ray that is defined by the line segment $A_{0k}A_{ik}$, such that:*

$$\|\sum_{j=1}^{4} B_{jk}\vec{u}(P_{ik},P_{jk})\| > B_{ik},$$

Then the corresponding weighted Fermat-Torricelli node P_{0k} that has got a subconscious of $\{P_{0k}P_{1k}, P_{0k}P_{2k}, P_{0k}P_{3k}, P_{0k}P_{4k}\}$ remains the same with A_{0k}, for $k = 1,2,3,...,n$.

The modified weighted Fermat-Torricelli problem for tetrahedra states that:

Problem 2 (Modified weighted Fermat-Torricelli problem [10])
Let $A_{1k}A_{2k}A_{3k}A_{4k}$ be a tetrahedron in $\mathbb{R}^3$, $\mathcal{B}_{ik}$ be a non-negative number (weight) which corresponds to each line segment $A_{0k}A_{ik}$, respectively. Find a point A_{0k} which minimizes the sum of the lengths of the line segments a_{0ik} that connect every vertex A_{ik} with A_{0k} multiplied by the positive weight $\mathcal{B}_{ik}$:

$$\sum_{i=1}^{4} \mathcal{B}_i a_{0ik} = minimum. \tag{20}$$

By letting $\mathcal{B}_{ik} = B_{ik}$, for $i = 1,2,3,4$, $k = 1,2,...,n$, the weighted Fermat-Torricelli problem for tetrahedra and the corresponding modified weighted Fermat-Torricelli problem for tetrahedra are equivalent by collecting instantaneous images of the weighted Fermat-Torricelli network via the geometric plasticity of tetrahedra in $\mathbb{R}^3$.

The geometric plasticity of tetrahedra connects the weighted Fermat-Torricelli problem for tetrahedra with the modified weighted Fermat-Torricelli problem for boundary tetrahedra by allowing a mass flow continuity for the weights, such that the corresponding weighted Fermat-Torricelli point remains the same in $\mathbb{R}^3$.

The weighted Fermat-Torricelli nodes remain the same $P_{0k} \equiv A_{0k}$ but different values of the subconscious (remaining weight) may occur.

We denote by B_{ji} a mass flow which is transferred from A_{ji} to A_{0i} for $j = 1,2$ by B_{0i} a residual weight which remains at A_0 and by B_{ki} a mass flow which is transferred from A_{0i} to A_{ki} for $k = 3,4$.

We denote by $\tilde{B}_{ji}$ a mass flow which is transferred from A_{0i} to A_{ji} for $i = 1,2$, by $\tilde{B}_{0i}$ a residual weight which remains at A_{0i} and by $\tilde{B}_{ki}$ a mass flow which is transferred from A_{ki} to A_{0i}, for $k = 3,4$.

Thus, we derive the weighted outward flow condition and weighted inward flow condition:

$$B_{1i} + B_{2i} = B_{3i} + B_{4i} + B_{0i} \tag{21}$$

and

$$\tilde{B}_{1i} + \tilde{B}_{2i} + \tilde{B}_{0i} = \tilde{B}_{3i} + \tilde{B}_{4i}. \tag{22}$$

By adding (21) and (22) and by setting $\bar{B}_{0i} = B_{0i} - \tilde{B}_{0i}$, we obtain:

$$\bar{B}_{1i} + \bar{B}_{2i} = \bar{B}_{3i} + \bar{B}_{4i} + \bar{B}_{0i} \tag{23}$$

such that:

$$\bar{B}_{1i} + \bar{B}_{2i} + \bar{B}_{3i} + \bar{B}_{4i} = c, \tag{24}$$

where c is a positive real number, for $i = 1,2,...,n$.

We denote by a_{0im} the length of the line segment $A_{0m}A_{im}$, $\alpha_{i0jm} \equiv \angle A_{im}A_0A_{jm}$ and $\alpha_{i,j0km}$ the angle which is formed by the line segment that connects A_{0m} with the trace of the orthogonal projection of A_{im} to the plane defined by $\triangle A_{jm}A_0A_{km}$ with a_{0im}, for $i,j,k,l = 1,2,3,4$, $i \neq j \neq k \neq i$ and $m = 1,2,3,...,n$

Lemma 3 (Determination of the position of A_{0i} on exactly five given angles [10, Proposition 2.9, p. 902], [12, Formulas (10), (11), p. 120])
Each angle $\alpha_{i,k0ml}$ depends on α_{102l}, α_{103l}, α_{104l}, α_{203l} and α_{204l}, for $i,k,m = 1,2,3,4$, $i \neq k \neq m$, and $l = 1,2,,...,n$

$$\cos^2(\alpha_{i,k0ml}) = \frac{\sin^2(\alpha_{k0ml}) - \cos^2(\alpha_{m0il}) - \cos^2(\alpha_{k0il})}{\sin^2(\alpha_{k0ml})} + \\ + \frac{2\cos(\alpha_{m0il})\cos(\alpha_{k0il})\cos(\alpha_{k0ml})}{\sin^2(\alpha_{k0ml})} \tag{25}$$

and

$$\cos\alpha_{304} = -\frac{1}{4}[2b + \\ + 4\cos\alpha_{102}\left(\cos\alpha_{104}\cos\alpha_{203} + \cos\alpha_{103}\cos\alpha_{204}\right) - \\ - 4\left(\cos\alpha_{103}\cos\alpha_{104} + \cos\alpha_{203}\cos\alpha_{204}\right)]\csc^2\alpha_{102} \tag{26}$$

or

$$\cos\alpha_{304} = \frac{1}{4}[4\cos\alpha_{103}(\cos\alpha_{104} - \cos\alpha_{102}\cos\alpha_{204}) + \\ + 2\left(b + 2\cos\alpha_{203}\left(-\cos\alpha_{102}\cos\alpha_{104} + \cos\alpha_{204}\right)\right)]\csc^2\alpha_{102} \tag{27}$$

where

$$b \equiv \sqrt{\prod_{i=3}^{4}} (1 + \cos(2\alpha_{102}) + \cos(2\alpha_{10i}) + \cos(2\alpha_{20i}) - 4\cos\alpha_{102}\cos\alpha_{10i}\cos\alpha_{20i}).$$

We denote by α_l the dihedral angle which is formed by the planes defined by $\triangle A_{1l}A_{0l}A_{2l}$ and $\triangle A_{1l}A_{2l}A_{3l}$, and by $\alpha_{g_{4l}}$ the dihedral angle formed by the planes defined by $\triangle A_{1l}A_{4l}A_{2l}$ and $\triangle A_{1l}A_{2l}A_{3l}$, for $l = 1,2,...,n$.

Lemma 4 [[10, Formula (27), p. 997]]
The variable length a_{04l} is given by

$$a_{04l}^2 = a_{02}^2 + a_{24l}^2 - 2a_{24l}\Bigg[\sqrt{a_{02l}^2 - h_{0,12l}^2}\cos\alpha_{124l} + \\ + h_{0,12l}\sin\alpha_{124l}\left(\cos\alpha_{g_{4l}}\left(\frac{\left(\frac{a_{02}^2 + a_{23}^2 - a_{03}^2}{2a_{23}}\right) - \sqrt{a_{02l}^2 - h_{0,12l}^2}\cos\alpha_{123l}}{h_{0,12l}\sin\alpha_{123l}}\right) + \\ + \sin\alpha_{g_{4l}}\sin\arccos\left(\frac{\left(\frac{a_{02l}^2 + a_{23l}^2 - a_{03l}^2}{2a_{23l}}\right) - \sqrt{a_{02l}^2 - h_{0,12l}^2}\cos\alpha_{123l}}{h_{0,12l}\sin\alpha_{123l}}\right)\right)\Bigg] \tag{28}$$

and

$$h_{0,12l} = \frac{a_{01l}a_{02l}}{a_{12l}}\sqrt{1 - \left(\frac{a_{01l}^2 + a_{02l}^2 - a_{12l}^2}{2a_{01l}a_{02l}}\right)^2}. \tag{29}$$

Theorem 3 [Dynamic Plasticity of weighted network with two roots and two growing branches]
Given the weighted Fermat-Torricelli point A_{0i} that has got a subconscious $\bar{B}_{0i}$ to be an interior point of the tetrahedron $A_{1i}A_{2i}A_{3i}A_{4i}$ with the vertices lie on four prescribed rays that meet at A_{0i} and from the five given values of α_{102i}, α_{103i}, α_{104i}, α_{203i}, α_{204i}, the positive real weights $\bar{B}_{ji}$ are given by:

$$\bar{B}_{1i} = \left(\frac{\sin\alpha_{4,203i}}{\sin\alpha_{1,203i}}\right)\frac{c - \bar{B}_{0i}}{2}, \tag{30}$$

$$\bar{B}_{2i} = \left(\frac{\sin\alpha_{4,103i}}{\sin\alpha_{2,103i}}\right)\frac{c - \bar{B}_{0i}}{2}, \tag{31}$$

$$\bar{B}_{3i} = \left(\frac{\sin\alpha_{4,102i}}{\sin\alpha_{3,102i}}\right)\frac{c - \bar{B}_{0i}}{2}, \tag{32}$$

$$\bar{B}_{4i} = \frac{c - \bar{B}_{0i}}{2}, \tag{33}$$

under the weighted conditions

$$\bar{B}_{1i} + \bar{B}_{2i} + \bar{B}_{3i} + \bar{B}_{4i} = c, \tag{34}$$

and

$$\bar{B}_{1i} + \bar{B}_{2i} = \bar{B}_{3i} + \bar{B}_{4i} + \bar{B}_{0i}. \tag{35}$$

Proof. By considering a two-way communication network and by assuming mass flow continuity the weights $\bar{B}_{ki}$, for $i = 1,2,3,4$, are determined by the weighted outward and inward flow conditions (21), (22), which yield the weighted conditions (34) and (35).
Thus, we obtain that:

$$\sum_{k=1}^{4} B_{ki}a_{0ki} + \sum_{k=1}^{4} \tilde{B}_{ki}a_{0ki} \to min, \tag{36}$$

which gives

$$\sum_{k=1}^{4} \bar{B}_{ki}a_{0ki} \to min. \tag{37}$$

By differentiating (37) w.r. to a_{01l}, a_{02l} a_{03l}, respectively, taking into account the derivative of a_{04l} w.r. to a_{01l}, a_{02l} a_{03l}, by lemma 4, we obtain (30), (31), (32) and (33). □

Remark 2 *We note that the dynamic plasticity equations of Theorem 3 have been derived in [10] for weighted Fermat-Torricelli trees, which consist of two roots one branch and one growing branch that have inherited a subconscious (weighted Fermat-Torricelli node) under different weighted (inflow - outflow conditions):*
$\bar{B_{1i}} + \bar{B_{2i}} + \bar{B_{3i}} = \bar{B_{0i}} + \bar{B_{4i}}$ *for* $i = 1,2,...,n$.

We assume that the common perpendicular line of $A_{1i}A_{2i}A_{3i}A_{4i}$ passes through the common midpoints m_{12} and m_{34} of $A_{1i}A_{2i}$ and $A_{4i}A_{3i}$, respectively and $m_{12}m_{34} >> A_{1i}A_{2i}$. We denote by φ_i the angle formed by $\overrightarrow{A_{1i}A_{2i}}$ and $\overrightarrow{A}_{4i}A_{3i}$ and by B_{ji} the weight (positive real number) which corresponds to the vertex A_{ji}, for $j = 1,2,3,4$, $i = 1,2,...,n$. Hence, by rotating $A_{1i}A_{2i}A_{3i}A_{4i}$ by φ_i with respect to $m_{12}m_{34}$, we obtain n weighted isosceles trapezoid $A'_{1i}A'_{2i}A'_{3i}A'_{4i}$ and $B'_{ji} = B_{ji}$. We denote by O_i the intersection point of the equal diagonals $A'_{1i}A'_{3i}$ and $A'_{2i}A'_{4i}$, by A_{0i} the corresponding weighted Fermat-Torricelli node with remaining weight B_{0i} (one node that has got a subconscious) and by O_{12i} and O_{34i} the two corresponding weighted Steiner nodes with remaining weights B_{12i} and B_{34i} (two nodes that got a subconscious) for $A'_{1i}A'_{2i}A'_{3i}A'_{4i}$.

Theorem 4 *If* A_{0i} *lies on the common perpendicular segment* $m_{12}m_{34}$, *then*

$$\bar{B_{1i}} = \bar{B_{2i}} \tag{38}$$

and

$$\bar{B_{3i}} = \bar{B_{4i}} \tag{39}$$

Proof. By substituting $\alpha_{4,102i} = \alpha_{3,102i}$ in (32) and (33), we obtain (39). By working cyclically with the indices and by exchanging the indices $3 \to 2$, $4 \to 1$ and $1 \to 4$, $2 \to 3$, we derive (38). □

We may consider that $\{A_{1i}, A_{2i}\}$ lie on a circular cone C_{012i}, having $m_{12}m_{34}$ as axis of rotation with vertex the weighted Fermat-Torricelli point A_{0i} and $\{A_{3i}, A_{4i}\}$ lie on a circular cone C_{034i}, having $m_{12}m_{34}$ as axis of rotation with vertex the weighted Fermat-Torricelli point A_{0i}. We note that C_{012i} and C_{034i} intersect only at A_{0i}.

Proposition 4 (Rotational plasticity of tetrahedra) *If we select* $\{R_{1i}, R_{2i}\}$ *two points with weights* B_{1i}, B_{2i}, *respectively, on* C_{012i}, *such that their midpoint* m_{12i} *lies on the line defined by* $m_{12}m_{34}$ *and* $\{R_{3i}, R_{4i}\}$ *two points with weights* B_{3i} *and* B_{4i}, *respectively, on* C_{034i}, *such that their midpoint* m_{34i} *lies on the line defined by* $m_{12}m_{34}$, *then the corresponding weighted Fermat-Torricelli point* R_{0i} *of* $R_{1i}R_{2i}R_{3i}R_{4i}$ *remains the same with* A_{0i} *for* $B_{1i} = B_{2i}$ *and* $B_{3i} = B_{4i}$, *for* $i = 1,2,...,n$.

Proof. It is a direct consequence of Theorem 4 and taking into account that
$R_{1i}R_{2i}R_{3i}R_{4i}$ are derived by rotating the two isosceles triangles $\triangle R_{1i}A_{0i}R_{2i}$ and $\triangle R_{3i}A_{0i}R_{4i}$ along $m_{12}m_{34}$. By rotating properly $R_{1i}R_{2i}R_{3i}R_{4i}$, we may derive a weighted isosceles trapezoid or a weighted rectangle ($R_{1i}R_{2i} = R_{3i}R_{4i}$) for $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$. Thus, the weighted balancing condition $\sum_{j=1}^{4} B_{ji}\overrightarrow{u(A_{0i}, A_{ji})} = \vec{0}$, yields $R_{0i} \equiv A_{0i}$. □

Definition 5 *A "botanological" thumb for a boundary symmetric tetrahedron* $A_{1i}A_{2i}A_{3i}A_{4i}$ *whose common perpendicular passes through the common midpoints* m_{12} *and* m_{34} *of* $A_{1i}A_{2i}$ *and* $A_{4i}A_{3i}$, *respectively and* $m_{12}m_{34} >> A_{1i}A_{2i}$ *is a "botanological" network, which is transformed to a botanological "thumb" for a boundary rectangle or a boundary isosceles trapezoid, by rotating properly* $A_{1i}A_{2i}$ *w.r.* $m_{12}m_{34}$.

Definition 6 *A "botanological" thumb is a collection of "botanological" thumbs for a finite number of boundary symmetric tetrahedra in* $\mathbb{R}^3$.

We will describe an evolutionary scheme for the creation of a "botanological" thumb in $\mathbb{R}^3$.

1. Evolutionary Phase 1
At time $t = 0$, we consider a point "seed" A_{0i} on the ground.

2. Evolutionary Phase 2
After time t, by assuming mass flow continuity two equal roots start to grow underground and two equal branches start to grow overground, such that their endpoints form a boundary rectangle $A'_{1i}A'_{2i}A'_{3i}A'_{4i}$. Taking into account Proposition 3, we derive that $B_{1i} = B_{2i}$ and $B_{3i} = B_{4i}$.

3. Evolutionary Phase 3
We consider two cases: (i) If A_{0i} is the intersection of the diagonals $A'_{1i}A'_{3i}$ and $A'_{2i}A'_{4i}$ the weighted Fermat-Torricelli node A_{0i} has acquired a subconscious $\bar{B}_{0i}$. (ii) If A_{0i} lies

on the midperpendicular line segment $m_{12}m_{34}$ the weighted Fermat-Torricelli node A_{0i} has acquired a subconscious $\bar{B}_{0i}$.

4. Evolutionary Phase 4
The subconscious $\bar{B}_{0i}$ may cause a geometric plasticity and/or a rotational plasticity of the weighted Fermat-Torricelli tree $\{A'_{1i}A_{0i}, A'_{2i}A_{0i}, A'_{3i}A_{0i}, A'_{4i}A_{0i}\}$.

(i) The geometric plasticity (Theorem 2) yields a weighted Fermat-Torricelli tree $\{R_{1i}A_{0i}, R_{2i}A_{0i}, R_{3i}A_{0i}, R_{4i}A_{0i}\}$, such that their endpoints form an isosceles trapezoid $R_{1i}R_{2i}R_{3i}R_{4i}$, $A''_{0i} \equiv A_{0i}$ and $\bar{B}_{ji}$ corresponds to R_{ji}, for $j = 1,2,3,4$ and $i = 1,2,...,n$.

(ii) The rotational plasticity (Proposition 4), the dynamic plasticity (Theorem 3) and the symmetry of boundary tetrahedra taken from Theorem 4, creates a "botanological" thumb for $i = 1,2,...,n$, having the corresponding weighted Fermat-Torricelli node A_{0i} constant on the ground (point "seed"), but with different subconscious quantities $\bar{B}_{0i}$, for $i = 1,2,...,n$.

5 Generalized regularity for tetrahedra in the three dimensional Euclidean Space

The weighted Steiner problem for a boundary weighted tetrahedron $A_1A_2A_3A_4$ in $\mathbb{R}^3$ having two subconscious nodes (weighted Fermat-Torricelli or weighted Steiner points) has been studied recently in [11].

We denote by $A_1A_2A_3A_4$ a tetrahedron in $\mathbb{R}^3$, with $A_i(x_i, y_i, z_i)$ $(i = 1,2,3,4)$, by b_i a positive real number(weight) which corresponds to the vertex A_i, O_{12}, O_{34} two interior points (nodes) of $A_1A_2A_3A_4$ in $\mathbb{R}^3$, by b_{12} the weight which corresponds to O_{12}, b_{34} the weight which corresponds to O_{34}, by H the length of the common perpendicular (Euclidean distance) between the two lines defined by A_1A_2, A_4A_3, by A_iA_j the Euclidean distance from A_i to A_j, by $O_{12}O_{34}$ the Euclidean distance from O_{12} to O_{34}, by A_iO_{12} the Euclidean distance from A_i to O_{12} and by A_jO_{34} the Euclidean distance from A_j to O_{34}, by T_{12} the intersection point of the line defined by $O_{12}O_{34}$ and the line defined by A_1A_2 and by T_{34} the intersection point of the line defined by $O_{12}O_{34}$ and the line defined by A_4A_3, M_{12} the midpoint of A_1A_2 and M_{34} the midpoint of A_4A_3, for $i, j = 1,2,3,4$.

We denote by A''_4 the intersection point of the line defined by the A_4A_3 and the line defined by the common perpendicular of A_1A_2 and A_4A_3 and by A''_1 the intersection point of the line defined by A_1A_2 and the line defined by the common perpendicular of A_1A_2

We set
$\vec{a}_{ij} \equiv \overrightarrow{A_iA_j}$, for $i, j = 1,2,3,4$, $i \neq j \neq k$, $\alpha_{12} \equiv \angle A_1O_{12}A_2$, $\alpha_{34} \equiv \angle A_3O_{34}A_4$, $\alpha_1 \equiv \angle A_2O_{12}O_{34}$, $\alpha_2 \equiv \angle A_1O_{12}O_{34}$, $\alpha_3 \equiv \angle A_4O_{34}O_{12}$, $\alpha_4 \equiv \angle A_3O_{34}O_{12}$, $\varphi \equiv \arccos(\frac{\vec{a}_{12}\cdot\vec{a}_{43}}{a_{12}a_{43}})$ and $b_{ST} = \frac{b_{12}+b_{34}}{2}$.

Furthermore, we denote by A_{12} the vertex of $\triangle A_1A_{12}A_2$, such that: $\angle A_1A_{12}A_2 = \pi - \alpha_{12}$, $\angle A_{12}A_1A_2 = \pi - \alpha_1$ and $\angle A_1A_2A_{12} = \pi - \alpha_2$, by A_{34} the vertex of $\triangle A_4A_{34}A_3$, such that: $\angle A_4A_{34}A_3 = \pi - \alpha_{34}$, $\angle A_{34}A_4A_3 = \pi - \alpha_4$ and $\angle A_4A_3A_{34} = \pi - \alpha_3$, by H_{12} the trace of the height of $\triangle A_1A_{12}A_2$ w.r to the base A_1A_2 and by A_{34} the vertex of $\triangle A_4A_{34}A_3$, such that: $\angle A_4A_{34}A_3 = \pi - \alpha_{34}$, $\angle A_{34}A_4A_3 = \pi - \alpha_4$ and $\angle A_4A_3A_{34} = \pi - \alpha_3$ and by H_{34} the trace of the height of $\triangle A_4A_{34}A_3$ w.r to the base A_4A_3.

We set $H \equiv A''_4A''_1$, $t_{34} \equiv A''_4T_{34}$ $t_{12} \equiv A''_1T_{12}$ $k_1 \equiv A''_1A_1$ and $k_2 \equiv A''_4A_4$, $m_{12} \equiv A''_1M_{12}$ and $m_{34} \equiv A''_4M_{34}$, $h'_{12} \equiv A''_1H_{12}$ and $h'_{34} \equiv A''_4H_{34}$.

We assume that: $A_1A_4 + A_2A_3 > A_1A_2 + A_3A_4$.

The weighted Steiner problem for $A_1A_2A_3A_4$ in $\mathbb{R}^3$ states that:

Problem 3 ([11, Problem 5]) *Find $O_{12}(x_0, y_0, z_0)$ and $O_{34}(x_{0'}, y_{0'}, z_{0'})$ with given weights b_{12} in O_{12} and b_{34} in O_{34}, such that*

$$f(O_{12}, O_{34}) = b_1A_1O_{12} + b_2A_2O_{12} + b_3A_3O_{34} + b_4A_4O_{34} + \\ + \frac{b_{12}+b_{34}}{2}O_{12}O_{34} \to min. \tag{40}$$

Theorem 5 ([11, Theorem 3]) *The solution of the weighted Steiner problem is a weighted Steiner tree in $\mathbb{R}^3$ whose nodes O_{12} and O_{34} (weighted Fermat-Torricelli points) are seen by the angles:*

$$\begin{aligned} \cos\alpha_{12} &= \frac{b_{ST}^2 - b_1^2 - b_2^2}{2b_1b_2}, \\ \cos\alpha_1 &= \frac{b_1^2 - b_2^2 - b_{ST}^2}{2b_2b_{ST}}, \\ \cos\alpha_{34} &= \frac{b_{ST}^2 - b_3^2 - b_4^2}{2b_3b_4}, \\ \cos\alpha_4 &= \frac{b_4^2 - b_3^2 - b_{ST}^2}{2b_3b_{ST}}. \end{aligned} \tag{41}$$

The inradius r_{12} is the radius of the inscribed circle of triangle $\triangle A_1A_{12}A_2$ with sides $A_1A_2 = \lambda\frac{b_{12}+b_{34}}{2}$, $A_1A_{12} = \lambda b_2$ and $A_2A_{12} = \lambda b_1$, where $\lambda = \frac{A_1A_2}{\frac{b_{12}+b_{34}}{2}}$.

The inradius r_{34} is the radius of the inscribed circle of triangle $\triangle A_3A_{34}A_4$ with sides $A_3A_4 = \lambda\frac{b_{12}+b_{34}}{2}$, $A_3A_{34} = \lambda b_4$ and $A_4A_{34} = \lambda b_3$, where $\lambda = \frac{A_3A_4}{\frac{b_{12}+b_{34}}{2}}$.

We use the substitutions for r_{12} and r_{34}, ([11, Section 2, p. 6]):

$$r_{12} \equiv \frac{A_1A_2}{(b_1 + b_2 + \frac{b_{12}+b_{34}}{2})(b_1 + b_2 - \frac{b_{12}+b_{34}}{2})(b_2 + \frac{b_{12}+b_{34}}{2} - b_1)(b_1 + \frac{b_{12}+b_{34}}{2} - b_2)},$$

$$r_{34} \equiv \frac{A_4A_3}{(b_3 + b_4 + \frac{b_{12}+b_{34}}{2})(b_3 + b_4 - \frac{b_{12}+b_{34}}{2})(b_3 + \frac{b_{12}+b_{34}}{2} - b_4)(b_4 + \frac{b_{12}+b_{34}}{2} - b_3)},$$

$$\beta_{12} = \arccos(\frac{A_1A_2}{2r_{12}}),$$

$$\beta_{34} = \arccos(\frac{A_4A_3}{2r_{34}}).$$

Theorem 6 ([11, Theorem 4]) *The following system of equations w.r. to t_{34} and t_{12} allows the computation of the position of the weighted Simpson line $O_{12}O_{34}$ of the weighted full Steiner tree for $A_1A_2A_3A_4$:*

$$\frac{t_{34}-t_{12}\cos\phi}{\sqrt{H^2+t_{12}^2\sin^2\phi}}=\frac{h'_{34}-t_{34}}{r_{34}} \tag{42}$$

and

$$\frac{t_{12}-t_{34}\cos\phi}{\sqrt{H^2+t_{34}^2\sin^2\phi}}=\frac{h'_{12}-t_{12}}{r_{12}} \tag{43}$$

Proposition 5 ([11, Proposition 1])

$$\frac{t_{34}-t_{12}\cos\phi}{\sqrt{H^2+t_{12}^2\sin^2\phi}}=\frac{m_{34}-t_{34}}{a_{34}\frac{\sqrt{3}}{2}} \tag{44}$$

and

$$\frac{t_{12}-t_{34}\cos\phi}{\sqrt{H^2+t_{34}^2\sin^2\phi}}=\frac{m_{12}-t_{12}}{a_{12}\frac{\sqrt{3}}{2}} \tag{45}$$

Theorem 7 ([11, Theorem 5]) *The following system of equations w.r. to t_{34}, t_{12} and $\angle A_4FA_3$ allows the computation of the position of the line defined by $T_{12}T_{34}$ of the (unweighted) Fermat-Torricelli tree of $A_1A_2A_3A_4$:*

$$\frac{t_{34}-t_{12}\cos\phi}{\sqrt{H^2+t_{12}^2\sin^2\phi}}=\frac{m_{34}-t_{34}}{\frac{a_{34}}{2}\tan\frac{\angle A_4FA_3}{2}}, \tag{46}$$

$$\frac{t_{12}-t_{34}\cos\phi}{\sqrt{H^2+t_{34}^2\sin^2\phi}}=\frac{m_{12}-t_{12}}{\frac{a_{12}}{2}\tan\frac{\angle A_4FA_3}{2}}, \tag{47}$$

$$\cot\frac{\angle A_4FA_3}{2}= \frac{2(H^2+k_1(t_{12}-t_{34}^2\cos\varphi))+k_2(t_{34}-t_{12}\cos\varphi)}{(t_{12}-k_1)\sqrt{H^2+t_{34}^2\sin^2\varphi}+(t_{34}-k_2)\sqrt{H^2+t_{12}^2\sin^2\varphi}}. \tag{48}$$

We denote by ω the dihedral angle (twist angle) formed by the planes $A_1A_2T_{12}T_{34}$ and $A_4A_3T_{34}T_{12}$, by $\varphi_{12}=\angle A_1T_{12}T_{34}$ and $\varphi_{34}=\angle A_4T_{34}T_{12}$.

Theorem 8 ([11, Theorem 6]) *The twist angle ω is given by*

$$\cos\omega=\frac{\cos\varphi-\cos\varphi_{12}\cos\varphi_{34}}{\sin\varphi_{12}\sin\varphi_{34}}. \tag{49}$$

Remark 3 *We correct two typographical errors that occur in [11] by replacing $\sqrt{H^2+t_{34}\sin^2\phi}$ by $\sqrt{H^2+t_{34}^2\sin^2\phi}$ and the angle φ_{34} by $\sin\varphi_{34}$ in [11, Formula (3.1)].*

Definition 7 *A generalized regular tetrahedron is a tetrahedron, which determines a generalized (weighted) regular quadrilateral, formed by rotating A_1A_2 or A_3A_4 by the twist angle ω, w.r. to the (weighted) Simpson line $A_{12}A_{34}$.*

We denote by ω_F the twist angle formed by the planes defined by $\triangle A_1FA_2$ and $\triangle A_3FA_4$ and by ω_S the twist angle formed by the planes $\triangle A_1O_{12}A_2$ and $\triangle A_3O_{34}A_4$.

Theorem 9 (Generalized regularity of tetrahedra) *If $A_1A_2A_3A_4$ is a generalized regular quadrilateral, then generalized regular tetrahedra are derived by:*
(i) rotating the twist angle ω_F w.r. to the line defined by $M_{12}M_{34}$

$$\cos\omega_F=\frac{\cos\varphi-\cos^2\angle A_1M_{12}F}{\sin^2\angle A_1M_{12}F}. \tag{50}$$

or (ii)rotating the twist angle ω_F w.r. to the Simpson line defined by $T_{12}T_{34}$

$$\cos\omega_S=\frac{\cos\varphi-\cos^2\angle A_1T_{12}O_{12}}{\sin^2\angle A_1T_{12}O_{12}}. \tag{51}$$

Proof. A generalized regular convex quadrilateral is a trapezoid having the property: $A_1A_2\parallel A_3A_4$. Thus, the Fermat-Torricelli point F is the intersection of diagonals A_1A_3 and A_2A_4 and lies on the line defined by $M_{12}M_{34}$, which yields $\angle A_1M_{12}F=\angle A_3M_{34}F$. By substituting $\angle A_1M_{12}F=\angle A_3M_{34}F$ in (49), we obtain (50). We recall that $A_1A_2A_{12}$ and $A_3A_4A_{34}$ are equilateral triangles outward from $A_1A_2A_3A_4$ and the Simpson line intersects A_1A_2 and A_3A_4 at T_{12} and T_{34}, respectively. By substituting $\angle A_1T_{12}O_{12}=\angle A_3T_{34}O_{34}$ in (49), we obtain (51). □

Remark 4 *The position of A_1'' and A_4'' may be calculated by Theorem 7.*

Definition 8 *A weighted regular tetrahedron is a tetrahedron in $\mathbb{R}^3$, such that the weighted Simpson line L passes through the weighted Fermat-Torricelli point F.*

We assume that $A_1A_2A_3A_4$ is a weighted regular tetrahedron $A_1A_2A_3A_3$, such that: $M_{12}M_{34}>>\max A_1A_2,A_3A_4$.

Theorem 10 (Weighted regularity of tetrahedra) *The common perpendicular line of A_1A_2 and A_3A_4 passes through the common midpoints M_{12} and M_{34}, respectively, if and only if $b_1=b_2$ and $b_3=b_4$.*

Proof. The weighted Simpson line passes through A_{12}, A_{34}, the weighted Steiner nodes O_{12}, O_{34}, the weighted Fermat-Torricelli point F and M_{12}, M_{34}. Therefore, $\triangle A_1A_2A_{12}$ and $A_3A_4A_{34}$ are isosceles triangles $A_1A_{12} = A_2A_{12}$ and $A_3A_{34} = A_4A_{34}$, which yield $b_1 = b_2$ and $b_3 = b_4$. Hence, it is shown one direction.
We assume that the common perpendicular line of A_1A_2 and A_3A_4 does not pass through the common midpoints M_{12} and M_{34}, $b_1 = b_2$ and $b_3 = b_4$. By substituting $b_1 = b_2$ and $b_3 = b_4$ and given a subconscious weight B_S in (41), we derive that $\angle A_1O_{12}O_{34} = \angle A_2O_{12}O_{34}$ and $\angle A_3O_{34}O_{12} = \angle A_4O_{34}O_{12}$. By substituting $b_1 = b_2$, $b_3 = b_4$ in (42) and (43) we obtain the values of t_{12} and t_{34}, in order to calculate the twist angle ω_S. By rotating A_1A_2 w.r. to $A_{12}A_{34}$ by ω_S, $A_1A_2 \parallel A_3A_4$, and $A_{12}A_{34}$ passes through M_{12}, M_{34}, otherwise O_{12}, O_{34} and F are not collinear. It proves another direction and the theorem as well. □

We may follow the same evolutionary scheme for a "botanological" thumb in $\mathbb{R}^3$. Taking into account that the point seed which has got a subconscious B_{ST} is located underground, an evolutionary two way communication network will start to grow having two roots one main branch and two branches. By assuming a constant mass flow continuity that corresponds to the two roots $b_1 = b_2$ (O_{12} is located underground) one main branch with remaining weight B_{ST} and two branches with weights $b_3 = b_4$ (O_{34} is located overground). Therefore, by applying Theorem 10 we obtain a boundary weighted regular tetrahedron formed by the endpoints of two symmetrical roots and two symmetrical branches, such that the main branch is perpendicular to the ground.

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