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A new characterization of fuzzy ideals of semigroups and its applications

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ABSTRACT
In this paper, we develop a new technique for constructing fuzzy ideals of a semigroup. By using generalized Green's relations, fuzzy star ideals are constructed. It is shown that the new fuzzy ideal of a semigroup can be used to investigate the relationship between fuzzy sets and abundance and regularity for an arbitrary semigroup. Appropriate examples of such fuzzy ideals are given in order to illustrate the technique. Finally, we explain when a semigroup satisfies conditions of regularity.

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1. Introduction

Many problems in engineering, economics and mathematics involve uncertainty. Zadeh [1] and others (see, [2, 3]) have proposed some theories to deal with uncertainty. Let X be a non-empty set. Following Zadeh [1], a *fuzzy subset* μ of X is a function of X into the closed interval $[0, 1]$. In the past few decades, due to the wide applications of fuzzy sets in many areas, such as fuzzy structure multi-agent systems, formation control, fuzzy automata, fuzzy algebra, fuzzy graph and so on, the fuzzy set theory has been a hot topic (see, [3–10]). In fuzzy algebra and fuzzy logic, many authors (see [11–16]) studied fuzzy groups, fuzzy subsemigroups, fuzzy ideals, fuzzy relations, etc. Budimirorić and others [12] studied fuzzy algebra identities with applications to fuzzy semigroups. It is shown that for every fuzzy subalgebra there is a least fuzzy equality such that a fuzzy identity holds on it. In contrast to the fuzzy identities, fuzzy ideals of semigroups play an important role in the characterization of semigroup structures. Recently, a lot of scholars investigated some classes of fuzzy ideals of a semigroup. For example, Khan et al. [15] defined generalized fuzzy ideals in ordered semigroups. Ibrar et al. [14] focussed on the characterization of an ordered semigroups in the frame work of generalized bipolar fuzzy interior ideals.

Motivated by the study of fuzzy ideals in ordered semigroups and regular semigroups in terms of fuzzy subsets, this paper shall give the notion of fuzzy star ideals of a semigroup to investigate conditions of abundance and regularity of an arbitrary semigroup.

We proceed as follows: Section 2 provides some basic concepts and some known results. Section 3 advances definitions of fuzzy left (right)star ideals and fuzzy star ideals of a semigroup and give some concrete constructions. The main result of this paper is presented in Section 4. In particular, some equivalent conditions for an arbitrary semigroup to be abundant are proved in this section. In Section 5, some applications are obtained and some examples are given. In the last section, some remarks are provided.

2. Preliminaries

We follow the notions adopted in [1, 17, 18]. First, we recall some known facts about the generalized Green relations $\mathcal{L}^*$ and $\mathcal{R}^*$.

Let S be a semigroup and $a, b \in S$. We say a, b are $\mathcal{L}$-related in S if there exist $x, y \in S^1$ satisfying $a = xb$ and $b = ya$. The relation $\mathcal{R}$ is defined dually. A semigroup S is *regular* if for all $a \in S$ there is $x \in S$ such that $a = axa$. It is well known that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are generalized Green's $\mathcal{L}$ and $\mathcal{R}$, respectively.

Lemma 2.1 ([17]): *Let S be a semigroup and $a, b, e = e^2 \in S$. Then the following statements are true:*

(1) $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ $\iff$ $(\forall\ x, y \in S^1)$ $ax = ay$ $[xa = ya]$ if and only if $bx = by$ $[xb = yb]$;
(2) $a\mathcal{L}^*e$ $[a\mathcal{R}^*e]$ $\iff$ $ae = a$ $[ea = a]$ and for all $x, y \in S^1, ax = ay[xa = ya]$ implies $ex = ey[xe = ye]$;

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(3) $\mathcal{L}^*$ *and* $\mathcal{R}^*$ *are right and left congruences on* S*, respectively.*

Evidently, in an arbitrary semigroup, we have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. But for regular elements a, b, it is easy to check that $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ if and only if $a\mathcal{L}b$ $[a\mathcal{R}b]$.

Recall from Fountain [17] that a^+ and a^* are the idempotents in $\mathcal{R}^*$-class and $\mathcal{L}^*$-class containing a, respectively. For convenience, we denote by $E(S)$ the set of idempotents of S; $\mathcal{L}^*{}_a$ and $\mathcal{R}^*{}_a$ denote the $\mathcal{L}^*$-class and $\mathcal{R}^*$-class of S containing a, respectively. A semigroup S is called *abundant* if each $\mathcal{L}^*$ class and each $\mathcal{R}^*$ class of S contains at least one idempotent. Let S be a semigroup. A left ideal I of S is said to be a *left* $*$*-ideal* of S if for all $a \in I$, $\mathcal{L}^*{}_a \subseteq I$. A *right* $*$*-ideal* of S can be defined in the dual way. In particular, a subset I of S is called a $*$*-ideal* of S if it is both a left $*$- and right $*$-ideal of S. Usually, for $a \in S$, we simply denote by $\mathcal{L}^*(a)$ and $\mathcal{R}^*(a)$ the smallest left and right $*$-ideal containing a, respectively. If $e \in E(S)$, then Se and eS are the smallest left and right $*$-ideal containing e, respectively. It is easy to check that a semigroup S is abundant if and only if for each $a \in S$, there are $e, f \in E(S)$ such that $\mathcal{L}^*(a) = Se$ and $\mathcal{R}^*(a) = fS$. Recently, many authors have considered the structure of abundant semigroups (see, [19–22]). However, it is not common to study abundant semigroups by fuzzy sets. Therefore, it is an interesting thing to characterize properties of abundant semigroups by fuzzy set theory.

Now, we recall some basic concepts and notations in the fuzzy set theory.

Let A be a subset of a nonempty set X. The *characteristic function* of A is the function C_A of X defined by $C_A(x) = 1$ if $x \in A$ and $C_A(x) = 0$ if $x \notin A$. Let μ and ν be two fuzzy subsets of a semigroup S. Then $\mu \subseteq \nu$ [resp., $\mu \cap \nu$] is defined by $\mu(x) \leq \nu(x)$ [resp., $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$] for all $x \in S$, and the product $\mu \circ \nu$ is defined by the rule $\mu \circ \nu(x) = \vee_{x=yz}\{\mu(y) \wedge \nu(z)\}$ if there are $y, z \in S$ such that $x = yz$; otherwise, $\mu \circ \nu(x) = 0$.

A fuzzy subset μ of a semigroup S is called a *fuzzy left (right) ideal* of S if $\mu(ab) \geq \mu(b)$ ($\mu(ab) \geq \mu(a)$) for all $a, b \in S$. In particular, μ is called a *fuzzy ideal* of S if it is both a fuzzy left ideal and a fuzzy right ideal of S.

Lemma 2.2 ([18]): *Let* μ, ν *be a fuzzy left ideal and a fuzzy right ideal of a semigroup* S*, respectively. Then, for all* $a, b \in S$*, the following statements are true:*

(1) $a\mathcal{L}b \iff \mu(a) = \mu(b)$;
(2) $a\mathcal{R}b \iff \nu(a) = \nu(b)$.

3. Definitions and constructions

In this section, we shall give definitions of a fuzzy left (right) star ideal, fuzzy star ideal of a semigroup and some concrete constructions. In particular, some examples are provided to explain such new fuzzy ideals.

Definition 3.1: Let μ be a fuzzy subset of a semigroup S. Then μ is called a *fuzzy left [resp., right] star ideal* of S if it satisfies:

(i) $(\forall\, a, b \in S)$ $\mu(ab) \geq \mu(b)$ [resp., $\mu(ab) \geq \mu(a)$];
(ii) $(\forall\, a \in S)$ $\mathcal{L}^*{}_a \cap E(S) \neq \varnothing$ [resp., $\mathcal{R}^*{}_a \cap E(S) \neq \varnothing$] $\Rightarrow \mu(a^*) = \mu(a)$ [resp., $\mu(a^+) = \mu(a)$].

Especially, we say that μ is *a fuzzy star ideal* of S if it is both a fuzzy left star ideal and a fuzzy right star ideal of S.

By Definition 3.1, it is easy to see that, in an arbitrary regular semigroup S, a fuzzy left [resp., right] ideal of S is a fuzzy left [resp., right] star ideal of S. Here, we give two non-regular semigroups which satisfy the conditions of Definition 3.1.

Example 3.1: Let $S = \{0.e, h, a\}$. Define a multiplication on S by the rule $e^2 = e, h^2 = h, eh = ea = ah = a, he = ha = ae = a^2 = 0$ and $0x = x0 = 0$ for all $x \in S$.

It is easy to check that S is a semigroup with $E(S) = \{0, e, h\}$. On the other hand, it is routine to check that the $\mathcal{L}^*$ classes of S are $\{0\}, \{e\}, \{h, a\}$, and that the $\mathcal{R}^*$ classes of S are $\{0\}, \{h\}, \{e, a\}$. These imply that $\mathcal{L}^*{}_x \cap E(S) \neq \varnothing$ and $\mathcal{R}^*{}_x \cap E(S) \neq \varnothing$ for all $x \in S$. Therefore, S is abundant. Moreover, for all $x \in S$, $a \neq axa = 0$. This means that S is non-regular. That is, S is a non-regular abundant semigroup. Consider two fuzzy subsets of S as follows:

$$\mu : S \longrightarrow [0, 1], \quad \mu(0) = 1, \quad \mu(e) = 1/2, \quad \mu(h) = \mu(a) = 1/3;$$
$$\nu : S \longrightarrow [0, 1], \quad \nu(0) = 1, \quad \nu(h) = 1/3, \quad \nu(e) = \nu(a) = 1/2.$$

It is evident that μ and ν are a fuzzy left star ideal and a fuzzy right star ideal of S, respectively.

Example 3.2: Let

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{R},\ 1 \geq a \geq 0,\ b \geq 0 \right\},$$

where $\mathbb{R}$ is the set of all real numbers. Then it is easy to see that S is a non-regular abundant semigroup with respect to the general matrix multiplication, and that the set of idempotents of S is

$$E(S) = \left\{ \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\} \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In fact, it is routine to check that the $\mathcal{L}^*$ classes of S are

$$S\backslash E(S)\bigcup\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}0&0\\b_1&1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}0&0\\b_2&1\end{pmatrix}\right\},\ldots$$

and that the $\mathcal{R}^*$ classes of S are

$$S\backslash E(S)\bigcup\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}0&0\\b&1\end{pmatrix}\,\middle|\, b\in\mathbb{R}\right\}.$$

On the other hand, we choose $1 > a > 0$ and $b > 0$, then

$$\begin{pmatrix}a&0\\b&1\end{pmatrix}\neq\begin{pmatrix}a&0\\b&1\end{pmatrix}\begin{pmatrix}s&0\\t&1\end{pmatrix}\begin{pmatrix}a&0\\b&1\end{pmatrix}=\begin{pmatrix}asa&0\\bsa+ta+b&1\end{pmatrix},$$

where $0\leq s\leq 1$ and $0\leq t$. In fact, if $0<a<1$ and $b>0$, then $a\neq asa$ for all $0\leq s\leq 1$. This means that S is not regular. Therefore, S is a non-regular abundant semigroup.

Note that for all $A\in S$, $A^*=A^+$ is not true. However, $|A^*|=|A^+|$ for all $A\in S$, where $|A|$ denotes the determinant of the matrix A. Consider a fuzzy subset ν of S as follows:

$$\nu : S\longrightarrow [0,1],$$
$$\nu(A)=1-|A^+|=\begin{cases}0 & \text{if } |A|\neq 0\\ 1 & \text{if } |A|=0\end{cases}. \qquad (1)$$

It is easy to check that ν is a fuzzy star ideal of S.

Remark 3.1: A fuzzy left (right) star ideal of a semigroup S is a fuzzy left (right) ideal of S, and the converse is not true. In fact, in our Example 3.2, consider a fuzzy subset μ of S as follows:

$$\mu : S\longrightarrow [0,1],\quad \mu(A)=1-|A|,$$

where $|A|$ denotes the determinant of the matrix A.

Obviously, μ is a fuzzy ideal of S. However, μ is not a fuzzy star ideal of S. In fact,

$$\mu\left[\begin{pmatrix}\frac{1}{2}&0\\1&1\end{pmatrix}\right]=1-\frac{1}{2}=\frac{1}{2}>0=\mu\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]=\mu\left[\begin{pmatrix}\frac{1}{2}&0\\1&1\end{pmatrix}^*\right],$$

which implies that $\mu(A^*)=\mu(A)$ is not true for all $A\in S$.

Next, we give some constructions of fuzzy left [right] star ideals of an arbitrary semigroup.

Theorem 3.2: *Let A be a left [right]$*$-ideal of an arbitrary semigroup S. Then C_A is a fuzzy left [right] star ideal of S.*

Proof: Let A be a left $*$-ideal of S. Then it is easy to see that C_A is a fuzzy left ideal of S. Suppose that $a\in S$ and $\mathcal{L}^*{}_a\cap E(S)\neq\emptyset$. Consider the following two cases:

(1) if $a\in A$, then $a^*\in\mathcal{L}^*{}_a\subseteq A$ since A is a left $*$-ideal of S, and so $C_A(a^*)=1=C_A(a)$;
(2) if $a\notin A$, then $C_A(a^*)\geq 0=C_A(a)$. In fact, $a\notin A$ implies $a^*\notin A$ since A is a left $*$-ideal of S. Hence, $C_A(a^*)=0=C_A(a)$.

Thus C_A is a fuzzy left star ideal of S. Dually, C_A is a fuzzy right star ideal of S for any right $*$-ideal A of S. ■

Remark 3.2: Generally, A is a left [resp., right]-ideal of a semigroup S if and only if C_A is a fuzzy left [resp., fuzzy right]-ideal of a semigroup S. In our Theorem 3.2, the converse is not true. However, if S is an abundant semigroup, then the converse holds (see, Theorem 3.7).

Proposition 3.3: *Let S be an arbitrary semigroup and $a\in S$. Define two fuzzy subsets μ and ν of S as follows:*

$$\mu(x)=\begin{cases}t & \text{if } x\in\mathcal{R}^*(a)\\ 0 & \text{otherwise}\end{cases},\quad \text{where } 0<t\leq 1; \qquad (2)$$

$$\nu(x)=\begin{cases}t & \text{if } x\in\mathcal{L}^*(a)\\ 0 & \text{otherwise}\end{cases},\quad \text{where } 0<t\leq 1. \qquad (3)$$

Then μ [resp., ν] is a fuzzy right star ideal [resp., a fuzzy left star ideal] of S.

Proof: Obviously, μ and ν are well-defined. Let $x,y\in S$. If $x\in\mathcal{R}^*(a)$, then $xy\in\mathcal{R}^*(a)$ since $\mathcal{R}^*(a)$ is a right $*$-ideal of S. Hence $\mu(xy)=t=\mu(x)$; If $x\notin\mathcal{R}^*(a)$, then $\mu(xy)\geq 0=\mu(x)$. Therefore, μ is a fuzzy right ideal of S.

Now, we prove that μ is a fuzzy right star ideal of S. Suppose $x\in S$ and $\mathcal{R}^*{}_x\cap E(S)\neq\emptyset$, then for all $x^+\in\mathcal{R}^*{}_x\cap E(S)$, we consider the following two cases:

(1) if $x\notin\mathcal{R}^*(a)$, then $x^+\notin\mathcal{R}^*(a)$ and so $\mu(x^+)=0=\mu(x)$;
(2) if $x\in\mathcal{R}^*(a)$, then $x\mathcal{R}^*a$, and so $x^+\mathcal{R}^*a$. Hence $x^+\in\mathcal{R}^*{}_a\subseteq\mathcal{R}^*(a)$ since $\mathcal{R}^*(a)$ is the smallest right $*$-ideal of S containing a. Therefore, $\mu(x^+)=t=\mu(x)$.

Summing the above arguments, we conclude that μ is a fuzzy right star ideal of S. Dually, ν is a fuzzy left star ideal of S. ■

Proposition 3.4: *Let μ_1 be a fuzzy right star ideal of an arbitrary semigroup S with idempotents and $e\in E(S^1)$ such that $\mu_1(e)>0$. Define a fuzzy subset μ of S as* *follows:*

$$\mu(x)=\begin{cases}\mu_1(x) & \text{if } x\in eS^1\\ 0 & \text{otherwise}\end{cases} \tag{4}$$

Then μ is a fuzzy right star ideal of S.

Proof: Clearly, for all $e\in E(S^1)$, $eS^1=eS\cup\{e\}$, and so μ is well-defined. We first show that μ is a fuzzy right ideal of S. To see it, let $x,y\in S$. If $x\in eS^1$, then $xy\in eS^1$. Hence $\mu(xy)=\mu_1(xy)\geq\mu_1(x)=\mu(x)$; If $x\notin eS^1$, then $\mu(xy)\geq 0=\mu(x)$. This means that μ is a fuzzy right ideal of S.

Next, we prove that μ is a fuzzy right star ideal of S. Suppose $x\in S$ and $\mathcal{R}^*{}_x\cap E(S)\neq\varnothing$. Consider the following two cases:

(1) if $x\notin eS^1$, then $\mu(x^+)=0=\mu(x)$;
(2) if $x\in eS^1$, then $x^+\in\mathcal{R}^*{}_x\subseteq eS^1$ since eS^1 is a right $*$-ideal of S. Hence $\mu(x^+)=\mu_1(x^+)=\mu_1(x)=\mu(x)$ since μ_1 is a fuzzy right star ideal of S.

From the above arguments, we conclude that μ is a fuzzy right star ideal of S. ■

Proposition 3.5: *Let μ_1 be a fuzzy right star ideal of an arbitrary semigroup S with idempotents and $e,h\in E(S^1)$ such that $\mu_1(e)>0,\mu_1(h)>0$. Define a fuzzy subset μ of S as follows:*

$$\mu(x)=\begin{cases}\mu_1(x) & \text{if } x\in eS^1\cup hS^1\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

Then μ is a fuzzy right star ideal of S.

Proof: It is similar to the proof of Proposition 3.4. ■

Proposition 3.6: *Let μ_1,μ_2 be two fuzzy right star ideals of an arbitrary semigroup S with idempotents and $e,h\in E(S^1)$ such that $\mu_1(e)>0,\mu_2(h)>0$ and $eS^1\cap hS^1=\varnothing$. Define a fuzzy subset μ of S as follows:*

$$\mu(x)=\begin{cases}\mu_1(x) & \text{if } x\in eS^1\\ \mu_2(x) & \text{if } x\in hS^1\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

Then μ is a fuzzy right star ideal of S.

Proof: Evidently, for all $e,h\in E(S^1)$, $eS^1=eS\cup\{e\}$ and $hS^1=hS\cup\{h\}$. Thus μ is well-defined. Let $x,y\in S$. If $x\in eS^1$, then $xy\in eS^1$. Hence, $\mu(xy)=\mu_1(xy)\geq\mu_1(x)=\mu(x)$ since μ_1 is a fuzzy right star ideal of S; If $x\in hS^1$, then $xy\in hS^1$. Hence, $\mu(xy)=\mu_2(xy)\geq\mu_2(x)=\mu(x)$ since μ_2 is a fuzzy right star ideal of S; If $x\notin eS$ and $x\notin hS$, then $\mu(xy)\geq 0=\mu(x)$. Thus μ is a fuzzy right ideal of S.

Now, we show that μ is a fuzzy right star ideal of S. To see it, let $x\in S$ and $\mathcal{R}^*{}_x\cap E(S)\neq\varnothing$. Then for all $x^+\in\mathcal{R}^*{}_x\cap E(S)$, we have the following three cases:

(1) if $x\in eS^1$, then $x^+\in\mathcal{R}^*{}_x\subseteq eS^1$ since eS^1 is a right $*$-ideal of S. Hence $\mu(x^+)=\mu_1(x^+)=\mu_1(x)=\mu(x)$ since μ_1 is a fuzzy right star ideal of S;
(2) if $x\in hS^1$, then $x^+\in\mathcal{R}^*{}_x\subseteq hS^1$ since hS^1 is a right $*$-ideal of S. Hence $\mu(x^+)=\mu_2(x^+)=\mu_2(x)=\mu(x)$ since μ_2 is a fuzzy right star ideal of S;
(3) if $x\notin eS^1$ and $x\notin hS^1$, then $\mu(x^+)=0=\mu(x)$.

From the above arguments, it is easy to observe that μ is a fuzzy right star ideal of S. ■

In the remaining of this section, we consider a construction of a fuzzy left [right] star ideal of an abundant semigroup.

Theorem 3.7: *Let A be a subset of an abundant semigroup S. Then A is a left [right] $*$-ideal of S if and only if C_A is a fuzzy left [right] star ideal of S.*

Proof: Obviously, A is a left [right] ideal of S if and only if C_A is a fuzzy left [right] ideal of S. The necessity is clear from Theorem 3.2, so we only show the sufficiency.

Let C_A be a fuzzy left star ideal of S and $a\in A$. Then $\mathcal{L}^*{}_a\subseteq A$. In fact, if $x\in\mathcal{L}^*{}_a$, then there exists $a^*\in\mathcal{L}^*{}_a\cap E(S)$ satisfying $x\mathcal{L}^*a\mathcal{L}^*a^*$ since S is an abundant semigroup. Hence, by Lemma 2.1(2), $C_A(x)=C_A(xa^*)\geq C_A(a^*)=C_A(a)=1$, that is, $C_A(x)=1$, and so $x\in A$. This means that $\mathcal{L}^*{}_a\subseteq A$. Therefore, A is a left $*$-ideal of S. Dually, it is easy to check that A is a right $*$-ideal of S if C_A is a fuzzy right star ideal of S. ■

4. Properties

In this section, we shall investigate properties of fuzzy left [resp., right] star ideals and fuzzy star ideals of abundant semigroups. Especially, we give sufficient and necessary conditions for an arbitrary semigroup to be abundant in view of fuzzy star ideals.

Proposition 4.1: *Let S be an abundant semigroup and $a,b\in S$. Then for all fuzzy left [resp., right] star ideals μ of S, $a\mathcal{L}^*b$ [resp., $a\mathcal{R}^*b$] implies $\mu(a)=\mu(b)$.*

Proof: Suppose that μ is a fuzzy left star ideal of S, and suppose that $a,b\in S$ with $a\mathcal{L}^*b$. Then there are $a^*\in\mathcal{L}^*{}_a\cap E(S)$ and $b^*\in\mathcal{L}^*{}_b\cap E(S)$ such that $a^*\mathcal{L}^*a\mathcal{L}^*b\mathcal{L}^*b^*$ since S is abundant. Hence, by Lemma 2.1(2), $\mu(a)=\mu(ab^*)\geq\mu(b^*)=\mu(b)=\mu(ba^*)\geq\mu(a^*)=\mu(a)$ since μ is a fuzzy left star ideal of S, which implies that $\mu(a)=\mu(b)$. Dually, $a\mathcal{R}^*b$ implies $\mu(a)=\mu(b)$ for all fuzzy right star ideals μ of S. ■

Theorem 4.2: *Let μ and ν be a fuzzy right star ideal and a fuzzy left star ideal of an abundant semigroup S, respectively. Then the following statements are true:*

(1) $\mu \circ \nu = \mu \cap \nu$;
(2) $\mu \circ \mu = \mu$ *and* $\nu \circ \nu = \nu$.

Proof: (1) Obviously, for all $a \in S$, $\mathcal{L}^*{}_a \cap E(S) \neq \varnothing$ and $\mathcal{R}^*{}_a \cap E(S) \neq \varnothing$ since S is an abundant semigroup. Hence

$$\begin{aligned}\mu \circ \nu(a) &= \vee_{a=xy}\{\mu(x) \wedge \nu(y)\}\\ &\geq \{\mu(a^+) \wedge \nu(a)\} \vee \{\mu(a) \wedge \nu(a^*)\}\\ &= \{\mu(a) \wedge \nu(a)\} \vee \{\mu(a) \wedge \nu(a)\}\\ &= \mu(a) \wedge \nu(a) = (\mu \cap \nu)(a),\end{aligned}$$

which implies that $\mu \circ \nu \supseteq \mu \cap \nu$. On the other hand, $\mu \circ \nu \subseteq \mu \circ S \subseteq \mu$ and $\mu \circ \nu \subseteq S \circ \nu \subseteq \nu$, that is, $\mu \circ \nu \subseteq \mu \cap \nu$. Therefore, (1) holds.

(2) By hypothesis, $\mathcal{R}^*{}_a \cap E(S) \neq \varnothing a \in S.\mathcal{R}^*{}_a \cap E(S) \neq \varnothing a \in S$. Hence, by Definition 3.1(ii), $(\mu \circ \mu)(a) = \vee_{a=xy}\{\mu(x) \wedge \mu(y)\} \geq \mu(a^+) \wedge \mu(a) = \mu(a) \wedge \mu(a) = \mu(a)$, which implies $\mu \circ \mu \supseteq \mu$. On the other hand, $\mu \circ \mu \subseteq \mu \circ S \subseteq \mu$ since μ is a fuzzy right star ideal of S. Therefore, $\mu \circ \mu = \mu$. Dually, $\nu \circ \nu = \nu$. ■

As an immediate result of Theorem 4.2, the following theorem holds.

Theorem 4.3: *Every fuzzy star ideal of an abundant semigroup is idempotent.*

Remark 4.1: (1) The condition "fuzzy star ideal" in Theorem 4.3 can not be weakened to "fuzzy ideal". In fact, in our Example 3.2, we have proved that S is an abundant semigroup. Define a fuzzy subset of S as follows:

$$\mu : S \longrightarrow [0,1], \quad \mu(A) = 1 - |A|,$$

where $|A|$ denotes the determinant of the matrix A. It is easily seen that μ is a fuzzy ideal of S. However, for all $A \in S$, if $|A| \neq 0$ and $|A| \neq 1$, then it is easy to check that

$$\begin{aligned}\mu \circ \mu(A) &= \bigvee_{A=BC} \{\mu(B) \wedge \mu(C)\}\\ &= 1 - \sqrt{|A|} \neq 1 - |A| = \mu(A),\end{aligned}$$

that is, $\mu \circ \mu \neq \mu$. Therefore, there exists a fuzzy ideal of an abundant semigroup which is not idempotent.

(2) By Theorem 4.3, it is evident that every fuzzy ideal of a regular semigroup is idempotent since an arbitrary regular semigroup is abundant and its fuzzy ideals are fuzzy star ideals.

Example 4.1: Let

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{R},\ a \geq 0,\ b \geq 0 \right\},$$

where $\mathbb{R}$ is the set of all real numbers. Then it can be easily seen that S is a semigroup with respect to the general matrix multiplication, and that

$$E(S) = \left\{ \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\} \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Moreover, it is routine to check that the $\mathcal{L}^*$ classes of S are

$$S \backslash E(S) \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & 0 \\ b_1 & 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & 0 \\ b_2 & 1 \end{pmatrix} \right\}, \ldots$$

and that the $\mathcal{R}^*$ classes of S are

$$S \backslash E(S) \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}.$$

Obviously, S is abundant. However, for all $0 \neq a \in \mathbb{R}$, $0 \neq b \in \mathbb{R}$, $x, y \in \mathbb{R}$,

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \neq \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

Therefore, S is a non-regular abundant semigroup. Define a fuzzy subset μ of S as follows:

$$\mu : S \longrightarrow [0,1], \quad \mu(A) = \begin{cases} 1 & \text{if } A = E_{2\times 2} \\ 1/3 & \text{otherwise} \end{cases} \tag{7}$$

where

$$E_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that μ is a fuzzy star ideal of S. On the other hand, let $A \in S$. If $A \in E(S)$ and $A \neq E_{2\times 2}$, then

$$\begin{aligned}\mu \circ \mu(A) &= \vee_{A=BC}\{\mu(B) \wedge \mu(C)\}\\ &= \left(\frac{1}{3} \wedge \frac{1}{3}\right) \vee \left(\frac{1}{3} \wedge 1\right) = \frac{1}{3} = \mu(A);\end{aligned}$$

If $A = E_{2\times 2}$, then $\mu \circ \mu(A) = \vee_{A=BC}\{\mu(B) \wedge \mu(C)\} = (\frac{1}{3}) \vee (1) = 1 = \mu(A)$; If $A \in S \setminus E(S)$, then $\mu \circ \mu(A) = \vee_{A=BC}\{\mu(B) \wedge \mu(C)\} = (\frac{1}{3}) \vee (\frac{1}{3}) = \frac{1}{3} = \mu(A)$. Therefore, $\mu \circ \mu = \mu$, as required.

Theorem 4.4: *Let S be an abundant semigroup and $a, b \in S$. Put $A = a^+S$ [resp., Sa^*] and $B = b^+S$ [resp., Sb^*]. Then the following statements are equivalent:*

(1) $a\mathcal{R}^*b$ *[resp.,* $a\mathcal{L}^*b$*];*
(2) $C_A(b) = C_B(a) = 1$;
(3) $C_I(a) = C_I(b)$ *for all right [resp., left]∗-ideals I of S;*
(4) $\mu(a) = \mu(b)$ *for all fuzzy right [resp., left] star ideals μ of S.*

Proof: By Theorem 3.7 and Proposition 4.1, (4) ⇒ (3) ⇒ (2) and (1) ⇒ (4) are evident. We only need to show that (2) ⇒ (1).

Let $a, b \in S$ be such that $C_A(b) = C_B(a) = 1$, where $A = a^+S$, $B = b^+S$. Then $a \in b^+S$ and $b \in a^+S$. Hence, $a^+ \in \mathcal{R}^*{}_a \subseteq b^+S$ and $b^+ \in \mathcal{R}^*{}_b \subseteq a^+S$ since a^+S and b^+S are right $*$-ideals of S. Thus $a^+ = b^+a^+$ and $b^+ = a^+b^+$, and so $a^+\mathcal{R}b^+$. Therefore, $a\mathcal{R}^*a^+\mathcal{R}b^+\mathcal{R}^*b$. That is, $a\mathcal{R}^*b$ since $\mathcal{R} \subseteq \mathcal{R}^*$. Dually, $a\mathcal{L}^*b$. This completes the proof. ■

Next, we shall study the relationship between fuzzy star ideals and abundance for an arbitrary semigroup with idempotents.

Theorem 4.5: *Let S be an arbitrary semigroup. Then S is abundant if and only if for every $a \in S$, there exist $e, h \in E(S)$ such that $C_I(a) = C_I(e)$ and $C_J(a) = C_J(h)$, where I and J are an arbitrary left $*$-ideal and an arbitrary right $*$-ideal of S, respectively.*

Proof: Let S be abundant and $a \in S$. Suppose that I and J are an arbitrary left $*$-ideal and an arbitrary right $*$-ideal of S, respectively. Then there are $e, h \in E(S)$ such that $a\mathcal{L}^*e$ and $a\mathcal{R}^*h$. Hence, by Theorem 4.4, $C_I(a) = C_I(e)$ and $C_J(a) = C_J(h)$.

Conversely, if for all $a \in S$, there are $e, h \in E(S)$ such that $C_I(a) = C_I(e)$ and $C_J(a) = C_J(h)$ for all left $*$-ideals I and all right $*$-ideals J of S. Choose Se to replace I. We have $a \in Se$. Hence $\mathcal{L}^*(a) \subseteq Se$ since $\mathcal{L}^*(a)$ is the smallest left $*$-ideal of S containing a. Choose $\mathcal{L}^*(a)$ to replace I. We have $e \in \mathcal{L}^*(a)$, and so $Se \subseteq \mathcal{L}^*(a)$ since Se is the smallest left $*$-ideal of S containing e. This means that $\mathcal{L}^*(a) = Se$. Dually, $\mathcal{R}^*(a) = hS$. Therefore, S is an abundant semigroup. ■

As an immediate corollary of Proposition 4.1, Theorems 4.4 and 4.5, we have

Corollary 4.6: *Let S be an arbitrary semigroup. Then S is abundant if and only if for every $a \in S$, there exist $e, h \in E(S)$ such that $\mu(a) = \mu(e)$ and $\nu(a) = \nu(h)$, where μ and ν are an arbitrary fuzzy left star ideal and an arbitrary fuzzy right star ideal of S, respectively.*

5. Applications

In this section, we shall consider some concrete applications of Corollary 4.6 and Theorem 4.4. In particular, some examples are given to explain the advantages of the described fuzzy ideals.

Corollary 5.1: *Let S be an arbitrary semigroup. Then S is regular if and only if for every $a \in S$, there exist $e, h \in E(S)$ such that $\mu(a) = \mu(e)$ and $\nu(a) = \nu(h)$, where μ and ν are an arbitrary fuzzy left star ideal and an arbitrary fuzzy right star ideal of S, respectively.*

Proof: The sufficiency is clear from Lemma 2.2, so we only show the necessity. Note that a regular semigroup is abundant, and that a fuzzy ideal of a regular semigroup is a fuzzy star ideal. We have that the necessity is true from Corollary 4.6. ■

Example 5.1: Let

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\},$$

where $\mathbb{R}$ is the set of all real numbers. Then it is easily seen that S is a semigroup with respect to the general matrix multiplication, and that

$$E(S) = \left\{ \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \middle| \; a \in \mathbb{R} \right\} \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Obviously, if $b \neq 0$, then

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix};$$

if $b = 0$, then

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix}.$$

Therefore, S is a regular semigroup. Let μ and ν be a fuzzy left ideal and a fuzzy right ideal of S, respectively. If $A \in E(S)$, then $\mu(A) = \mu(A)$ and $\nu(A) = \nu(A)$; If $A \in S \setminus E(S)$, then there is $B \in S$ such that $A = ABA$ and $AB = BA = E_{2\times 2}$, where

$$E_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $\mu(A) = \mu(ABA) \geq \mu(BA) \geq \mu(A)$ since μ is a fuzzy left ideal. Similarly, $\nu(A) = \nu(ABA) \geq \nu(AB) \geq \nu(A)$ since ν is a fuzzy right ideal. This means that $\mu(A) = \mu(BA) = \mu(E_{2\times 2})$ and $\nu(A) = \nu(AB) = \nu(E_{2\times 2})$. Therefore, S satisfies the conditions of Corollary 5.1.

Note that a regular semigroup S is *inverse* if each $\mathcal{L}$ class and each $\mathcal{R}$ class of S contains a unique idempotent. As an immediate corollary of Corollary 5.1 and Theorem 4.4, we have

Corollary 5.2: *Let S be an arbitrary semigroup. Then S is an inverse semigroup if and only if for every $a \in S$, there exists a unique element $e \in E(S)$ such that $\mu(a) = \mu(e)$ and there exists a unique element $h \in E(S)$ such that $\nu(a) = \nu(h)$, where μ and ν are an arbitrary fuzzy left star ideal and an arbitrary fuzzy right star ideal of S, respectively.*

Example 5.2: Let

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \middle| \; a, b \in \mathbb{R} \text{ and } b \neq 0 \right\},$$

where $\mathbb{R}$ is the set of all real numbers. Then it is easy to see that S is a semigroup with respect to the general matrix multiplication, and that $E(S) = \{E_{2\times 2}\}$, where

$$E_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously,

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix}.$$

However, for all

$$\begin{pmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \in S,$$

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}.$$

Therefore, S is an inverse semigroup. Let μ and ν be a fuzzy left ideal and a fuzzy right ideal of S, respectively. Let $A \in S$. By a similar proof of Example 5.1, there exists only one idempotent $E_{2\times 2}$ of S such that $\mu(A) = \mu(E_{2\times 2})$ and $\nu(A) = \nu(E_{2\times 2})$. Therefore, S satisfies the conditions of Corollary 5.2.

6. Conclusions

Recently, considerable attention has been paid to study of fuzzy semigroups such as fuzzy ordered semigroups, fuzzy regular semigroups, and so on. As a generalized regular semigroup, the class of abundant semigroups encompasses a wide variety of semigroups, from regular semigroups and abundant monoids first arose in connection with the theory of S-systems. It is natural in this paper to consider these semigroups which satisfies abundance. Here we develop an approach to fuzzy semigroups, in the framework of fuzzy star ideals. In particular such as an approach based on the use of $\mathcal{L}^*$ and $\mathcal{R}^*$ is strong and completable. By fuzziness of certain ideal of a semigroup, it is possible to deal with fuzzy semigroup structure and to establish the relationships between fuzzy star ideals and semigroup abundance and semigroup regularity for an arbitrary semigroup. This is also the highlight of this paper. As a concrete application, we advance sufficient and necessary conditions for an arbitrary semigroup to be regular and inverse, respectively. Fuzzy star ideals introduced in this paper could improve applications of the theory of regular semigroups, and might be important for further study of fuzzy semigroup theory.

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References

[1] Zadeh LA. Fuzzy sets. Inform Control. 1965;8:338–353.
[2] Hila K, Abdulah S. A study on intuitionistic fuzzy sets in Γ-semihypergroups. J Intell Fuzzy Syst. 2014;26: 1695–1710.
[3] Karaaslan F. Possibility neutrosophic soft sets and PNS-decision making method. Appl Soft Comput. 2017;54:403–414.
[4] Chen JX, Li JM, Zhang R, et al. Distributed fuzzy consensus of uncertain topology structure multi-agent systems with non-identical partially unknown control directions. Appl Math Comput. 2019;362:Article ID 124581.
[5] Mesiarová-zemánková A. Decomposable and k-additive multi-capacities and multi-polar capacities. Fuzzy Sets Syst. 2016;287(3):22–36.
[6] Onasanya B, Hošková-Mayerová Š. Some topological and algebraic properties of alpha-level subsets' topology of a fuzzy subset. An Stiint Univ Ovidius Constanta. 2018;26(3):213–227.
[7] Onasanya B, Hošková-Mayerová Š. Multi-fuzzy group induced by multisets. Ital J Pure Appl Math. 2019;41: 597–604.
[8] Qiu JB, Sun KK, Wang T, et al. Observer-based fuzzy adaptive event-triggered control for pure-feedback nonlinear systems with prescribed performance. IEEE Trans Fuzzy Syst. 2019;27(11):2152–2162.
[9] Singh PK. m-polar fuzzy graph representation of concept lattice. Eng Appl Art Intell. 2018;67(1):52–62.
[10] Sun KK, Mou SS, Qiu JB, et al. Adaptive fuzzy control for non-triangular structural stochastic switched nonlinear systems with full state constraints. IEEE Trans Fuzzy Syst. 2019;27(8):1587–1601.
[11] Borzooei RA, Rezaei GR, Muhiuddin G, et al. Multipolar fuzzy α-ideals in BCI-algebras. Int J Mach Learn Cyber. 2021;12:2339–2348.
[12] Budimirorić B, Budimirorić V, Šešelia B, et al. Fuzzy identies with application to fuzzy semigroups. Inform Sci. 2014;266(15):148–159.
[13] Hakim A, Khan H, Ahmad I, et al. Fuzzy bipolar soft semiprime ideals in ordered semigroups. Heliyon. 2021;7:Article ID e06618 (13 pages).
[14] Ibrar M, Khan A, Abbas F. Generalized bipolar fuzzy interior ideals in ordered semigroups. Honam Math J. 2019;41(2):285–300.

[15] Khan A, Jun YB, Shabir M. A study of generalized fuzzy ideals in oredered semigroups. Neural Comput Appl. 2012;21:69–78.
[16] Li CH, Xu BG, Huang HW. Bipolar fuzzy abundant semigroups with applications. J Intell Fuzzy Syst. 2020;39(1):167–176.
[17] Fountain JB. Abundant semigroups. Proc London Math Soc. 1982;44:103–129.
[18] Mordeson JN, Malik DS, Kuroki N. Fuzzy semigroups. Berlin Heidelberg New York: Springer-verlag; 2010.
[19] Fountain JB, Gomes G, Gould V. Membership of $A \vee G$ for classes of finite weakly abundant semigroups. Period Math Hungar. 2009;59:9–36.
[20] Li CH, Xu BG, Huang HW. Cayley graphs over Green* relations of abundant semigroups. Graphs Combin. 2019;35:1609–1617.
[21] Li CH, Pei Z, Xu BG. A new characterization of a proper type B semigroup. Open Math. 2020;18:1590–1600.
[22] Sun L, Wang LM. Abundance of the semigroup of all transformations of a set that reflect an equivalence relation. J Algebra Appl. 2014;13(2):Article ID 1350088.