A SECTOR IN-PLANE FINITE ELEMENT FOR ANALYSIS OF MULTI-LAYER CYLINDERS EXPOSED TO PRESSURE AND THERMAL LOADING

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Abstract:

In this paper, analysis of multi-layer cylinders exposed to pressure and thermal loading is considered using a sector element based on the strain approach. Materials in cylinder layers have been assumed to be linearly elastic, homogeneous, and isotropic. The sector in-plane finite element employed is defined in the polar coordinate system with three degrees of freedom at each corner node (the two translations and the in-plane rotation). The results of displacement and stresses obtained in the present paper are compared with the analytical solutions, and a good accuracy shown.

1 Introduction

The strain-based approach is a new approach based on assumed polynomial strain components which are integrated to obtain the exact terms representing all the rigid body modes and the other components of the displacement functions.

Many researchers have used this approach to the development of new finite elements. Djoudi and Bahai [1] have developed a newly strain based finite element for the vibration of cylindrical panels with openings, and excellent results have been obtained. Himeur and Guenfoud [2] have developed a new plate bending triangular finite element in perspective to building shell elements. Rebiai and Belounar [3] have formulated a new membrane finite element for linear and material nonlinear analysis.

The strain-based approach has been used to develop new elements [4-7] for linear analysis of plate bending and dynamic analysis of structures.

In the case of circular boundaries, it is appropriate to develop a sector element having a system of polar coordinates, some sector elements have been developed by Sabir and Salhi [8], Sabir and Sfendji [9] for general plane elasticity in polar coordinates.

Belarbi and Charif [10], Bouzrira et al. [11] have developed new sector membrane elements with three degrees of freedom at each node, including in-plane rotation as an additional degree of freedom for analysis of problems having a circular contour. The performance of the developed elements is tested by applying them to the analysis of cylinders under pressure only.

Bouzriba and Bouzrira [12] have developed a new sector element for analysis of cylinders exposed to internal pressure and logarithmic gradient of temperature, where the analysis of the cylinder under temperature constraints has never been tested before using this approach.

Due to the importance of multi-layered composite cylinder in various technological fields, several investigations have been done to determine the stresses and displacement of multi-layered cylinders exposed to pressure and temperature such as Xiang et al. [13], Shi et al. [14], Zhang et al. [15], and Bahoum et al. [16]. In the present study, the sector element developed by Bouzriba and Bouzrira [12] has been used for

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analysis of multi-layer cylinders exposed to pressure and logarithmic gradient of temperature in order to prove the accuracy and robustness of the strain based approach.

2 Strain-based finite element

The sector in-plane finite element developed by Bouzriba and Bouzrira [12] has three degrees of freedom in each node:

$$u\,,\,v \text{ and } \phi = 1/2\left(-\frac{1}{r}\partial u/\partial\theta + \partial v/\partial r + v/r\right)$$

With u, v and ϕ are the displacement in the r direction, the displacement in the θ direction and the in-plane drilling rotation respectively.

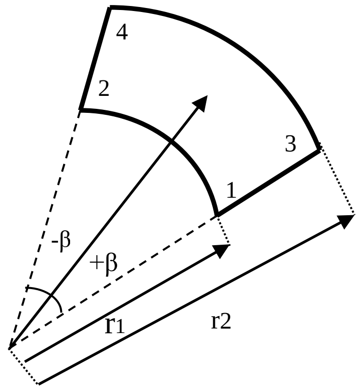

Figure 1. Sector element in polar coordinates.

The displacements u, v and ϕ are represented by the following functions:

$$u = \cos\theta\, a_1 + \sin\theta\, a_2 + r a_4 + (1+r^2)a_6 + (1+\theta r)a_{10} \\ + r^3 a_{11} \tag{1.a}$$

$$v = -\sin\theta\, a_1 + \cos\theta\, a_2 + r a_3 - r\theta a_4 + r\theta a_5 \\ -\theta a_6 - \frac{r^4\theta^2}{6}a_8 + \frac{r\theta^2}{2}a_9 - \left(\frac{r\theta^2}{2} + \theta\right)a_{10} \\ - r\log(r)\left(a_{10} - a_7\right) - 2r^2 a_{12} \tag{1.b}$$

$$\phi = a_3 - \theta a_4 + \theta a_5 - \frac{\theta}{2r}a_6 + \left(\frac{1}{2} + \log(r)\right)a_7 \\ -\frac{5r^3\theta^2}{12}a_8 + \frac{\theta^2}{2}a_9 - \left(1 + \frac{\theta^2}{2} + \frac{\theta}{2r} + \log(r)\right)a_{10} \\ -3r a_{12} \tag{1.c}$$

These can be written as:

$$\{u_i\} = [C]\{A\} \tag{2}$$

Where: $\{A\}$ is the constant parameters vector, and $[C]$ is the transformation matrix as given in the Appendix.

The stiffness matrix [K^e] can be numerically calculated by:

$$\left[K^e\right]=\left[C^{-1}\right]^T\left[\int_{-1}^{+1}\int_{-1}^{+1} r[B]^T[D][B]\det|J|d\zeta\, d\eta\right]\left[C^{-1}\right] \tag{3}$$

Nodal force due to temperature gradient is given as follow:

$$\begin{aligned}\{F\}&=\left[C^{-1}\right]^T\iint[B]^T(1+v)[D\ ]\alpha T\begin{Bmatrix}1\\1\\0\end{Bmatrix}r\,dr\,d\theta\\ &=\left[C^{-1}\right]^T\left[\int_{-1}^{+1}\int_{-1}^{+1} r[B]^T(1+v)[D\ ]\alpha T\begin{Bmatrix}1\\1\\0\end{Bmatrix}\det|J|d\xi d\eta\right]\end{aligned} \tag{4}$$

Where:
T : Temperature gradient.
α : Thermal expansion coefficient.

In the case of plane strain problems the elasticity matrix [D] is given by:

$$[D\ ]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & 0\\ v & 1-v & 0\\ 0 & 0 & \dfrac{1-2v}{2}\end{bmatrix} \tag{5}$$

The stress field can be evaluated by:

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \tau_{r\theta}\end{Bmatrix}=[D]\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \gamma_{r\theta}\end{Bmatrix}-(1+v)[D]\alpha T\begin{Bmatrix}1\\1\\0\end{Bmatrix} \tag{6}$$

Where:

$$\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\\ \gamma_{r\theta}\end{Bmatrix}=[B][C]^{-1}\{U_i\} \tag{7}$$

3 Multi-layer cylinders subjected to internal and external pressure

An exact solution for an n-layered hollow cylinder subjected to uniform pressures on the inner and outer surfaces was developed by Xiang et al. [13].

In Figure 2, a circular hollow cylinder with n-layers subjected to the internal pressures P_i and external pressure P_o has been shown, the Young's modulus and the Poisson's ratio of the layer i are taken by Ei and v_i respectively.

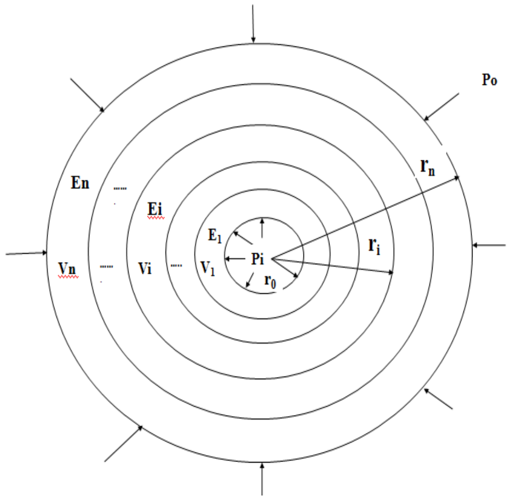

Figure 2. The cross-section of the hollow cylinder with n-layers.

The component of the normal stresses and the displacement in the radial direction of the layer *i* can be written as:

$$\begin{cases} \sigma_{r,i} = \dfrac{A_i}{r^2} + C_i \\ \sigma_{\theta,i} = -\dfrac{A_i}{r^2} + C_i \\ u_i = \alpha_i \dfrac{A_i}{r} + \beta_i C_i r + I_i \cos(\theta) + K_i(\theta) \end{cases} \tag{8}$$

With:

$$\begin{cases} \alpha_i = -\dfrac{1+v_i}{E_i} \\ \beta_i = \dfrac{(1-2v_i)(1+v_i)}{E_i} \, for\ plane\ strain \\ I_i = K_i = 0 \end{cases} \tag{9}$$

The constants (A_i, C_i) are obtained using the following boundary conditions:

$$\begin{gathered} \sigma_{r,i}\big|_{r=r_i} = \sigma_{r,i+1}\big|_{r=r_i} \\ u_i\big|_{r=r_i} = u_{i+1}\big|_{r=r_i} \\ \frac{A_i}{r_{i-1}^2} + C_i = -q_{i-1} \\ \frac{A_{i+1}}{r_{i+1}^2} + C_{i+1} = -q_{i+1} \end{gathered} \tag{10}$$

The relationship between the constants (A_i, C_i) and the extrusion stresses (q_{i-1},q_i) obtained using the boundary conditions is given by the following equations:

$$\begin{cases} \dfrac{A_i}{r_i^2} = \dfrac{\gamma_i}{\gamma_i - 1}(q_{i-1} - q_i) \\ C_i = -\dfrac{1}{\gamma_i - 1}(\gamma_i q_{i-1} - q_i) \end{cases} \tag{11}$$

Where:

$q_{i-1}\ (i = 2.....n.)$ is the extrusion stress between the layer *i-1* and the layer *i*.

According to Xiang et al. [13], the extrusion stress in term of P_i and P_o can be expressed by:

$$q_i = \frac{a_i}{a_n} P_0 + \left(b_i - \frac{b_n}{a_n} a_i \right) P_i \tag{12}$$

The recurrence relations a_i and b_i are given by:

$$\begin{cases} a_{i+1} = \dfrac{(S_i - T_i)\, a_i - (S_i - \gamma_i T_i) a_{i-1}}{(\alpha_{i+1} - \beta_{i+1})(\gamma_i - 1)} \\ b_{i+1} = \dfrac{(S_i - T_i) b_i - (S_i - \gamma_i\, T_i) b_{i-1}}{(\alpha_{i+1} - \beta_{i+1})(\gamma_i - 1)} \end{cases} \tag{13}$$

i=1, 2,……..(n-1)

With initial values: W

$$a_0 = 0, a_1 = 1, b_0 = 1, b_1 = 0 \tag{14}$$

Where:

$$\begin{cases} S_i = (\alpha_i - \beta_{i+1})\gamma_{i+1}\gamma_i + (\alpha_{i+1} - \alpha_i)\gamma_i \\ T_i = (\alpha_{i+1} - \beta_i) + (\beta_i - \beta_{i+1})\gamma_{i+1} \\ \gamma_{i+1} = r_i^2 / r_{i+1}^2 \end{cases} \tag{15}$$

4 Compound cylinders subjected to internal pressure and change of temperature

In this section, the analytical model of a compound cylinder under thermo-mechanical loading proposed by Bahoum et al. [16] is used.

Considering a two-layer compound cylinder subjected to pressure and change of temperature shown in Figure 3, in which, each cylinder C_{yi} (i=1,2) is subjected to the radial distribution function of temperature represented by:

$$T_{cyi} = (T_{in} - T_{ou})\frac{\ln\left(\frac{r_{ou}}{r}\right)}{\ln\left(\frac{r_{ou}}{r_{in}}\right)} + T_{ou} \tag{16}$$

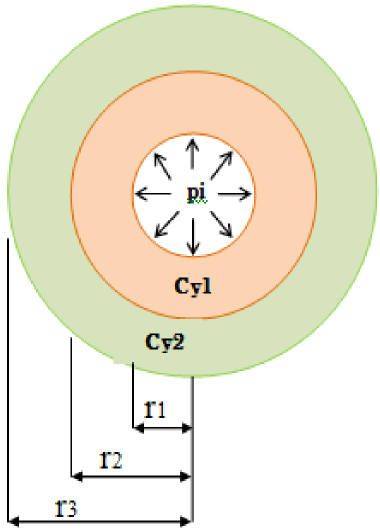

Figure 3. Compound cylinder geometry.

The radial stress σ_r^i , the tangential stress σ_θ^i and the radial displacement u_i of each cylinder C_{yi} are obtained by the following equations:

$$\sigma_r^i(r) = E_i\left[\begin{array}{l} -\frac{\alpha_i}{(1-v_i)\,r^2}\int_{r\,\mathrm{int}}^{r}\rho\,T_{cyi}d\rho + \frac{C_{i1}}{(1+v_i)(1-2v_i)} \\ -\frac{C_{i2}}{(1+v_i)\,r^2} \end{array}\right] \tag{17.a}$$

$$\sigma_\theta^i(r) = E_i\left[\begin{array}{l} \frac{\alpha_i}{(1-v_i)\,r^2}\int_{r\,\mathrm{int}}^{r}\rho\,T_{cyi}d\rho + \frac{C_{i1}}{(1+v_i)(1-2v_i)} \\ +\frac{C_{i2}}{(1+v_i)\,r^2} - \frac{\alpha_i T_{cyl}}{(1-+v_i)} \end{array}\right] \tag{17.b}$$

$$u_i(r) = \left(\frac{1+v_i}{1-v_i}\right)\left(\frac{\alpha_i}{r}\right)\int_{r\,\mathrm{int}}^{r}\rho T_{cyi}d\rho + C_{i1}r + \frac{C_{i2}}{r} \tag{17.c}$$

Where: C_{i1} and C_{i2} are the constants of integration for the cylinder $C_{yi=1,2}$ obtained by applying the following boundary conditions:

$$\sigma_r^1(r_1) = -P_i\,,\ \sigma_r^2(r_3) = 0 \tag{18}$$

$$\sigma_r^1(r_2) = \sigma_r^2(r_2)\,,\ u_1(r_2) = u_2(r_2)$$

5 Examples

Two examples are considered. The first is concerned with a three-layered hollow cylinder submitted to internal pressure P_i=100 MPa, the inner and outer layer are made from steel and the middle layer is concrete. The radius of the inner layer is taken as 0.05 m. Table 1 gives the thickness and material properties of each layer. The distribution of displacement u, and normal stresses (σ_r , σ_θ) are plotted in Figure 4, 5 and 6. These results are compared with the theoretical solution obtained by Xiang *et al.* [13] and a good accuracy is observed.

Table 1. Material properties and thickness of the layers.

Layer	Material	Thickness (mm)	E_i (kN/mm²)	v_i
1	Steel	50	210	0.28
2	Concrete	430	23	0.18
3	Steel	20	210	0.28

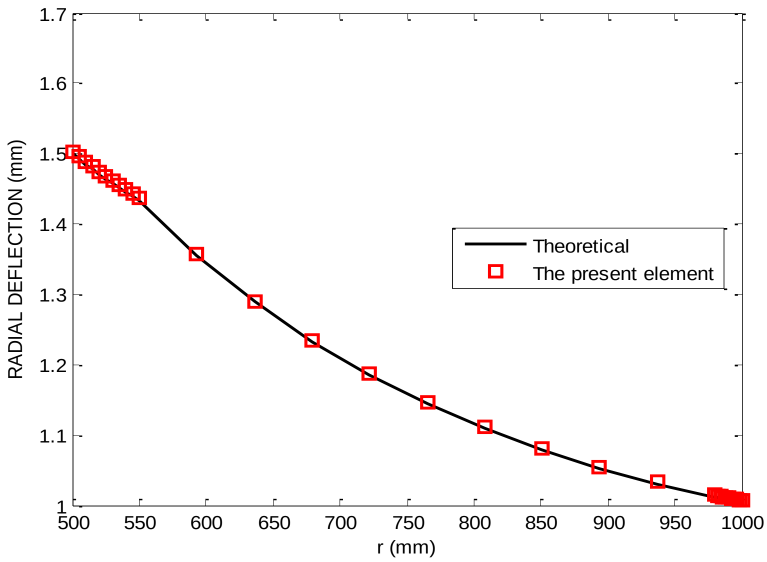

Figure 4. Distribution of displacement u in the three-layered cylinder.

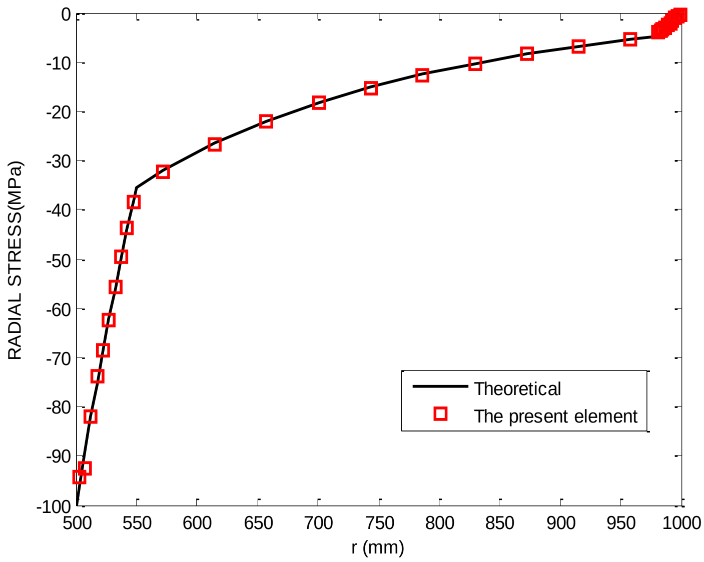

Figure 5. Distribution of radial stress σr in the three-layered cylinder.

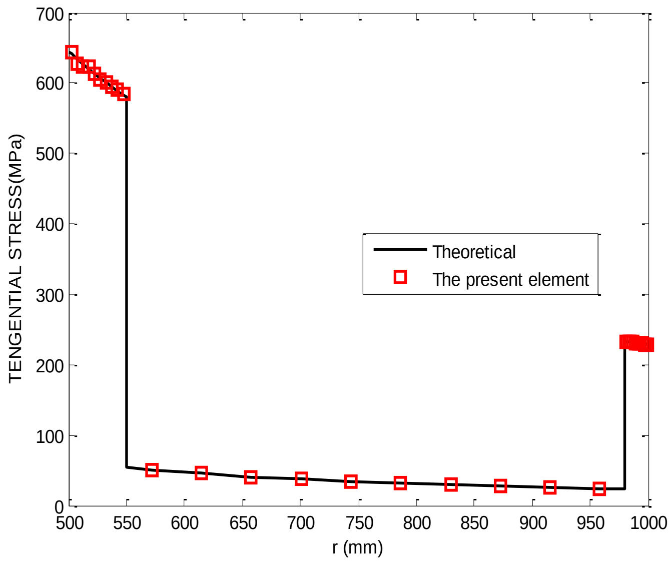

Figure 6. Distribution of tangential stress σ_θ in the three-layered cylinder.

The second model is a two-layer compound cylinder under combined pressure and temperature loading. It is constructed of a steel inner layer and an aluminum outer layer. The inner radius of the inner cylinder is taken as 20 mm.

The thickness and material properties of the layers are given in table 2. The internal pressure was 1 MPa, and the outer pressure 0 MPa. The inner temperature is taken 100 °C, and the outer temperature 25 °C. Due to rotationally symmetric behavior of the problem, only one quarter of the cylinder is considered for the finite element idealization (Figure 7).

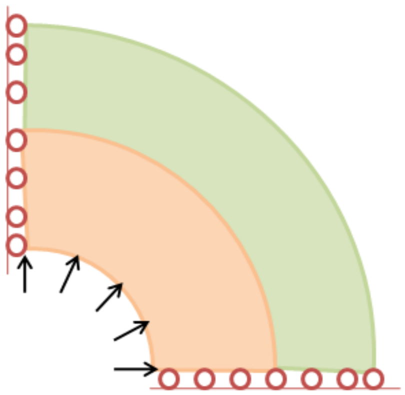

Figure 7. Finite element idealization.

Figure 8 to 11, show temperature distribution, variation of radial deflection, and normal stresses through the compound cylinder thickness compared with theoretical values given by Bahoum *et al.* [16].
From these results, it can be seen that the convergence of the solution to the analytical value is ensured, and this element is very powerful for this type of problem.

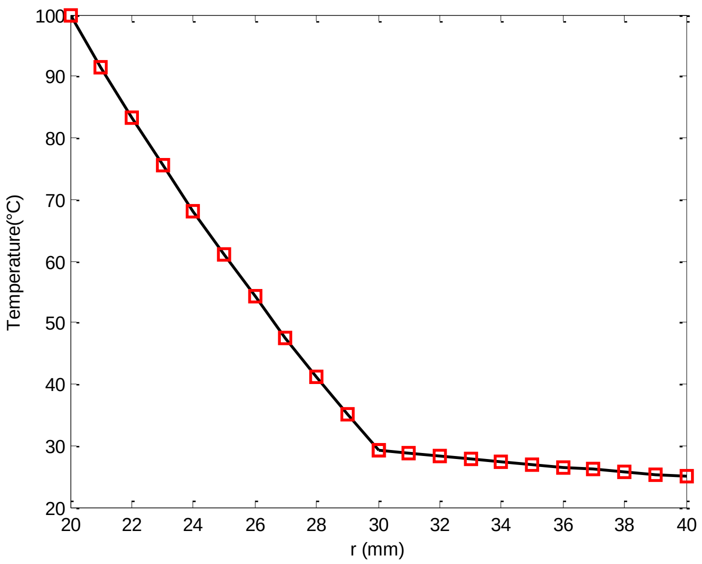

Figure 8. Temperature Distribution Through the Cylinder Thickness.

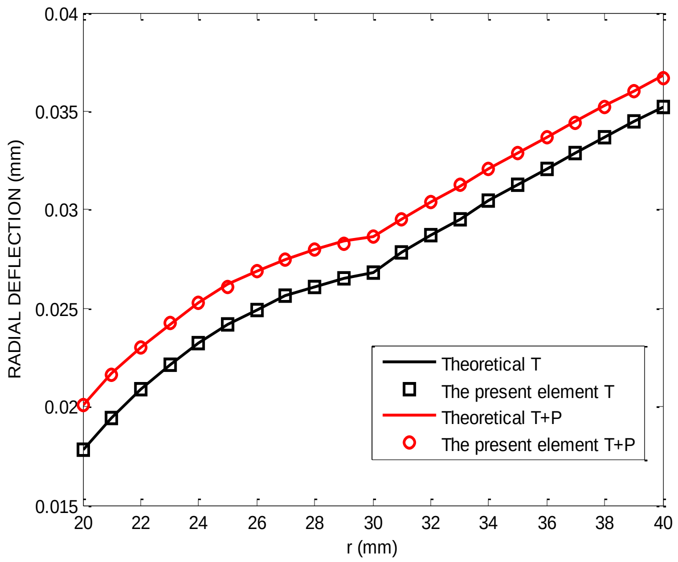

Figure 9. Radial deflection distribution.

Table 2. Material properties and thickness of the layers.

Layer	Material	Thickness (mm)	*E* (kN/mm²)	*ν*	*A* (1/°C)	*λ* (W/mK)
1	Steel ASTM A564 H1150	20-30	210	0.3	11×10^{-6}	19.5
2	Aluminum	30-40	72	0.33	24×10^{-6}	234

Finally, it can be mentioned that the analysis of multi-layer cylinders submitted to thermo- mechanical loading has never been tested before using this approach.

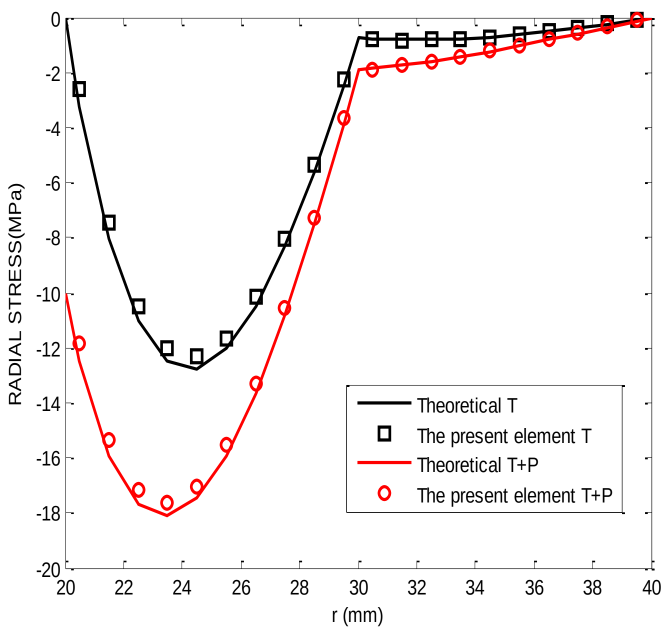

Figure 10. radial stress distribution.

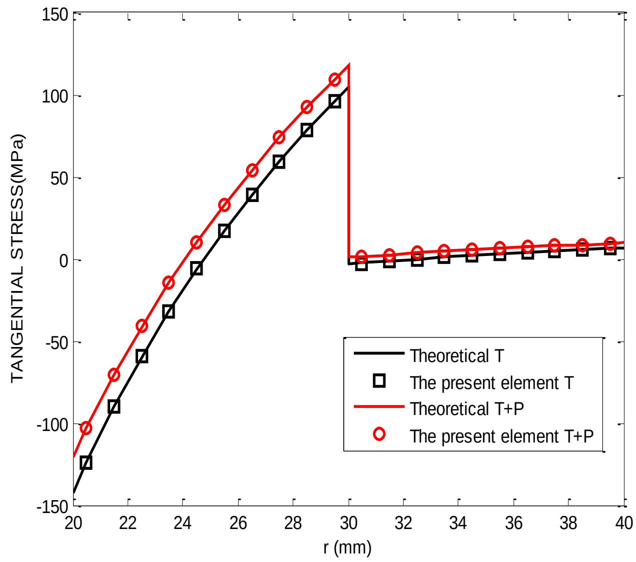

Figure 11. Tangential stress distribution.

5 Conclusion

In this paper, the analysis of multi-layer cylinders under uniform pressure and thermal loading by a sector in-plane finite element was shown. The results of analysis, compared with theoretical values, show that the proposed approach can be efficiently used in the analysis of multi-layered composite cylinder under thermo-mechanical loading with a great economy and it can be incorporated into finite element code.

References

[1] Djoudi, M. S., Bahai, H.: *Strain based finite element for the vibration of cylindrical panels with opening*, Thin-Walled Structures, 42 (2004), 4, 575–588.

[2] Himeur, M., Guenfoud, M.: *Bending triangular finite element with a fictious fourth node based on the strain approach*, European Journal of computational Mechanics, 20 (2011), 7-8, 455 – 485.

[3] Rebiai, C., Belounar, L.: *A new strain based rectangular finite element with drilling rotation for linear and nonlinear analysis*, Archives of Civil and Mechanical Engineering, 13 (2013),1, 72– 81.

[4] Belounar, L., Guerraiche, K.: *A new strain based brick element for plate bending*, Alexandria Engineering Journal, 53 (2014),1, 95-105.

[5] Abderrahmani , S.,Maalem,T., Zatar,A., Hamadi,D.: *A New Strain Based Sector Finite Element for Plate Bending Problems*, In : International Journal of Engineering Research in Africa. Trans TechPublications, 31(2017), 1, 1-13.

[6] Rebiai, C.: *Finite Element Analysis of 2-D Structures by New Strain Based Triangular Element*, Journal of Mechanics, (2018), 1-9.

[7] Belounar, A., Benmebarek, S., Houhou, M. N.,Belounar, L.: *Static, free vibration, and buckling analysis of plates using strain-based Reissner–Mindlin elements*. International Journal of Advanced Structural Engineering, 11(2019), 2, 211-230.

[8] Sabir, A. B., Salhi, H. Y.: *A strain based finite element for general plane elasticity in polar coordinates*, Res. mechanica, 19 (1986), 1, 1-16.

[9] Sabir, A.B., Sfendji, A.: f*inite element for plane elasticity problems*, proceeding of conference femcad crash, Institute for industrial technology transfer, Paris, 1993.

[10] Belarbi, M.T, Charif, A.: *Nouvel élément secteur basé sur le modèle de déformation avec rotation dans le plan,* Revue Européenne des Éléments finis, 7(1998),4,439-458.

[11] Bouzrira, C., Sabir, A. B., Nemouchi, Z.: *A sector in-plane finite element in polar coordinates with rotational degree of freedom*, Archives of civil engineering, 51 (2005),4, 471-483.

[12] Bouzriba, A., Bouzrira, C.: *Sector element for analysis of thick cylinders exposed to internal pressure*

and change of temperature, Građevinar, 67(2015), 6, 547-555.

[13] Xiang, H., Shi, Z., Zhang, T.: *Elastic analyses of heterogeneous hollow cylinders*, Mechanics Research Communications, 33(2006), 5, 681-691.

[14] Shi, Z., Zhang, T., Xiang, H.: *Exact solutions of heterogeneous elastic hollow cylinder*, Composite structures, 79(2007),1, 140-147.

[15] Zhang, Q., Wang, Z. W., Tang, C. Y., Hu, D. P., Liu, P. Q., Xia, L. Z.: *Analytical solution of the thermo-mechanical stresses in a multilayered composite pressure vessel considering the influence of the closed ends*, International Journal of Pressure Vessels and Piping, 98, (2012), 102-110.

[16] Bahoum K., Dinay,M., Mabrouki,M.: *Stress analysis of compound cylinders subjected to thermomechanical loads*, J. Mechanical Science and Technology, 31 (2017), 4, 1805-1811.

Appendix

Components of the matrix [C] of this element are

$$[C]=\begin{bmatrix} \cos\theta & \sin\theta & 0 & r & 0 & 1+r^2 & 0 & 0 & 0 & 1+\theta r & r^3 & 0 \\ -\sin\theta & \cos\theta & r & -r\theta & r\theta & -\theta & r\log(r) & -\frac{r^4\theta^2}{6} & \frac{r\theta^2}{2} & -\frac{r\theta^2}{2}-\theta-r\log(r) & 0 & -2r^2 \\ 0 & 0 & 1 & -\theta & \theta & -\frac{\theta}{2r} & \frac{1}{2}+\log(r) & \frac{-5r^3\theta^2}{12} & \frac{\theta^2}{2} & -\left(1+\frac{\theta^2}{2}+\frac{\theta}{2r}+\log(r)\right) & 0 & -3r \end{bmatrix}$$

Where: r_i and θ_i are the coordinates of node i (i=1, 4), the matrix [C] is given by:

$$[C]=\big[[C_1][C_2][C_3][C_4]\big]^T$$

$$[B]=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 2r & 0 & 0 & 0 & \theta & 3r^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & r & 0 & -\frac{\theta r^3}{3} & \theta & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\theta}{r} & 1 & -\frac{\theta^2 r^3}{2} & 0 & \frac{\theta}{r} & 0 & -2r \end{bmatrix}$$