# (1,*N*)-Arithmetic Labelling of Arbitrary Supersubdivision of disconnected graphs

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## Abstract

A $(p, q)$ -graph $G$ is said to be $(1, N)$ -Arithmetic if there is a function $\varphi$ from the vertex set $V(G)$ to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ . In this paper we prove that the arbitrary supersubdivision of disconnected paths $Pn \cup Pr$ and disconnected path and cycle $Pn \cup Cr$ are $(1, N)$ -Arithmetic Labelling for all positive integers $N > 1$ .

**Keywords:** $(1, N)$ -Arithmetic Labelling, Supersubdivisions, Arbitrary supersubdivision, dis- connected paths, disconnected path and cycle.

## 1 Introduction and Definitions

Let $G$ be a graph with $q$ edges. A graceful labelling of $G$ is an injection from the set of its vertices to the set $\{0, 1, 2, \ldots, q\}$ such that the values of the edges are all integers from 1 to $q$ , the value of an edge being the absolute value of the difference between the integers attributed to its end vertices. Recently G. Sethuraman and P. Selvaraju[8] have introduced a new method of construction called supersubdivision of a graph and showed that arbitrary supersubdivisions of paths are graceful. They conjectured that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [2] disproved this conjecture by proving that every supersubdivision of a y-tree is graceful (recall that a y-tree is obtained from a path by appending an edge to a vertex of a path adjacent to an end point). Sethuraman and Selvaraju [8] proved that every connected graph has some supersubdivision that is graceful.They pose the question as to whether some supersubdivision is valid for disconnected graphs.

V. Ramachandran and C. Sekar [7] worked on these conjectures and got an affirmative answer for their question. V. Ramachandran and C. Sekar[7] proved that arbitrary Supersubdivision of dis- connected graph is graceful.

In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of $m$ vertices the $m$ -vertices part of $K_{2,m}$.

Let G be a graph with n vertices and t edges. A graph H is said to be a supersubdivision of $G$ if $H$ is obtained by replacing every edge $ei$ of $G$ by the complete bipartite graph $K_{2,m}$ for some positive integer $m$ in such a way that the ends of $ei$ are merged with the two vertices part of $K_{2,m}$ after removing the edge $ei$ from $G$. A supersubdivision $H$ of a graph $G$ is said to be an arbitrary supersubdivision of the graph $G$ if every edge of $G$ is replaced by an arbitrary $K_{2,m}$ ($m$ may vary for each edge arbitrarily).

A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph. Joseph A. Gallian [3] surveyed numerous graph labelling methods. B. D. Acharya and S. M. Hegde [1] introduced $(k, d)$ - arithmetic graphs. A $(p, q)$ - graph $G$ is said to be $(k, d)$ - arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression $k, k + d, k + 2d, \dots, k + (q - 1)d$.

V. Ramachandran and C. Sekar [6] introduced a new concept $(1, N)$ - arithmetic Labelling of Graphs. A $(p, q)$ -graph $G$ is said to be $(1, N)$ - arithmetic if there is a function $\varphi : V(G) \to \{0, 1, N, (N + 1), 2N, (2N + 1), \dots, N(q - 1), N(q - 1) + 1\}$. In this situation the induced mapping $\varphi^*$ to the edges is given by $\varphi^*(uv) = \varphi(u) + \varphi(v)$. If the values of $\varphi(u) + \varphi(v)$ are $1, N + 1, 2N + 1, \dots, N(q - 1) + 1$ all distinct, then we call the labelling of vertices as $(1, N)$ - arithmetic labelling. In case if the induced mapping $\varphi^*$ is defined as $\varphi^*(uv) = |\varphi(u) - \varphi(v)|$ and if the resulting edge labels are are distinct and equal to $\{1, N+1, 2N+1, \dots, N(q-1)+1\}$. We call it as one modulo $N$ graceful. V. Ramachandran and C. Sekar [4,5] proved that the arbitrary supersubdivisions of paths, disconnected paths, cycles, stars, arbitrary supersubdivision of disconnected path and cycle is one modulo N graceful for all positive integer N.

**Definition 1.1.** [4] *A graph $G$ with $q$ edges is said to be one modulo $N$ graceful (where $N$ is a positive integer) if there is a function $\varphi$ from the vertex set of $G$ to* $\{0, 1, N, (N + 1), 2N, (2N + 1), \dots, N(q - 1), N(q - 1) + 1\}$ *in such a way that*
$(i)$ *$\varphi$ is $1 - 1$*
$(ii)$ *$\varphi$ induces a bijection $\varphi^*$ from the edge set of $G$ to* $\{1, N+1, 2N+1, \dots, N(q-1)+1\}$ *where* $\varphi^*(uv) = |\varphi(u) - \varphi(v)|$.

**Definition 1.2.** [6] *A $(p,q)$-graph $G$ is said to be $(1, N)$ -Arithmetic if there is a function $\varphi$ from the vertex set $V(G)$ to* $\{0, 1, N, (N + 1), 2N, (2N + 1), \dots, N(q - 1), N(q - 1) + 1\}$ *so that the values obtained as the sums of the labelling assigned to their end vertices, can be arranged in the arithmetic progression* $\{1, N + 1, 2N + 1, \dots, N(q - 1) + 1\}$.

**Definition 1.3.** [1] *A $(p, q)$ - graph $G$ is said to be $(k, d)$ - arithmetic if its vertices can be assigned distinct nonnegative integers so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices, can be arranged in the arithmetic progression* $k, k+d, k+2d, \dots, k+(q-1)d$.

**Definition 1.4.** [3] *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of $m$ vertices the $m$ -vertices part of $K_{2,m}$. Let $G$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a supersubdivision of $G$ if $H$ is obtained by replacing every edge $e_i$ of $G$ by the complete bipartite graph $K_{2,m}$ for some positive integer $m$ in such a way that the ends of ei are merged with the two vertices part of $K_{2,m}$ after removing the edge $e_i$ from $G$. $H$ is denoted by $SS(G)$.*

**Definition 1.5.** *A supersubdivision $H$ of a graph $G$ is said to be an arbitrary supersubdivision of the graph $G$ if every edge of $G$ is replaced by an arbitrary $K_{2,m}$ ($m$ may vary for each edge arbitrarily). $H$ is denoted by $ASS(G)$.*

**Definition 1.6.** *Cycle $C_n$ with $n$ points is a graph given by $(V, E)$ where $V(C_n) = \{v_1; v_2; \dots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$.*

## 2 Main Results

In this paper we prove that arbitrary supersubdivision of disconnected paths $P_n \cup P_r$ and discon- nected path and cycle $P_n \cup C_r$ are $(1, N)$ -Arithmetic Labelling for all positive integers $N > 1$.

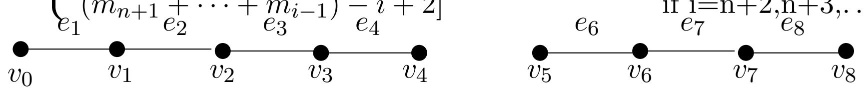
**Theorem 2.1.** *Arbitrary supersubdivision of disconnected paths $P_n \cup P_r$ is $(1, \mathrm{N})$ -Arithmetic* *labelling for every positive integer $N > 1$ provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geq 2$.*

*Proof.* Let $P_n$ be a path with successive vertices $v_0, v_1, v_2, ......v_n-1$ and let $e_i$ $(1 \leq i \leq n-1)$ denote the edge $vi-1vi$ of $P_n$. Let $P_r$ be a path with successive vertices $vn, vn+1, vn+2, ......$ $vn+r-1$ . Let $H$ be an arbitrary supersubdivision of the disconnected graph $Pn \cup Pr$ where each edge $ei$ of $P_n \cup Pr$ is replaced by a complete bipartite graph $K_{2,m_i}$ with $mi > 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ . We observe that $H$ has

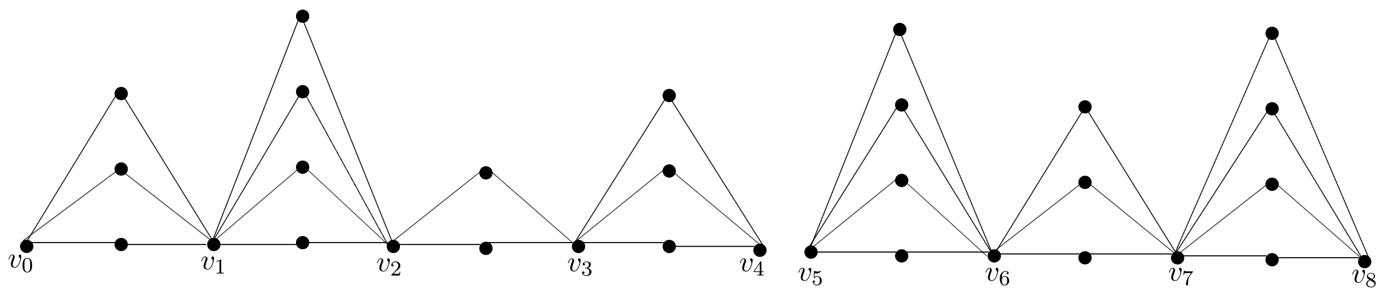
Define

$\varphi(u_i) = N(i-1), i = 1, 2, 3, \ldots, n$
$\varphi(u_i) = N(i), i = n+1, n+2, n+3, \ldots, n+r$
For $k = 1, 2, 3, \ldots, m_i$

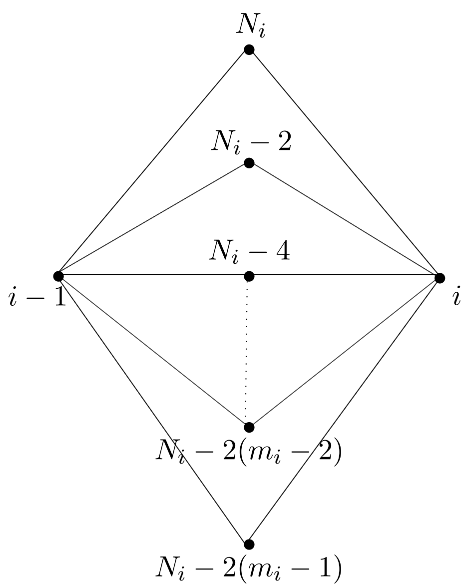
$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} 2N(k-1)+1 & \text{if i=1} \\ N(2k+i-1)+2N(m_1+m_2+\cdots+m_{i-1}-i)+1 & \\ & \text{if i=2,3,... n-1} \\ N(2k+n-5)+1+2N(m_1+m_2+\cdots+m_{n-1}-n+1) & \text{if i=n+1} \\ N(2k+i-6)+1+2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{i-1})-i+2] & \text{if i=n+2,n+3,... n+r-1} \end{cases}$$




Path $P_5$ Path $P_4$




Arbitrary supersubdivision of $P_5 \cup P_4$

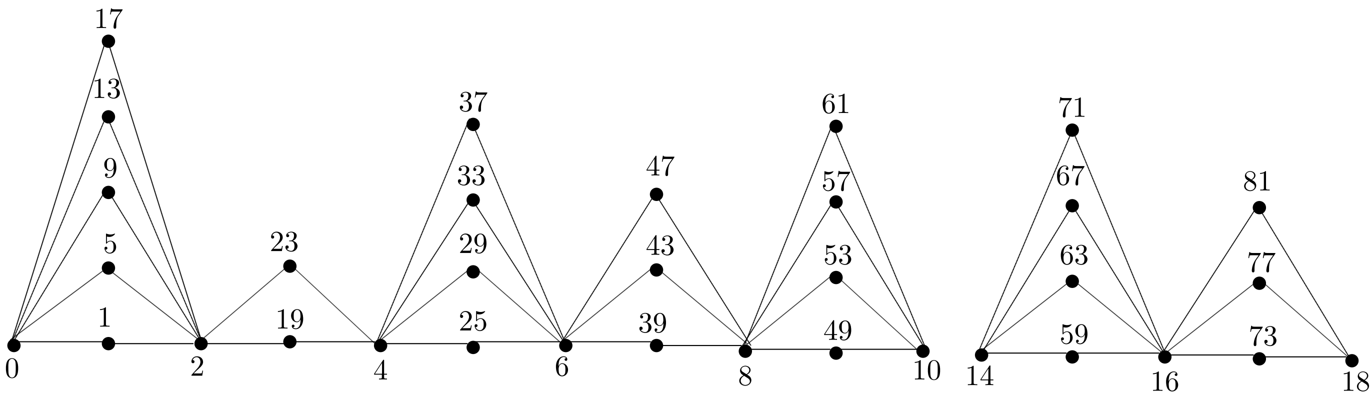



Graceful Labelling of $K_{2,m_i}$ of $H$

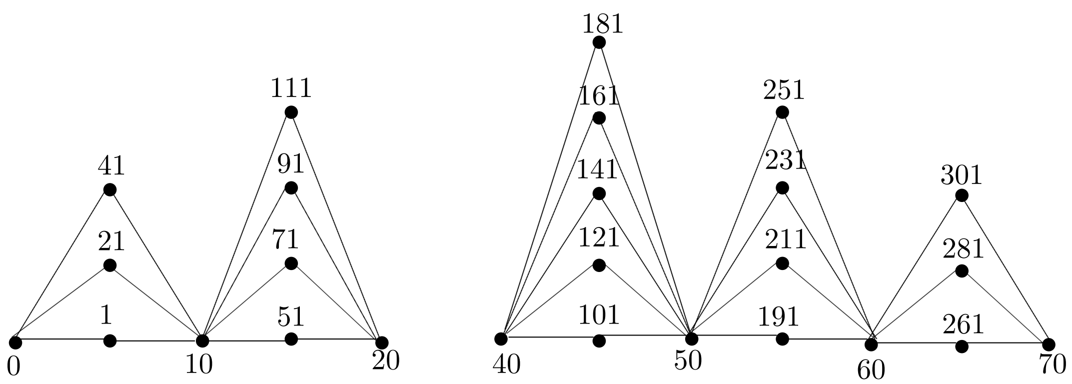
Now for each $i$, $1 \le i \le n-1$, the two vertices of the 2− vertices part of $K_{2,m_i}$ get the labels $i-1$ and $i$ and the $m_i$ vertices of the $m_i$− vertices part of $K_{2,m_i}$. For $n+1 \le i \le n+r-1$, the two vertices of the 2 −vertices part of $K_{2,m_i}$ and the $m_i$ vertices of the $m_i$− vertices part of $K_{2,m_i}$ It is clear from the above labelling that the $m_i+2$ vertices of $K_{2,m_i}$ have distinct labels and the $2m_i$ edges of $K_{2,m_i}$ also have distinct labels for $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$.Therefore the vertices of each $K_{2,m_i}$, $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$ in the arbitrary supersubdivision $H$ of $P_n \cup P_r$ have distinct labels and also the edges of each $K_{2,m_i}$, $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$ in the arbitrary supersubdivision graph $H$ of $P_n \cup P_r$ have distinct labels. Therefore $\varphi$ is $1-1$.

Also the induced mapping $\varphi^*$ to the edges is given by $\varphi^*(uv) = \varphi(u) + \varphi(v)$. If the values of $\varphi(u) + \varphi(v)$ are $1, N+1, 2N+1, \ldots, N(q-1)+1$ all distinct. Hence $H$ is $(1, N)$-Arithmetic labelling for every positive integer $N > 1$. Clearly $\varphi$ defines a $(1, N)$-Arithmetic labelling of arbitrary supersubdivision of disconnected paths $P_n \cup P_r$.

**Example 2.2.** $(1,2)$-*Arithmetic labelling of* $ASS(P_6 \cup P_3)$



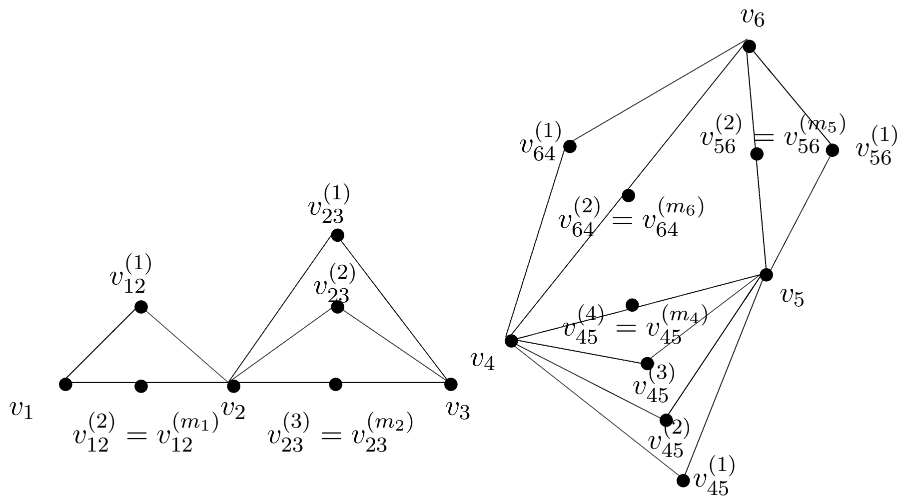

**Example 2.3.** (1, 10) -*Arithmetic labelling of* $ASS(P_3 \cup P_4)$



**Theorem 2.4.** *Arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$ is $(1, N)$ - Arithmetic labelling for every positive integer $N > 1$, provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m > 2$.*

*Proof.* Let $P_n$ be a path with successive vertices $v_1, v_2, ......v_n$ and let $e_i$ $(1 \le i \le n-1)$ denote the edge $v_i v_{i+1}$ of $P_n$. Let $C_r$ be a cycle with successive vertices $v_{n+1}, v_{n+2}, ......v_{n+r}$ and let $e_i$ $(n + 1 \le i \le n + r)$ denote the edge $v_i v_{i+1}$.

Let $H$ be an arbitrary supersubdivision of the disconnected graph $P_n \cup C_r$ where each edge $e_i$ of $P_n \cup$ Cr is replaced by a complete bipartite graph $K_{2,m_i}$ with $m_i > 2$ for $1 \le i \le n-1$ and $n + 1 \le i \le n + r$. Here the edge $v_{n+r}\, v_{n+1}$ is replaced by $k_{2,r-1}$. [5] We observe that $H$ has $M = 2(m_1 + m_2 + ........ + m_{n-1} + m_{n+1} + ....... + m_{n+r})$ edges.




*Arbitrary* Supersubdivision of $P_3 \cup C_3$

Define

$\varphi(v_i) = N(i-1), i = 1, 2, 3, \ldots, n$
$\varphi(v_i) = N(i), i = n+1, n+2, n+3, \ldots, n+r$
For $k = 1, 2, 3, \ldots, m_i$

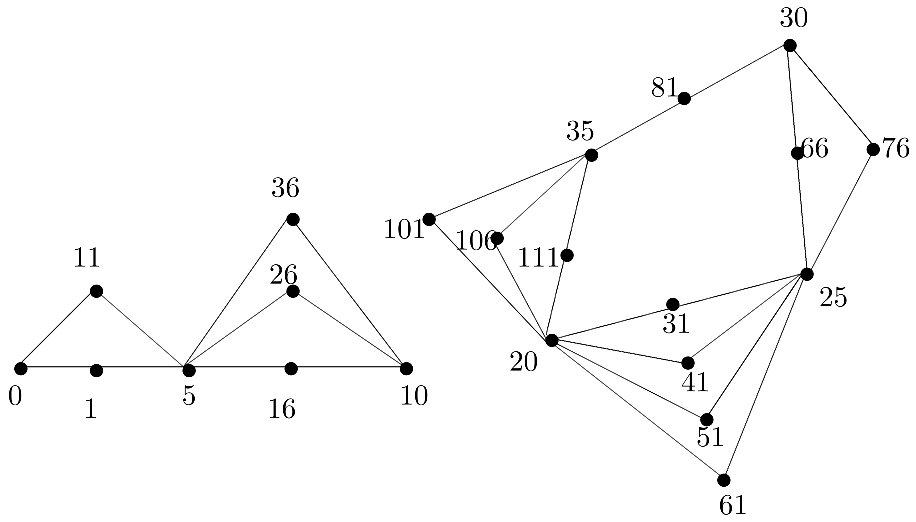
$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} 2N(k-1)+1 & \text{if } i=1 \\ N(2k+i-1)+2N(m_1+m_2+\cdots+m_{i-1}-i)+1 & \\ & \text{if } i=2,3,\ldots n-1 \\ N(2k+n-5)+1+2N(m_1+m_2+\cdots+m_{n-1}-n+1) & \text{if } i=n+1 \\ N(2k+i-6)+1+2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{i-1})-i+2] & \text{if } i=n+2,n+3,\ldots n+r-2 \\ N(k+n+2r-8)+1+2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{n+r-2})-n-r+4] & \text{if } i=n+r-1 \end{cases}$$

Thus it is clear that the vertices have distinct labels. Therefore $\varphi$ is $1 - 1$.

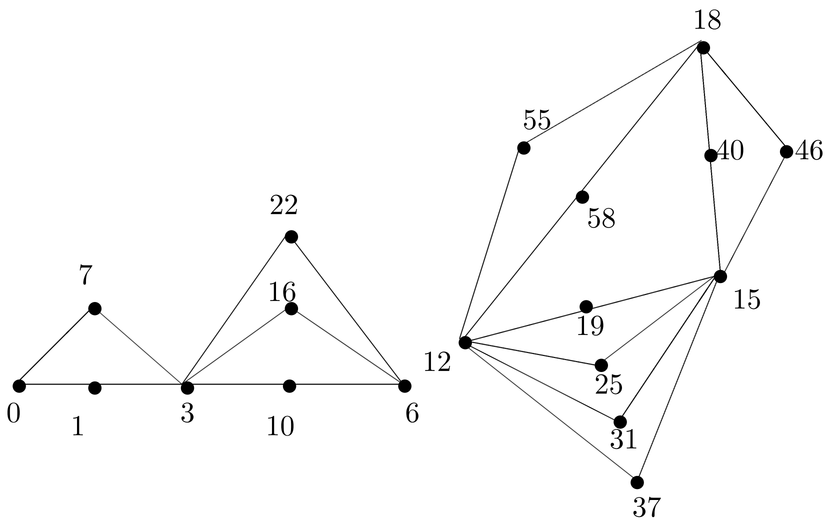
It is clear from the above labelling that the $m_i + 2$ vertices of $K_{2, m_i}$ have distinct labels and the $2_{m_i}$ edges of $\varphi$ also have distinct labels for $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$. Therefore the vertices of each $K_{2, m_i}$, $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$ in the arbitrary supersubdivision $H$ of $P_n \cup C_r$ have distinct labels and also the edges of each $K_{2, m_i}$, $1 \le i \le n-1$ and $n+1 \le i \le n+r-1$ in the arbitrary supersubdivision graph $H$ of $P_n \cup C_r$ have distinct labels.

Also the induced mapping $\varphi^*$ to the edges is given by $\varphi^*(uv) = \varphi(u) + \varphi(v)$. If the values of $\varphi(u) + \varphi(v)$ are $1, N + 1, 2N + 1, \ldots, N(q - 1) + 1$ all distinct. Hence $H$ is $(1, N)$ -Arithmetic labelling for every positive integer $N > 1$. Clearly $\varphi$ defines a $(1, N)$ -Arithmetic labelling of arbitrary supersubdivision of disconnected path and cycle $P_n \cup C_r$. Hence arbitrary supersubdivisions of disconnected path and cycle $P_n \cup C_r$ is $(1, N)$ -Arithmetic labelling, for every positive integer $N > 1$.

**Example 2.5.** (1, 5) -*Arithmetic labelling of ASS*($P_3 \cup C_4$)




**Example 2.6.** (1, 3) -Arithmetic labelling of ASS(P3 $\cup$ C3)



## 3 Conclusion

In this paper, we proved that the arbitrary supersubdivision of disconnected paths $P_n \cup C_r$ and disconnected path and cycle $P_n \cup C_r$ are $(1, N)$ -Arithmetic Labelling for all positive integers $N > 1$. Subdivision or supersubdivision or arbitrary supersubdivision or arbitrary supersubdivision of disconnected of certain graphs which are not graceful may be graceful. The method adopted in making a graph $(1, N)$ -Arithmetic Labelling will provide a new approach to have graceful labelling of graphs and it will be helpful to attack standard conjectures and unsolved open problems.

## References

[1] B. D. Acharya and S. M. Hegde, Arithmetic graphs, *Journal of Graph Theory*, 14 (1990) 275-299.

[2] C. Barrientos, Graceful arbitrary supersubdivisions of graphs *Indian J. Pure Appl. Math.*, 38 (2007), 445–450.

[3] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combi- natorics*, 18 (2011), #DS6.

[4] V. Ramachandran, C. Sekar, One modulo N gracefullness of arbitrary supersubdivisions of graphs, International J. Math. Combin., Vol.2 (2014) 36-46.

[5] V. Ramachandran and C.Sekar, One Modulo N Gracefulness of Some Arbitrary Supersubdivi- sion and Removal Graphs, International J.Math. Combin. Vol.1(2015), 126-135.

[6] V. Ramachandran and C.Sekar, (1, N)-arithmetic graphs, International Journal of Computers and Applications Vol.38(1)(2016), 55-59.

[7] C. Sekar and V. Ramachandran, Graceful labelling of arbitrary Supersubdivision of disconnected graph, Ultra Scientist,Vol. 25(2)A, 315-318 (2013).

[8] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary supersubdivision of graphs, *Indian J. Pure Appl. Math.*, (2001), 1059–1064.