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Abelian theorems in the framework of the distributional index Whittaker transform

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Abstract. Our aim is to derive the Abelian theorems for the index Whittaker transform over distributions of compact support and over certain function spaces of generalized functions.

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1. Introduction and preliminaries

In 1964, Wimp [16] introduced an integral transform of index type called index Whittaker transform defined by

$$F(\tau) = \int_0^\infty W_{\mu,i\tau}(x) f(x) x^{-2} dx, \quad \tau > 0,$$

where i is the imaginary unit, $\mu < \frac{1}{2}$ is a parameter and $W_{\mu,i\tau}$ is the Whittaker function. This transformation first appeared in [16] as a particular case of an integral transform having the Meijer-G function in the kernel. Its L_p theory was well studied in [14]. For $\mu = 0$ it can be reduced to the familiar Kontorovich-Lebedev transform [11], which is one of the most well-known index transforms [17] and has a wide range of applications.

The space $L_p^a(\mathbb{R}_+)$ is defined by [13] as the space of all those real-valued measurable functions f on $\mathbb{R}_+ = (0, \infty)$, such that

$$\|f\|_{L_p^a} = \begin{cases} \left[\int_0^\infty |f(x)|^p m_a(x) dx \right]^{\frac{1}{p}}, & 1 \le p < \infty, \quad 0 \le a < \infty \\ ess. \sup\limits_{x \in \mathbb{R}_+} |f(x)|, & p = \infty \end{cases}$$

where $m_a(x) = x^{-2a-1} e^{-x}$.

The index integral transform with the kernel Whittaker function has been studied by various authors; for instance, Srivastava et al. [14], Al-Musallam [1], Al-Musallam

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and Tuan [2], and Sousa et al.[13, 12]. Further, Becker has extended the theory in mathematical physics [3].

Several definitions of the index Whittaker transform are present in the literature. In this paper, we use the definition of the index Whittaker transform of the function $f \in L_2^a(\mathbb{R}_+)$ defined as [13]:

$$F(\tau) = (\Psi_a f)(\tau) = \int_0^\infty f(x) x^{a+i\tau} \Psi(a+i\tau, 1+2i\tau; x) m_a(x) dx, \quad \tau \geq 0.$$

From [13], we recall integral representations of $x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)$ as:

$$\begin{aligned} x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x) &= \frac{x^{a+\frac{1}{2}}}{2} \int_1^\infty e^{-\frac{x}{2}(t-1)} \left(\frac{t-1}{t+1}\right)^{\frac{a}{2}-\frac{1}{4}} P^{\frac{1}{2}-a}_{-\frac{1}{2}+i\tau}(t) dt \\ &= \frac{1}{\Gamma(a+i\tau)} \int_0^\infty e^{-s} s^{a+i\tau-1} \left(1+\frac{s}{x}\right)^{-a+i\tau} ds, \end{aligned}$$

where $P^{\frac{1}{2}-a}_{-\frac{1}{2}+i\tau}(t)$ is the associated Legendre function of the first kind [10, 14.12.4]. Also from [13], we have

$$|x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)| \leq x^a \Psi(a, 1; x).$$

Asymptotic behaviour near infinity and the origin are as follows [10, 13.14]:

$$\begin{aligned} &x^a \Psi(a, 1; x) \sim 1 \quad \text{as} \quad x \to \infty, \\ &x^a \Psi(a, 1; x) \sim O\left(x^a e^{\frac{x}{2}} \log\left(\frac{1}{x}\right)\right) \qquad \text{as} \quad x \to 0. \end{aligned}$$

In order to be useful, an integral transform needs to be invertible, and for $a > 0$, the inversion of the index Whittaker transform of $\Psi_a \in L_2(\mathbb{R}_+; \rho_a(\tau)d\tau)$ is given by [13]:

$$(\Psi_a^{-1} F)(x) = \int_0^\infty x^{a+i\tau} \Psi(a+i\tau, 1+2i\tau; x) F(\tau) \rho_a(\tau) d\tau,$$

where $\rho_a(\tau) = \pi^{-2}\tau \sinh(2\pi\tau)|\Gamma(a+i\tau)|^2$.
Abelian theorems for the distributional transforms were first studied by Zemanian in [18]. Later, Hayek and González [8] and Gonzlez and Negrin [5] studied these theorems for the ${}_2F_1$- transform and the distributional Kontorovich-Lebedev and Mehler-Fock transforms of general order, respectively. Moreover, Abelian theorems for Laplace and Mehler-Fock transforms of generalized functions are well discussed by González and Negrin in [6]. We note that these results determine the way that, knowing the distribution or generalized function, one can guess the behaviour of the transform of this distribution or generalized function as its domain variable in terms of the behaviour of this transform near the origin and at infinity.
Motivated by the above, in this paper we study Abelian theorems for the index Whittaker transform over the space of distributions of compact support and over

certain function spaces of generalized functions.
For any open set $\Omega \subset \mathbb{R}$, the space $E(\Omega)$ is defined as the vector space of all infinitely differentiable complex-valued functions ψ defined in Ω. This space equipped with the locally convex topology arising from the family of seminorms

$$\Gamma_{n,K} = \sup_{x\in K} |D_x^n \psi(x)|,$$

for all $n \in \mathbb{N}_0$, all compact sets $K \subset \Omega$, and with D_x^n denoting the nth derivative with respect to variable x, becomes a Fréchet space. As usual, we denote the space $E'(\Omega)$ as the dual space of $E(\Omega)$. This $E'(\Omega)$ agrees with the space of distributions on Ω of compact support.

2. Abelian theorems for the distributional index Whittaker transform

The index Whittaker transform of a distribution f of compact support on $\mathbb{R}_+$ is defined by the kernel method in [5] as

$$F(\tau) = \left\langle f(x), x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right\rangle_{m_a(x)}, \quad \tau > 0. \tag{1}$$

In this section, we establish Abelian theorems for the distributional index Whittaker transform. To do this, we demonstrate some previous outcomes. The expression included in the following lemma is obtained in [13].

Lemma 1. *Let $\mathcal{L}_{a,x}$ be the differential operator given by*

$$\mathcal{L}_{a,x} = x^2\frac{d^2}{dx^2} - ((2a-1)x + x^2)\frac{d}{dx}. \tag{2}$$

Then for all $k \in \mathbb{N}_0$, there exist polynomials $P_j^k(x)$ such that

$$\mathcal{L}_{a,x}^k = \sum_{j=1}^{2k} P_j^k(x) D_x^j,$$

where $P_{2k}^k(x) = x^{2k}$, $P_j^k(x)$ are polynomials of degree $k+j$ for $1 \le j \le k$ and of degree $2k$ for $k+1 \le j \le 2k-1$.

Now, to prove the next result we follow analogously to the proof of Zemanian [19, p. 140] and González and Negrín [4, p. 83] in their respective cases for the Kontorovich-Lebedev transform.

Lemma 2. *For each compact set $K \subset \mathbb{R}_+$ and each $k \in \mathbb{N}_0$, let $\Gamma_{k,K}$ be the seminorm on $E(\mathbb{R}_+)$ given by*

$$\Gamma_{k,K} = \sup_{x\in K} |\mathcal{L}_{a,x}^k \psi(x)|, \quad \psi \in E(\mathbb{R}_+),$$

where $\mathcal{L}_{a,x}^k$ is the k^{th} iteration of differential operator $\mathcal{L}_{a,x}$ given by (2). Then $\{\Gamma_{k,K}\}$ generates a topology on $E(\mathbb{R}_+)$ which agrees with the usual topology of this space.

Proof. The expression for $\mathcal{L}_{a,x}^k$ yields that any sequence $\{\varphi_n\} \subset E(\mathbb{R}_+)$ which tends to zero for the usual topology on $E(\mathbb{R}_+)$ also tends to zero for the topology generated by the family of seminorms $\{\Gamma_{k,K}\}$. Conversely, let $\{\varphi_n\}$ be a sequence on $E(\mathbb{R}_+)$ which tends to zero with respect to the topology generated by $\{\Gamma_{k,K}\}$. It is clear that $\{\varphi_n\}$, $\{\mathcal{L}_{a,x}\varphi_n\}$ tends to zero as $n \to \infty$, uniformly in each compact subset $K \subset \mathbb{R}_+$. Moreover, for (2)

$$\mathcal{L}_{a,x}\varphi_n(x) - x\varphi_n(x) = xD_x[x(D_x - 1)\varphi_n(x) - 2a\varphi_n(x)].$$

The left-hand side of the above equation tends to zero as $n \to \infty$, uniformly in each compact subset $K \subset \mathbb{R}_+$, which concludes the right-hand side of the above equation $xD_x[x(D_x - 1)\varphi_n(x) - 2a\varphi_n(x)]$, and thus $D_x[x(D_x - 1)\varphi_n(x) - 2a\varphi_n(x)]$ tends to zero as $n \to \infty$, uniformly in each compact subset $K \subset \mathbb{R}_+$.
Since for any $b \in \mathbb{R}_+$, $b \notin K$

$$\int_b^x D_t[t(D_t - 1)\varphi_n(t) - 2a\varphi_n(t)]dt = -2a[\varphi_n(x) - \varphi_n(b)] + x(D_x - 1)\varphi_n(x) \\ -b(D_x - 1)\varphi_n(b).$$

It follows that $xD_x\varphi_n(x) - bD_x\varphi_n(b) - (2a + x)\varphi_n(x) - (2a - b)\varphi_n(b)$, and thus $xD_x\varphi_n(x) - bD_x\varphi_n(b)$ tends to zero as $n \to \infty$, uniformly in each compact subset $K \subset \mathbb{R}_+$. Dividing by x and integrating once again, we get

$$\int_b^x [D_t\varphi_n(t) - \frac{b}{t}D_b\varphi_n(b)]dt = \varphi_n(x) - \varphi_n(b) - bD_x\varphi_n(b)(\ln x - \ln b),$$

and thus, noting that $\ln t - \ln b$ is bounded away from zero for all $x \in K$, we see that $D_x\varphi_n(b) \to 0$ as $n \to 0$. Consequently, $D_x\varphi_n(x)$, and therefore $D_x^2\varphi_n(x)$ tends to zero as $n \to 0$, uniformly on each compact subset $K \subset \mathbb{R}_+$. Upon repeatedly applying this argument to $\mathcal{L}_{a,x}^2$, $\mathcal{L}_{a,x}^3, \ldots$, we conclude that these terms tend to zero as $n \to 0$, uniformly on each compact subset $K \subset \mathbb{R}_+$. Finally, taking into account that the topologies on $E(\mathbb{R}_+)$ for both families of seminorms, i.e. the usual and the $\Gamma_{k,K}$, are metrizable, the conclusion follows. □

Lemma 3. *Let f be in $E'(\mathbb{R}_+)$, and let F be defined by* (1)*. Then there exist $M > 0$ and a non-negative integer q, all depending on f, such that*

$$|F(\tau)| \leq M \max_{0\leq k\leq q} (\tau^2 + a^2)^k, \quad \forall\tau > 0,\ a \in [0, \infty).$$

Proof. We know that $x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x)$ is an eigenfunction of $\mathcal{L}_{a,x}$ [13] as

$$\mathcal{L}_{a,x}\ x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x) = (-1)(\tau^2 + a^2)\ x^{a+i\tau}\Psi(a + i\tau, 1 + 2i\tau; x).$$

From Lemma 2, we may consider the space $E(\mathbb{R}_+)$ equipped with the topology arising from the family of seminorms $\Gamma_{k,K}$. As per [9, Proposition 2, p. 97],there exist $C>0$, a compact set $K\subset\mathbb{R}_+$, and a non-negative integer q, all depending on f, such that

$$|\left\langle f,\psi\right\rangle|\leq C\max_{0\leq k\leq q}\max_{x\in K}|\mathcal{L}^k_{a,x}\psi(x)|. \tag{3}$$

Now,

$$\begin{aligned}
|F(\tau)| &= |\left\langle f(x),x^{a+i\tau}\Psi(a+i\tau,1+2i\tau;x)\right\rangle_{m_a(x)}| \\
&\leq C\max_{0\leq k\leq q}\max_{x\in K}|\mathcal{L}_{a,x}x^{a+i\tau}\Psi(a+i\tau,1+2i\tau;x)| \\
&= C\max_{0\leq k\leq q}\max_{x\in K}|(\tau^2+a^2)^k x^{a+i\tau}\Psi(a+i\tau,1+2i\tau;x)| \\
&\leq C\max_{0\leq k\leq q}\max_{x\in K}|(\tau^2+a^2)^k x^{a}\Psi(a,1;x)| \\
&\leq M\max_{0\leq k\leq q}(\tau^2+a^2)^k,\quad\forall\tau>0
\end{aligned}$$

for certain $M>0$, since x ranges on the compact set $K\subset\mathbb{R}_+$. □

The smallest integer q, which tests the inequality in (3), is defined as the order of the distribution f [15, Théoréme XXIV, p. 88]. Now, we promote Abelian theorems for the distributional index Whittaker transform (1).

Theorem 1. *(Abelian theorem). Let f be a member of $E'(\mathbb{R}_+)$ of order $p\in\mathbb{N}_0$, and let F be given by (1). Then*
(i) for any $\beta>0$ one has

$$\lim_{\tau\to0^+}\left(\tau^\beta F(\tau)\right)=0,$$

(ii) for any $\beta>0$ one has

$$\lim_{\tau\to+\infty}\left(\tau^{-2p-\beta}F(\tau)\right)=0.$$

Proof. From Lemma 3 we have

$$|F(\tau)|\leq M\max_{0\leq k\leq p}(\tau^2+a^2)^k,\quad\forall\tau>0,\ a\in[0,\infty),$$

for some $M>0$, therefore the proof is complete. □

Now, if f is a locally integrable function on $\mathbb{R}_+$ and f has compact support on $\mathbb{R}_+$, then f gives rise to a regular member T_f of $E'(\mathbb{R}_+)$ of order $p=0$ by means of

$$\langle T_f,\psi\rangle=\int_0^\infty f(x)\psi(x)dx,\quad\forall\psi\in E(\mathbb{R}_+).$$

In fact, taking into account that

$$\begin{aligned}
|\left\langle T_f,\psi\right\rangle| &= \Big|\int_0^\infty f(x)\psi(x)dx\Big|\leq\sup_{x\in supp(f)}|\psi(x)|\int_{supp(f)}|f(x)|dx \\
&= \Gamma_{0,supp(f)}\int_{supp(f)}|f(x)|dx,
\end{aligned}$$

where supp(f) represents the support of the function f, it follows that T_f has the order $p = 0$. So, we have

$$F(\tau) = \left\langle T_f(x), x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right\rangle_{m_a(x)}$$
$$= \int_0^\infty f(x)x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)m_a(x)dx, \quad \tau > 0. \tag{4}$$

By using Theorem 1 for the index Whittaker transform of these regular members of $E'(\mathbb{R}_+)$, we have the following.

Lemma 4. *Let f be a locally integrable function on $\mathbb{R}_+$ and such that f has compact support on $\mathbb{R}_+$. Then the function F given by* (4) *satisfies the following:*
(i) for any $\beta > 0$ one has

$$\lim_{\tau\to 0^+}\left(\tau^\beta F(\tau)\right) = 0,$$

(ii) for any $\beta > 0$ one has

$$\lim_{\tau\to +\infty}\left(\tau^{-\beta} F(\tau)\right) = 0.$$

3. Abelian theorems for the index Whittaker transform of generalized functions

In this section, the space $W_\alpha(\mathbb{R}_+)$, $\alpha > 0$, is defined as the vector space of all infinitely differentiable complex-valued functions ψ defined on $\mathbb{R}_+$ such that

$$\Gamma_{k,\alpha}(\psi) = \sup_{x\in\mathbb{R}_+} |\omega_\alpha(x)m_a(x)\mathcal{L}^k_{a,x}\psi(x)| < \infty, \tag{5}$$

where $k \in \mathbb{N}_0$, $\mathcal{L}_{a,x}$ is the differential operator defined as (2) and $\omega_\alpha(x)$ denotes the continuous function [7]

$$\omega_\alpha(x) = \begin{cases} e^{-\frac{\alpha}{x}} & \text{for } x \in (0,1] \\ e^{-\alpha x} & \text{for } x \in [1,\infty). \end{cases}$$

We see that

$$\Gamma_{k,\alpha}(x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x))$$
$$= \sup_{x\in\mathbb{R}_+}\left|\omega_\alpha(x)m_a(x)(\tau^2+a^2)^k x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right|$$
$$\leq (\tau^2+a^2)^k \sup_{x\in\mathbb{R}_+}\left|\omega_\alpha(x)m_a(x)x^a\Psi(a, 1; x)\right|.$$

Now, invoking the asymptotic behaviours of the kernel, we arrive at

$$\Gamma_{k,\alpha}(x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)) < \infty, \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Therefore, $x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x) \in W_\alpha$. Moreover, W_α is the Fréchet space, generated by the family of seminorms (5). As usual, W'_α is denoted as the dual

space of W_α.
For $f \in W'_\alpha$, $\alpha > 0$, the generalized index Whittaker transform is defined by

$$F(\tau) = \left\langle f(x), x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right\rangle_{m_a(x)}, \quad \tau > 0. \tag{6}$$

From [9], for all $f \in W'_\alpha$, $\alpha > 0$, there exist $C > 0$ and a non-negative integer q, all depending on f, such that

$$|\langle f, \psi\rangle| \leq C \max_{0\leq k\leq q} \Gamma_{k,\alpha}(\psi) = C \max_{0\leq k\leq q} \sup_{x\in\mathbb{R}_+} |\omega_\alpha(x) m_a(x) \mathcal{L}^k_{a,x}\psi(x)| \tag{7}$$

for all $\psi \in W_\alpha$.

Lemma 5. *Let f be in W'_α, $\alpha > 0$, and let F be defined by* (6)*. Then there exist $M > 0$ and a non-negative integer q, all depending on f, such that*

$$|F(\tau)| \leq M \max_{0\leq k\leq q} (\tau^2 + a^2)^k, \quad \forall \tau > 0.$$

Proof. From (7) we have

$$\begin{aligned}
|F(\tau)| &= |\left\langle f(x), x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right\rangle_{m_a(x)}| \\
&\leq C \max_{0\leq k\leq q} \sup_{x\in\mathbb{R}_+} \left|\omega_\alpha(x) m_a(x)(\tau^2+a^2)^k x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right| \\
&\leq C \max_{0\leq k\leq q} \sup_{x\in\mathbb{R}_+} \left|\omega_\alpha(x) m_a(x)(\tau^2+a^2)^k x^{a}\Psi(a, 1; x)\right|.
\end{aligned}$$

Now, invoking the asymptotic behaviour of the kernel and noting that $\alpha > 0$, we have that

$$|F(\tau)| \leq M \max_{0\leq k\leq q} (\tau^2 + a^2)^k, \quad \forall \tau > 0,$$

for certain $M > 0$. □

The smallest integer q, which verifies the inequality in (7), is called the order of the generalized function f. Next we establish Abelian theorems for the index Whittaker transform of generalized functions in W'_α.

Theorem 2 (Abelian theorem)**.** *Let f be a member of W'_α, $\alpha > 0$ of order $p \in \mathbb{N}_0$, and let F be given by* (1)*. Then*
(i) for any $\beta > 0$ one has

$$\lim_{\tau\to 0^+} \left(\tau^\beta F(\tau)\right) = 0,$$

(ii) for any $\beta > 0$ one has

$$\lim_{\tau\to +\infty} \left(\tau^{-2p-\beta} F(\tau)\right) = 0.$$

Proof. From Lemma 5 one has

$$|F(\tau)| \leq M \max_{0\leq k\leq p} (\tau^2 + a^2)^k, \quad \forall \tau > 0,$$

for some $M > 0$, therefore the proof is complete. □

Now, a function f defined on $\mathbb{R}_+$, such that $\omega_\alpha^{-1}(x)m_a^{-1}(x)f(x)$, $\alpha > 0$, is Lebesgue-integrable on $\mathbb{R}_+$, gives rise to a regular generalized function T_f on W'_α of order $p = 0$ through

$$\langle T_f, \psi\rangle = \int_0^\infty f(x)\psi(x)dx, \quad \forall \psi \in W_\alpha.$$

In fact, taking into account that

$$\begin{aligned} |\langle T_f, \psi\rangle| &= |\int_0^\infty f(x)\psi(x)dx| \\ &= |\int_0^\infty \omega_\alpha^{-1}(x)m_a^{-1}(x)\omega_\alpha(x)m_a(x)f(x)\psi(x)dx| \\ &\leq \sup_{x\in\mathbb{R}_+} |\omega_\alpha(x)m_a(x)\psi(x)| \int_0^\infty |\omega_\alpha^{-1}(x)m_a^{-1}(x)f(x)|dx \\ &= \Gamma_{0,\alpha}(\psi) \int_0^\infty |\omega_\alpha^{-1}(x)m_a^{-1}(x)f(x)|dx, \end{aligned}$$

it follows that T_f has the order $p = 0$. So, we have

$$\begin{aligned} F(\tau) &= \left\langle T_f(x), x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)\right\rangle_{m_a(x)} \\ &= \int_0^\infty f(x)x^{a+i\tau}\Psi(a+i\tau, 1+2i\tau; x)m_a(x)dx, \ \tau > 0, \ a \in [0,\infty). \end{aligned} \quad (8)$$

The following lemma directly follows from the above theorem.

Lemma 6. *Let f be the function defined on $\mathbb{R}_+$ and such that $\omega_\alpha^{-1}(x)m_a^{-1}(x)f(x)$ is Lebesgue-integrable on $\mathbb{R}_+$, $\alpha > 0$, and let F be given by* (8). *Then*
(i) for any $\beta > 0$ one has

$$\lim_{\tau\to 0^+} \left(\tau^\beta F(\tau)\right) = 0,$$

(ii) for any $\beta > 0$ one has

$$\lim_{\tau\to +\infty} \left(\tau^{-\beta} F(\tau)\right) = 0.$$

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