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The structure of the algebra $(\mathcal{U}(\mathfrak{g})\otimes C(\mathfrak{p}))^K$ for the groups $\mathrm{SU}(n,1)$ and $\mathrm{SO}_e(n,1)$*

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Abstract. The structure of the algebra of $K-$invariants in $\mathcal{U}(\mathfrak{g})\otimes C(\mathfrak{p})$ is important for constructing $(\mathfrak{g},K)-$modules by means of algebraic Dirac induction as developed in [5] and its variants in [8] and [10]. We show that for the groups $\mathrm{SU}(n,1)$ and $\mathrm{SO}_e(n,1)$ this algebra is a free $\mathcal{U}(\mathfrak{g})^K-$module of rank $\dim C(\mathfrak{p})=2^{\dim\mathfrak{p}}$. We also indicate a way of constructing a $\mathcal{U}(\mathfrak{g})^K-$basis in $(\mathcal{U}(\mathfrak{g})\otimes C(\mathfrak{p}))^K$.

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1. Introduction

Throughout the whole paper, for any group H and any $H-$module V we denote by V^H the subspace of $H-$invariant vectors in V. Furthermore, for a compact group K we denote by $\hat{K}$ its unitary dual. The elements of $\hat{K}$ are called $K-$**types**. The degree (dimension) of a $K-$type δ will be denoted by $d(\delta)$.

Let $\mathfrak{g}_0$ be a real simple Lie algebra of noncompact type. Denote by G its adjoint group and choose its maximal compact subgroup K. Let $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$. Denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The Killing form B of $\mathfrak{g}$ restricts to a nondegenerate $K-$invariant symmetric bilinear form on $\mathfrak{p}\times\mathfrak{p}$. Denote by $C(\mathfrak{p})$ the corresponding Clifford algebra over $\mathfrak{p}$. An important element of the algebra $(\mathcal{U}(\mathfrak{g})\otimes C(\mathfrak{p}))^K$ of $K-$invariants is the so-called **Dirac operator** D defined by

$$D=\sum_i b_i\otimes d_i,$$

where $\{b_i\}$ is a basis of $\mathfrak{p}$ and $\{d_i\}$ is the dual basis with respect to $B|\mathfrak{p}\times\mathfrak{p}$. Dirac operators were first introduced into representation theory in [6] as a tool for constructing discrete series representations. The above algebraic version of the Dirac operator has been investigated in [11]. For a $(\mathfrak{g},K)-$module X, Dirac operator D

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acts on $X \otimes S$, where S is the spin–module over the Clifford algebra $C(\mathfrak{p})$. The Vogan–Dirac cohomology of X is defined in [11] by

$$H_V^D(X) = \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

Unfortunately, the Vogan–Dirac cohomology defined in this way is not a cohomology theory: it is a functor which is neither left nor right exact and admits no adjoints. Two alternative definitions were given in [5]: the Dirac cohomology which is left exact and admits a right adjoint, and the Dirac homology which is right exact and admits a left adjoint. Both functors coincide with the Vogan's definition for unitary and for finite dimensional representations. In [5], certain ways to construct representations with prescribed Dirac (co)homology W are described. It is shown that in this way one obtains all holomorphic (and antiholomorphic) discrete series representations. The constructions are by tensoring (or taking Hom) of $\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$ with W over a subalgebra of $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ containing the Dirac operator D. In [8], it is proved that non(anti)holomorphic discrete series representations of the group $\mathrm{SU}(2,1)$ can also be obtained by choosing a slightly bigger subalgebra of $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, and in [10], the same is done in the case of the group $\mathrm{SO}_e(4,1)$.

These results show the importance of investigating the structure of the algebra $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. In this paper, we prove that in the case of groups $\mathrm{SU}(n,1)$ and $\mathrm{SO}_e(n,1)$ the algebra $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is a free $\mathcal{U}(\mathfrak{g})^K$–module of finite rank $\dim C(\mathfrak{p}) = 2^{\dim \mathfrak{p}}$. In fact, we get more generally that for any finite dimensional K–module V the space of K–invariants $(\mathcal{U}(\mathfrak{g}) \otimes V)^K$ is free $\mathcal{U}(\mathfrak{g})^K$–module of rank $\dim V$. The proof will show how one can explicitly construct a $\mathcal{U}(\mathfrak{g})^K$–basis of $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.

2. K–types in $\mathcal{U}(\mathfrak{g})$

To prove the results on the K–invariants in $\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$ we shall first investigate the K–structure of $\mathcal{U}(\mathfrak{g})$ considered as a $\mathcal{U}(\mathfrak{g})^K$–module. Denote by $\mathcal{U}(\mathfrak{k}) \subseteq \mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{k}$. Furthermore, denote by $S(\mathfrak{g})$ and $S(\mathfrak{k}) \subseteq S(\mathfrak{g})$ the symmetric algebras over $\mathfrak{g}$ and $\mathfrak{k}$, and by $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ the polynomial algebras over $\mathfrak{g}$ and $\mathfrak{k}$. Then $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ can be identified with the symmetric algebras $S(\mathfrak{g}^*)$ and $S(\mathfrak{k}^*)$ over dual spaces $\mathfrak{g}^*$ and $\mathfrak{k}^*$ of $\mathfrak{g}$ and $\mathfrak{k}$. The Killing form B on $\mathfrak{g}$ allows us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{k}$ with $\mathfrak{k}^*$. Thus the algebras $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ are identified with $S(\mathfrak{g})$ and $S(\mathfrak{k})$. Considering polynomials as complex functions on $\mathfrak{g}$ and $\mathfrak{k}$, the inclusion $\mathcal{P}(\mathfrak{k}) \subseteq \mathcal{P}(\mathfrak{g})$ is obtained via the projection $pr : \mathfrak{g} \longrightarrow \mathfrak{k}$ along $\mathfrak{p}$. The group G acts by automorphisms on the algebras $\mathcal{U}(\mathfrak{g})$, $S(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{g})$, and the subgroup K also acts by automorphisms on the algebras $\mathcal{U}(\mathfrak{k})$, $S(\mathfrak{k})$ and $\mathcal{P}(\mathfrak{k})$. The algebra $\mathcal{U}(\mathfrak{g})^G$ is the center $\mathfrak{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$, and $\mathcal{U}(\mathfrak{k})^K$ is the center $\mathfrak{Z}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{k})$. Obviously, the multiplication defines algebra homomorphisms

$$\mathfrak{Z}(\mathfrak{g}) \otimes \mathfrak{Z}(\mathfrak{k}) \longrightarrow \mathcal{U}(\mathfrak{g})^K, \quad S(\mathfrak{g})^G \otimes S(\mathfrak{k})^K \longrightarrow S(\mathfrak{g})^K, \quad \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K \longrightarrow \mathcal{P}(\mathfrak{g})^K.$$

In [2], Knop proved the following highly nontrivial results:

Theorem 1.

(*a*) $\mathfrak{Z}(\mathfrak{g})\otimes\mathfrak{Z}(\mathfrak{k})\longrightarrow\mathcal{U}(\mathfrak{g})^K$ *is an isomorphism onto the center of the algebra* $\mathcal{U}(\mathfrak{g})^K$.

(*b*) *The algebra* $\mathcal{U}(\mathfrak{g})^K$ *is commutative* (*i.e.* $\mathcal{U}(\mathfrak{g})^K=\mathfrak{Z}(\mathfrak{g})\mathfrak{Z}(\mathfrak{k})$) *if and only if* $\mathfrak{g}_0$ *is either* $\mathfrak{su}(n,1)$ *or* $\mathfrak{so}(n,1)$. *In these cases,* $\mathcal{U}(\mathfrak{g})$ *is free as a* $\mathcal{U}(\mathfrak{g})^K-$*module.*

The symmetrization $S(\mathfrak{g})\simeq\mathcal{P}(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})$ is an isomorphism of vector spaces and of $G-$modules and (*a*) implies that the homomorphism

$$\mathcal{P}(\mathfrak{g})^G\otimes\mathcal{P}(\mathfrak{k})^K\longrightarrow\mathcal{P}(\mathfrak{g})^K$$

is always injective and by (*b*), in the cases $\mathfrak{g}_0=\mathfrak{su}(n,1)$ and $\mathfrak{g}_0=\mathfrak{so}(n,1)$, this is an isomorphism; furthermore, the last sentence in (*b*) implies that in these two cases $\mathcal{P}(\mathfrak{g})$ is free as a $\mathcal{P}(\mathfrak{g})^K-$module.

Let $\partial:S(\mathfrak{g})\longrightarrow\mathcal{D}(\mathfrak{g})$ be the usual isomorphism of the symmetric algebra $S(\mathfrak{g})$ onto the algebra $\mathcal{D}(\mathfrak{g})$ of linear differential operators on $\mathcal{P}(\mathfrak{g})$ with constant coefficients: for any $x\in\mathfrak{g}$ $\partial(x)$ is the derivation in the direction x. Let $S_+(\mathfrak{g})^K$ and $\mathcal{P}_+(\mathfrak{g})^K$ denote the maximal ideals (of codimension 1) of the algebras of $K-$invariants $S(\mathfrak{g})^K$ and $\mathcal{P}(\mathfrak{g})^K$ given by

$$S_+(\mathfrak{g})^K=\bigoplus_{k>0}S^k(\mathfrak{g})^K,\quad \mathcal{P}_+(\mathfrak{g})^K=\bigoplus_{k>0}\mathcal{P}^k(\mathfrak{g})^K=\{f\in\mathcal{P}(\mathfrak{g})^K;\ f(0)=0\}.$$

Let us define the (graded) space of $K-$**harmonic polynomials** on $\mathfrak{g}$ as follows:

$$\mathcal{H}_K(\mathfrak{g})=\{f\in\mathcal{P}(\mathfrak{g});\ \partial(u)f=0\ \forall u\in S_+(\mathfrak{g})^K\}.$$

Now, Proposition 1 in [3], the last sentence in (*b*) in Theorem 1, and the obvious analogues of Propositions 3 and 4 in [3] imply immediately:

Theorem 2. *For* $\mathfrak{g}_0=\mathfrak{su}(n,1)$ *and for* $\mathfrak{g}_0=\mathfrak{so}(n,1)$ *we have:*

(*a*) $\mathcal{P}(\mathfrak{g})=\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K\oplus\mathcal{H}_K(\mathfrak{g})$.

(*b*) *The multiplication defines an isomorphism* $\mathcal{P}(\mathfrak{g})^K\otimes\mathcal{H}_K(\mathfrak{g})\simeq\mathcal{P}(\mathfrak{g})$.

Let $\mathcal{N}$ be the zero set in $\mathfrak{g}$ of the ideal $\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K$ generated by $\mathcal{P}_+(\mathfrak{g})^K$ in $\mathcal{P}(\mathfrak{g})$:

$$\mathcal{N}=\{x\in\mathfrak{g};\ f(x)=0\ \forall f\in\mathcal{P}_+(\mathfrak{g})^K\}.$$

By Proposition 16 in [3] the zero set

$$\mathcal{N}_G=\{x\in\mathfrak{g};\ f(x)=0\ \forall f\in\mathcal{P}_+(\mathfrak{g})^G\}$$

is exactly the set of all nilpotent elements in $\mathfrak{g}$. Analogously,

$$\mathcal{N}_K=\{x\in\mathfrak{k};\ f(x)=0\ \forall f\in\mathcal{P}_+(\mathfrak{k})^K\}$$

is the set of all nilpotent elements in the reductive Lie algebra $\mathfrak{k}$. Now, $\mathcal{P}(\mathfrak{g})^K=\mathcal{P}(\mathfrak{g})^G\otimes\mathcal{P}(\mathfrak{k})^K$ by the Knop's theorem, so we get

Proposition 1. *$\mathcal{N}$ is the set of all nilpotent elements in $\mathfrak{g}$ whose projection to $\mathfrak{k}$ along $\mathfrak{p}$ is nilpotent in the reductive Lie algebra $\mathfrak{k}$:*

$$\mathcal{N} = \{x \in \mathfrak{g};\ x \in \mathcal{N}_G,\ pr\,x \in \mathcal{N}_K\}.$$

The elements of $\mathcal{N}$ will be called $K-$**nilpotent elements** of $\mathfrak{g}$.

By the Harish−Chandra isomorphism and by Chevalley's theorem on Weyl group invariants we know that the algebra $\mathcal{P}(\mathfrak{g})^G$ is generated by $\ell = \operatorname{rank}\mathfrak{g}$ homogeneous algebraically independent $G-$invariant polynomials $f_1, \dots, f_\ell$ and the algebra $\mathcal{P}(\mathfrak{k})^K$ is generated by $k = \operatorname{rank}\mathfrak{k}$ homogeneous algebraically independent $K-$invariant polynomials $\varphi_1, \dots, \varphi_k$. Since

$$\mathcal{P}(\mathfrak{g})^K = \mathcal{P}(\mathfrak{g})^G \mathcal{P}(\mathfrak{k})^K \simeq \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K,$$

we conclude that the algebra $\mathcal{P}(\mathfrak{g})^K$ is generated by $\ell + k$ homogeneous algebraically independent polynomials $f_1, \dots, f_\ell, \varphi_1, \dots, \varphi_k$. Thus

$$\mathcal{N} = \{x \in \mathfrak{g};\ f_1(x) = \cdots = f_\ell(x) = \varphi_1(x) = \cdots = \varphi_k(x) = 0\},$$

so $\mathcal{N}$ is a Zariski closed subset of $\mathfrak{g}$ of dimension

$$\dim \mathcal{N} = \dim \mathfrak{g} - \ell - k.$$

More generally, for any $(\xi, \eta) = (\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_k) \in \mathbb{C}^{\ell+k}$, we define a $K^{\mathbb{C}}-$stable Zariski closed set ($K^{\mathbb{C}}$ being the complexification of the group K) :

$$\mathcal{N}(\xi, \eta) = \{x \in \mathfrak{g};\ f_j(x) = \xi_j,\ j = 1, \dots, \ell,\ \varphi_i(x) = \eta_i,\ i = 1, \dots, k\}.$$

Obviously,

$$\dim \mathcal{N}(\xi, \eta) = \dim \mathfrak{g} - \ell - k \qquad \forall (\xi, \eta) \in \mathbb{C}^{\ell+k}.$$

As in [3] and [4], we conclude from Theorem 2(a):

Proposition 2. *For every $(\xi, \eta) \in \mathbb{C}^{\ell+k}$ the restriction of polynomials on $\mathfrak{g}$ to the set $\mathcal{N}(\xi, \eta)$ induces an isomorphism of $K-$modules*

$$\mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{P}(\mathcal{N}(\xi, \eta)) = \mathcal{R}(\mathcal{N}(\xi, \eta)).$$

Here for any subset $S \subseteq \mathfrak{g}$ we denote $\mathcal{P}(S) = \{f|S;\ f \in \mathcal{P}(\mathfrak{g})\}$, and $\mathcal{R}(T)$ denotes the algebra of regular functions on any algebraic variety T.

The dimensions and the ranks in our cases are the following:

$\mathfrak{g}_0$	$\dim \mathfrak{g}$	$\dim \mathfrak{k}$	$\operatorname{rank}\mathfrak{g}$	$\operatorname{rank}\mathfrak{k}$
$\mathfrak{su}(n,1)$	$n^2 + 2n$	n^2	n	n
$\mathfrak{so}(2n,1)$	$2n^2 + n$	$2n^2 - n$	n	n
$\mathfrak{so}(2n+1,1)$	$2n^2 + 3n + 1$	$2n^2 + n$	$n + 1$	n

So we see that in each case we have the equality of dimensions:

$$\dim \mathcal{N}(\xi, \eta) = \dim \mathfrak{k} = \dim K^{\mathbb{C}}. \tag{1}$$

Consider the action of the complex group $K^{\mathbb{C}}$ on $\mathfrak{g}$. For $x \in \mathfrak{g}$ denote by $\mathcal{O}_x$ its $K^{\mathbb{C}}-$orbit. Then, of course,

$$\dim \mathcal{O}_x = \dim K^{\mathbb{C}}/K_x^{\mathbb{C}} = \dim K^{\mathbb{C}} - \dim K_x^{\mathbb{C}}, \tag{2}$$

where $K_x^{\mathbb{C}}$ denotes the stabilizer of the point x in the group $K^{\mathbb{C}}$. So, if $K_x^{\mathbb{C}}$ is trivial, from (1) and (2) we get

$$\dim \mathcal{O}_x = \dim K^{\mathbb{C}} = \dim \mathcal{N}(\xi,\eta). \tag{3}$$

Lemma 1. *There exists $x \in \mathfrak{g}_0$ such that the stabilizer $K_x^{\mathbb{C}}$ is trivial. In this case, let $(\xi,\eta) = (f_1(x),\dots,f_\ell(x),\varphi_1(x),\dots,\varphi_k(x))$. The orbit $\mathcal{O}_x$ is open in $\mathcal{N}(\xi,\eta)$.*

Proof. By induction on n one directly verifies that the stabilizer is trivial, e.g. for the 3−diagonal matrix $x \in \mathfrak{g}_0$ with zeroes on diagonal, the upper parallel $(1,\dots,1,1)$ and the lower parallel $(-1,\dots,-1,1)$. □

Now, we can prove our main result referring to the structure of the $K-$module $\mathcal{H}_K(\mathfrak{g})$:

Theorem 3. *The $K-$module $\mathcal{H}_K(\mathfrak{g})$ of $K-$harmonic polynomials on $\mathfrak{g}$ is equivalent to the regular representation of K. In other words, the multiplicity of every $K-$type $\delta \in \hat{K}$ in the $K-$module $\mathcal{H}_K(\mathfrak{g})$ is equal to its degree $d(\delta)$.*

Proof. Let $x \in \mathfrak{g}_0$ be as in Lemma 1, i.e. such that its stabilizer in $K^{\mathbb{C}}$ is trivial. Set

$$(\xi,\eta) = (f_1(x),\dots,f_\ell(x),\varphi_1(x),\dots,\varphi_k(x)) \in \mathbb{C}^{\ell+k}.$$

The $K^{\mathbb{C}}-$orbit $\mathcal{O}_x$ is contained in $\mathcal{N}(\xi,\eta)$, and by (3) it is open in $\mathcal{N}(\xi,\eta)$. Thus, the restriction to $\mathcal{O}_x$ is an isomorphism of $\mathcal{P}(\mathcal{N}(\xi,\eta)) = \mathcal{R}(\mathcal{N}(\xi,\eta))$ onto $\mathcal{P}(\mathcal{O}_x)$. So we get the isomorphism $\mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{P}(\mathcal{O}_x)$ as $K-$modules. Now, $\mathcal{P}(\mathcal{O}_x) \subseteq \mathcal{R}(\mathcal{O}_x) \simeq \mathcal{R}(K^{\mathbb{C}})$. Using the Frobenius reciprocity we find that the multiplicity of any $K-$type $\delta \in \hat{K}$ in $\mathcal{R}(\mathcal{O}_x)$ is equal to its degree $d(\delta)$. Since we do not know a priori that $\mathcal{P}(\mathcal{O}_x) = \mathcal{R}(\mathcal{O}_x)$, we get only the inclusion of $K-$modules $\mathcal{H}_K(\mathfrak{g}) \hookrightarrow \mathcal{R}(K^{\mathbb{C}})$, and so if $m(\delta)$ denotes the multiplicity of $K-$type δ in $\mathcal{H}_K(\mathfrak{g})$, one has the inequalities:

$$m(\delta) \le d(\delta), \qquad \delta \in \hat{K}. \tag{4}$$

To prove the equalities we use the compact form K of the complex group $K^{\mathbb{C}}$. Denote by $\mathcal{P}(Kx)$ the restriction of the polynomial algebra $\mathcal{P}(\mathfrak{g})$ to the $K-$orbit Kx. Note that the fact that $K^{\mathbb{C}}$ is the complexification of K easily implies that the restriction $\mathcal{O}_x \longrightarrow Kx$ induces an isomorphism of $K-$modules $\mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{P}(\mathcal{O}_x)$ onto $\mathcal{P}(Kx)$. Thus, as a $K-$module we have

$$\mathcal{P}(Kx) = \bigoplus_{\delta\in\hat{K}} m(\delta)\delta. \tag{5}$$

The subalgebra $\mathcal{P}(Kx)$ of the algebra $C(Kx)$ of all complex continuous functions on the compact space Kx evidently distinguishes the points of Kx. Furthermore, this subalgebra is closed under complex conjugation. This follows from the fact that Kx is contained in the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Finally, the algebra $\mathcal{P}(Kx)$ obviously contains constants. Thus, by the Stone−Weierstrass theorem, the subalgebra $\mathcal{P}(Kx)$ is uniformly dense in $C(Kx)$. The Peter−Weyl theorem implies that in (4) we have the equalities $m(\delta) = d(\delta)$ for all $\delta \in \hat{K}$. This proves Theorem 3. □

The symmetrization $\mathcal{P}(\mathfrak{g}) \simeq S(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$ is a $K-$module isomorphism. Let H_K be the image of $\mathcal{H}_K(\mathfrak{g})$ in $\mathcal{U}(\mathfrak{g})$. The immediate consequence of Theorems 2 and 3 is

Theorem 4. *The multiplication induces an isomorphism of $K-$modules $\mathcal{U}(\mathfrak{g})^K \otimes H_K \simeq \mathcal{U}(\mathfrak{g})$. The multiplicity of every $K-$type $\delta \in \hat{K}$ in the $K-$module H_K is equal to its degree $d(\delta)$.*

Remark 1. *We note that it is easy to see (as in [1], [3] or [4]) that H_K is the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by all powers x^k, $x \in \mathcal{N}$, $k \in \mathbb{Z}_+$.*

Remark 2. *H_K is equivalent to $\mathcal{R}(K^{\mathbb{C}})$ as a $K-$module. The* Ad*−action of K on $\mathcal{U}(\mathfrak{g})$ restricted to H_K corresponds to the left regular action of K on $\mathcal{R}(K^{\mathbb{C}})$. But $\mathcal{R}(K^{\mathbb{C}})$ also carries the right regular action of K commuting with the left one. In fact, $\mathcal{R}(K^{\mathbb{C}})$ is a multiplicity free $K \times K-$module. The right regular action of K on $\mathcal{R}(K^{\mathbb{C}})$ by isomorphism $\mathcal{U}(\mathfrak{g})^K \otimes \mathcal{R}(K^{\mathbb{C}}) \longrightarrow \mathcal{U}(\mathfrak{g})$ gives rise to an action of K on $\mathcal{U}(\mathfrak{g})$ which commutes with both the* Ad*−action of K and the $\mathcal{U}(\mathfrak{g})^K-$module structure of $\mathcal{U}(\mathfrak{g})$. We note that this other $K-$action is not independent of the choice of $x \in \mathfrak{g}_0$ with the property from Lemma 1 that its* Ad*−stabilizer in $K^{\mathbb{C}}$ is trivial. Furthermore, this other $K-$action on $\mathcal{U}(\mathfrak{g})$ is not by automorphisms − we only get automorphisms on the localization $\left(\mathcal{U}(\mathfrak{g})^K \setminus \{0\}\right)^{-1}\mathcal{U}(\mathfrak{g})$ considered as an algebra over the quotient field of the ring $\mathcal{U}(\mathfrak{g})^K$. In a subsequent paper, we will investigate this other $K-$action in the simplest nontrivial case $\mathfrak{g}_0 = \mathfrak{so}(3,1)$.*

3. Freeness of $(\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ as a $\mathcal{U}(\mathfrak{g})^K-$module

Theorem 5. *Let V be a finite dimensional $K-$module. Then the space of $K-$invariants $(\mathcal{U}(\mathfrak{g}) \otimes V)^K$ is a free $\mathcal{U}(\mathfrak{g})^K-$module of finite rank* $\dim V$.

Proof. By Theorem 4 we have

$$(\mathcal{U}(\mathfrak{g}) \otimes V)^K \simeq (\mathcal{U}(\mathfrak{g})^K \otimes H_K \otimes V)^K = \mathcal{U}(\mathfrak{g})^K \otimes (H_K \otimes V)^K.$$

Thus, $(\mathcal{U}(\mathfrak{g}) \otimes V)^K$ is a free $\mathcal{U}(\mathfrak{g})^K-$module of rank $\dim(H_K \otimes V)^K$. If $n(\varepsilon)$ denotes the multiplicity of a $K-$type $\varepsilon \in \hat{K}$ in V, we have

$$(H_K \otimes V)^K \simeq \left(\left(\oplus_{\delta \in \hat{K}} d(\delta)\delta\right) \otimes \left(\oplus_{\varepsilon \in \hat{K}} n(\varepsilon)\varepsilon\right) \right)^K = \oplus_{\delta,\varepsilon \in \hat{K}} d(\delta) n(\varepsilon) (\delta \otimes \varepsilon)^K.$$

Thus,

$$\dim(H_K \otimes V)^K = \sum_{\delta,\varepsilon \in \hat{K}} d(\delta) n(\varepsilon) \dim(\delta \otimes \varepsilon)^K.$$

By Schur's Lemma $\dim(\delta\otimes\varepsilon)^K$ is 1 if δ and ε are contragredient to each other and 0 otherwise. Since the degrees of contragredient representations are equal, we finish the proof:

$$\dim(H_K\otimes V)^K=\sum_{\delta\in\hat{K}}n(\delta)d(\delta)=\dim V.$$

□

This proof also gives a way to find a $\mathcal{U}(\mathfrak{g})^K-$basis in $(\mathcal{U}(\mathfrak{g})\otimes V)^K$ if the $K-$structure of V is not too complicated and well known. This is the case for $V=C(\mathfrak{p})$, which is as a $K-$module isomorphic to the exterior algebra $\Lambda(\mathfrak{p})$. The isomorphism is given by the Chevalley map $\tau:\Lambda(\mathfrak{p})\longrightarrow C(\mathfrak{p})$, which is obtained by composing the antisymmetrisation map

$$v_1\wedge\cdots\wedge v_k\mapsto\frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)},\qquad v_1,\ldots,v_k\in\mathfrak{p},$$

from $\Lambda(\mathfrak{p})$ into the tensor algebra $\mathcal{T}(\mathfrak{p})$ with the canonical epimorphism $\mathcal{T}(\mathfrak{p})\longrightarrow C(\mathfrak{p})$. For $\mathrm{SO}_e(n,1)$ the $K-$module $\mathfrak{p}$ is irreducible, and for $\mathrm{SU}(n,1)$ it is a direct sum of two mutually contragredient irreducible $K-$types. The $K-$module $\Lambda(\mathfrak{p})$ is multiplicity free and for small values of n we can rather easily write down some canonical bases (e.g. Gelfand−Zeitlin's bases, or bases obtained from the highest weight vectors) for $K-$types δ appearing in $\Lambda(\mathfrak{p})$ (see calculations in [7] for $\mathrm{SU}(2,1)$ and in [9] for $\mathrm{SO}_e(4,1)$). Now one has to find the canonical bases in the contragredient $K-$types in $\mathcal{H}_K(\mathfrak{g})$ by solving systems of linear differential equations with constant coefficients. Finally, one has to combine these bases to write down the $K-$invariants in $\mathcal{H}_K(\mathfrak{g})\otimes\Lambda(\mathfrak{p})$, thus obtaining the basis of $(\mathcal{H}_K(\mathfrak{g})\otimes\Lambda(\mathfrak{p}))^K$. The basis of $(\mathcal{U}(\mathfrak{g})\otimes C(\mathfrak{p}))^K$ over $\mathcal{U}(\mathfrak{g})^K$ is obtained by the isomorphism $(\mathcal{H}_K(\mathfrak{g})\otimes\Lambda(\mathfrak{p}))^K\longrightarrow(H_K\otimes C(\mathfrak{p}))^K$. The second step is somewhat more complicated than the first one but the complete computations in the cases $\mathrm{SU}(2,1)$ and $\mathrm{SO}_e(4,1)$ seem to be considerably shorter than those in [7] and [9].

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