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On the investigation of a discontinuous Sturm-Liouville operator of scattering theory

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Abstract. This paper deals with the investigation of the direct scattering problem for the Sturm-Liouville operator containing both the discontinuous coefficient and discontinuity conditions at some point on the positive half-line. The integral representation of the Jost solution is obtained and the properties of its kernel function are given. A total collection of the scattering data is constructed and the behavior of the scattering function at infinity is examined.

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Key words: Sturm-Liouville equation, Jost solution, scattering data, discontinuous coefficient, discontinuity conditions

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1. Introduction

In this paper, we consider the Sturm-Liouville equation with the discontinuous coefficient

$$-y'' + q(x)y = \lambda^2 \rho(x) y, \qquad x \in (0,a) \cup (a,+\infty), \tag{1}$$

and discontinuity conditions at the point $a \in (0,+\infty)$

$$y(a-0) = \alpha y(a+0), \qquad y'(a-0) = \alpha^{-1} y'(a+0), \tag{2}$$

and the boundary condition

$$y(0) = 0, \tag{3}$$

where $\alpha > 0$, λ is a complex parameter, $\rho(x)$ is a piecewise-constant function

$$\rho(x) = \begin{cases} \beta^2, \ 0 < x < a, \\ 1, \quad a < x < \infty \end{cases}$$

with $0 < \beta \neq 1$, and the function $q(x)$ is real and satisfies the condition

$$\int_0^\infty x|q(x)|dx < \infty. \tag{4}$$

It is well known in quantum mechanics that the scattering of particles by a potential field is completely determined by the asymptotic form of the wave functions

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at infinity. Therefore, the following question arises: is it possible to reconstruct the potential from a knowledge of the asymptotic form of the wave functions at infinity? Then, if possible, indicate a method for constructing the potential. This question is discussed in detail in the work of V. A. Marchenko [19]; namely, when $\rho(x) \equiv 1$ and $\alpha = 1$ in the boundary value problem (1)-(3) with assumption (4), this boundary value problem has bounded solutions $u(x, \lambda)$ for $\lambda^2 > 0$ and $\lambda = i\lambda_k$ $(k = 1, 2, \ldots, n)$; moreover, as $x \to \infty$,

$$u(x, \lambda) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1) \quad (0 < \lambda^2 < \infty),$$
$$u(x, i\lambda_k) = m_k e^{-\lambda_k x}(1 + o(1)), \quad (k = 1, 2, \ldots, n),$$

respectively. Thus, the collection $\{S(\lambda)(-\infty < \lambda < \infty); \lambda_k; m_k(k = 1, 2, \ldots, n)\}$ provides a complete description of the behavior at infinity of all wave functions $u(x, \lambda)$ and is referred to as the scattering data of this problem (note that for this subject work [1] can be examined). In the present paper, we examine the properties of the scattering data of the boundary value problem (1)-(3). In contrast to other studies, the boundary value problem (1)-(3) contains both discontinuous coefficient and discontinuity conditions on the positive half-line. The scattering theory for the Sturm-Liouville operator with the discontinuous coefficient is studied in [4, 5, 6, 8, 9, 15, 16, 17, 20], and the Sturm-Liouville operator with discontinuity conditions on the positive half-line is discussed in [2, 10, 11, 12, 18].

In this paper, firstly we construct the integral representation of the Jost solution of the Sturm-Liouville equation (1) with discontinuity conditions (2). In the classic case ($\rho(x) \equiv 1$ and $\alpha = 1$), the transformation operator (or the Jost solution representation) which preserves the asymptotics of the solutions at infinity is obtained in [14] (see also [19]). The existence of the discontinuous coefficient $\rho(x)$ and discontinuity conditions (2) strongly influences the structure of the representation of the Jost solution because the triangular property of the Jost solution representation is lost and the kernel function has a discontinuity along the line $t = \beta(a - x) + a$ for $x \in (0, a)$. We state that the Jost solution representation of the Sturm-Liouville equation with discontinuous coefficient $\rho(x)$ is given in [9] and the Sturm-Liouville equation with discontinuity conditions (2) is obtained in [10].

Note that in physical and mathematical literature, there are numerous studies on scattering theory because of its applications in the quantum mechanics (see [3, 7, 13] and the references therein); therefore, we give the works close to the subject of this paper in the references.

2. Jost solution of the discontinuous Sturm-Liouville equation

We denote by $e(x, \lambda)$ the solution of the equation (1) with discontinuity conditions (2) and the condition at infinity $\lim_{x\to\infty} e(x, \lambda)e^{-i\lambda x} = 1$. In the case of $q(x) \equiv 0$ in the equation (1), the Jost solution of the equation (1) with discontinuity conditions (2) is as follows:

$$e_0(x, \lambda) = \begin{cases} e^{i\lambda x}, & x > a, \\ \theta^+ e^{i\lambda(\beta(x-a)+a)} + \theta^- e^{i\lambda(-\beta(x-a)+a)}, & 0 < x < a, \end{cases} \tag{5}$$

where $\theta^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha\beta}\right)$. Assume that $\theta^{+}+|\theta^{-}|>1$.

Theorem 1. *Let condition (4) holds. Then, for all λ from the closed upper half-plane, the equation (1) with discontinuity conditions (2) has the Jost solution $e(x,\lambda)$ that can be represented in the form*

$$e(x,\lambda)=e_0(x,\lambda)+\int_{\sigma(x)}^{\infty}K(x,t)e^{i\lambda t}dt, \tag{6}$$

where $\sigma(x)=\begin{cases}x, & x>a,\\ \beta(x-a)+a, & 0<x<a,\end{cases}$ the kernel $K(x,.)$ belongs to the space $L_1(\sigma(x),\infty)$ for each fixed $x\in(0,a)\cup(a,+\infty)$ and satisfies the inequality

$$\int_{\sigma(x)}^{\infty}|K(x,t)|dt\le e^{cp(x)}-1, \tag{7}$$

with $p(x)=\int_x^{\infty}s|q(s)|ds$ and $c=\theta^{+}+|\theta^{-}|$.

Proof. Consider the integral equation obtained by using the method of variation of constants for $e(x,\lambda)$:

$$e(x,\lambda)=e_0(x,\lambda)+\int_{x}^{\infty}s(x,t,\lambda)q(t)e(t,\lambda)dt, \tag{8}$$

where

$$s(x,t,\lambda)=\begin{cases}\frac{\sin\lambda(t-x)}{\lambda}, & a<x<t,\\ \frac{\sin\lambda\beta(t-x)}{\lambda\beta}, & x<t<a,\\ \frac{\theta^{+}\sin\lambda(t-(\beta(x-a)+a))}{\lambda}+\frac{\theta^{-}\sin\lambda(t-(-\beta(x-a)+a))}{\lambda}, & x<a<t.\end{cases} \tag{9}$$

It is known from [19] that when $x>a$, the Jost solution $e(x,\lambda)$ can be expressed in the form

$$e(x,\lambda)=e^{i\lambda x}+\int_{x}^{\infty}K(x,t)e^{i\lambda t}dt, \tag{10}$$

where

$$K(x,t)=K_0(x,t)+\frac{1}{2}\int_{x}^{\infty}q(s)ds\int_{t-(s-x)}^{t+(s-x)}K(s,u)duds, \tag{11}$$

with

$$K_0(x,t)=\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}q(s)ds. \tag{12}$$

Now, when $0<x<a$, let us seek the Jost solution $e(x,\lambda)$. Substituting expression (6) for $e(x,\lambda)$ in (8), we get

$$\int_{\beta(x-a)+a}^{\infty}K(x,t)e^{i\lambda t}dt=\int_{x}^{\infty}s(x,t,\lambda)q(t)e_0(t,\lambda)dt+$$
$$+\int_{x}^{\infty}s(x,t,\lambda)q(t)\int_{\beta(t-a)+a}^{\infty}K(t,s)e^{i\lambda s}dsdt. \tag{13}$$

Take into account the first term on the right-hand side of the equality (13). Using (5) and (9), then changing the order of integration respectively, we find

$$\begin{aligned}\int_x^\infty s(x,t,\lambda)q(t)e_0(t,\lambda)dt &= \frac{\theta^+}{2\beta}\int_{\beta(x-a)+a}^{\beta(a-x)+a} e^{i\lambda t}\left\{\int_{\frac{t+\beta(x+a)-a}{2\beta}}^{a} q(u)du\right\}dt \\ &+\frac{\theta^-}{2\beta}\int_{\beta(x-a)+a}^{\beta(a-x)+a} e^{i\lambda t}\left\{\int_{\frac{\beta(x+a)+a-t}{2\beta}}^{a} q(u)du\right\}dt \\ &+\frac{\theta^+}{2}\int_{\beta(x-a)+a}^{\beta(a-x)+a} e^{i\lambda t}\left\{\int_a^\infty q(u)du\right\}dt \\ &+\frac{\theta^+}{2}\int_{\beta(a-x)+a}^{\infty} e^{i\lambda t}\left\{\int_{\frac{t+\beta(x-a)+a}{2}}^{\infty} q(u)du\right\}dt \\ &-\frac{\theta^-}{2}\int_{\beta(x-a)+a}^{\beta(a-x)+a} e^{i\lambda t}\left\{\int_a^{\frac{t+\beta(a-x)+a}{2}} q(u)du\right\}dt \\ &+\frac{\theta^-}{2}\int_{\beta(a-x)+a}^{\infty} e^{i\lambda t}\left\{\int_{\frac{t+\beta(a-x)+a}{2}}^{\infty} q(u)du\right\}dt. \quad (14)\end{aligned}$$

Consider the second term on the right-hand side of the equality (13). Using (9) once more, we obtain

$$\begin{aligned}&\int_x^\infty s(x,t,\lambda)q(t)\int_{\beta(t-a)+a}^\infty K(t,s)e^{i\lambda s}dsdt \\ &= \frac{1}{2\beta}\int_x^a q(t)\int_{\beta(t-a)+a}^\infty K(t,s)\left\{\int_{s-\beta(t-x)}^{s+\beta(t-x)} e^{i\lambda\xi}d\xi\right\}dsdt \\ &+\frac{\theta^+}{2}\int_a^\infty q(t)\int_{\beta(t-a)+a}^\infty K(t,s)\left\{\int_{s-t+\beta(x-a)+a}^{s+t+\beta(a-x)-a} e^{i\lambda\xi}d\xi\right\}dsdt \\ &-\frac{\theta^-}{2}\int_a^{\beta(a-x)+a} q(t)\int_{\beta(t-a)+a}^\infty K(t,s)\left\{\int_{s+\beta(x-a)-a+t}^{s+\beta(a-x)+a-t} e^{i\lambda\xi}d\xi\right\}dsdt \\ &+\frac{\theta^-}{2}\int_{\beta(a-x)+a}^\infty q(t)\int_{\beta(t-a)+a}^\infty K(t,s)\left\{\int_{s-t+\beta(a-x)+a}^{s+t+\beta(x-a)-a} e^{i\lambda\xi}d\xi\right\}dsdt.\end{aligned}$$

Now, extending the function $K(t,s)$ by zero for $s<t$ for any $t \geq \beta(x-a)+a$, we have

$$\begin{aligned}&\int_x^\infty s(x,t,\lambda)q(t)\int_{\beta(t-a)+a}^\infty K(t,s)e^{i\lambda s}dsdt \\ &= \frac{1}{2\beta}\int_{-\infty}^\infty e^{i\lambda t}\left\{\int_x^a q(\xi)\int_{t-\beta(\xi-x)}^{t+\beta(\xi-x)} K(\xi,s)dsd\xi\right\}dt\end{aligned}$$

$$+\frac{\theta^+}{2}\int_{-\infty}^{\infty} e^{i\lambda t}\left\{\int_a^{\infty} q(\xi)\int_{t-\xi+\beta(x-a)+a}^{t+\xi+\beta(a-x)-a} K(\xi,s)dsd\xi\right\}dt$$
$$-\frac{\theta^-}{2}\int_{-\infty}^{\infty} e^{i\lambda t}\left\{\int_a^{\beta(a-x)+a} q(\xi)\int_{t+\beta(x-a)-a+\xi}^{t+\beta(a-x)+a-\xi} K(\xi,s)dsd\xi\right\}dt$$
$$+\frac{\theta^-}{2}\int_{-\infty}^{\infty} e^{i\lambda t}\left\{\int_{\beta(a-x)+a}^{\infty} q(\xi)\int_{t-\xi+\beta(a-x)+a}^{t+\xi+\beta(x-a)-a} K(\xi,s)dsd\xi\right\}dt. \tag{15}$$

Thus, substituting equalities (14) and (15) into the equality (13), we find

$$\begin{aligned} K(x,t) = {} & K_0(x,t)+\frac{1}{2\beta}\int_a^x q(\xi)\int_{t-\beta(\xi-x)}^{t+\beta(\xi-x)} K(\xi,s)dsd\xi \\ & +\frac{\theta^+}{2}\int_a^{\infty} q(\xi)\int_{t-\xi+\beta(x-a)+a}^{t+\xi+\beta(a-x)-a} K(\xi,s)dsd\xi \\ & -\frac{\theta^-}{2}\int_a^{\beta(a-x)+a} q(\xi)\int_{t+\xi+\beta(x-a)-a}^{t-\xi+\beta(a-x)+a} K(\xi,s)dsd\xi \\ & +\frac{\theta^-}{2}\int_{\beta(a-x)+a}^{\infty} q(\xi)\int_{t-\xi+\beta(a-x)+a}^{t+\xi+\beta(x-a)-a} K(\xi,s)dsd\xi, \end{aligned} \tag{16}$$

where for $\beta(x-a)+a<t<\beta(a-x)+a:$

$$\begin{aligned} K_0(x,t) = {} & \frac{\theta^+}{2\beta}\int_{\frac{t+\beta(x+a)-a}{2\beta}}^{a} q(u)du+\frac{\theta^-}{2\beta}\int_{\frac{\beta(x+a)+a-t}{2\beta}}^{a} q(u)du \\ & +\frac{\theta^+}{2}\int_a^{\infty} q(u)du-\frac{\theta^-}{2}\int_a^{\frac{t+\beta(a-x)+a}{2}} q(u)du, \end{aligned} \tag{17}$$

for $\beta(a-x)+a<t<\infty:$

$$K_0(x,t)=\frac{\theta^+}{2}\int_{\frac{t+\beta(x-a)+a}{2}}^{\infty} q(u)du+\frac{\theta^-}{2}\int_{\frac{t+\beta(a-x)+a}{2}}^{\infty} q(u)du. \tag{18}$$

In order to complete the proof of the theorem, it suffices to verify that for each fixed $x\in(0,a)\cup(a,\infty)$, the system of equations (11), (16) has the solution $K(x,.)\in L_1(\sigma(x),\infty)$ which satisfies the inequality (7). Let us use the successive approximation method. Set

(i) for $x\in(a,\infty)$:

$$K_n(x,t)=\frac{1}{2}\int_x^{\infty} q(\xi)\int_{t-(\xi-x)}^{t+(\xi-x)} K_{n-1}(\xi,s)dsd\xi, \quad n=1,2,\ldots \tag{19}$$

(ii) for $x \in (0,a)$:

$$\begin{aligned}
K_n(x,t) = & \frac{1}{2\beta}\int_a^x q(\xi)\int_{t-\beta(\xi-x)}^{t+\beta(\xi-x)} K_{n-1}(\xi,s)dsd\xi \\
& +\frac{\theta^+}{2}\int_a^\infty q(\xi)\int_{t-\xi+\beta(x-a)+a}^{t+\xi+\beta(a-x)-a} K_{n-1}(\xi,s)dsd\xi \\
& -\frac{\theta^-}{2}\int_a^{\beta(a-x)+a} q(\xi)\int_{t+\xi+\beta(x-a)-a}^{t-\xi+\beta(a-x)+a} K_{n-1}(\xi,s)dsd\xi \\
& +\frac{\theta^-}{2}\int_{\beta(a-x)+a}^\infty q(\xi)\int_{t-\xi+\beta(a-x)+a}^{t+\xi+\beta(x-a)-a} K_{n-1}(\xi,s)dsd\xi,\ n=1,2,\ldots
\end{aligned} \tag{20}$$

and in the case of $n=0$, $K_0(x,t)$ is determined by formulas (12) for $x \in (a,\infty)$ and (17), (18) for $x \in (0,a)$. Consider the case of $x \in (a,\infty)$. It follows from (12) and (19) that

$$\int_x^\infty |K_0(x,t)|dt \le \int_x^\infty s|q(s)|ds := p(x),$$
$$\int_x^\infty |K_n(x,t)|dt \le \frac{(p(x))^{n+1}}{(n+1)!}.$$

This implies that for $x \in (a,\infty)$ the series $K(x,t) = \sum_{n=0}^\infty K_n(x,t)$ converges to $L_1(x,\infty)$ and its sum $K(x,t)$ satisfies the inequality

$$\int_x^\infty |K(x,t)|dt \le e^{p(x)} - 1.$$

Now, take into account the case of $x \in (0,a)$. It is obtained from (17), (18) and (20) that

$$\int_{\beta(x-a)+a}^\infty |K_0(x,t)|dt \le (\theta^+ + |\theta^-|)\int_x^\infty s|q(s)|ds = cp(x),$$
$$\int_{\beta(x-a)+a}^\infty |K_n(x,t)|dt \le \frac{c^{n+1}(p(x))^{n+1}}{(n+1)!}.$$

This implies that for $x \in (0,a)$ the series $K(x,t) = \sum_{n=0}^\infty K_n(x,t)$ converges to $L_1(\beta(x-a)+a,\infty)$ and its sum $K(x,t)$ satisfies the inequality

$$\int_{\beta(x-a)+a}^\infty |K(x,t)|dt \le e^{cp(x)} - 1.$$

As a result, it is shown that the system of equations (11), (16) has the solution $K(x,.) \in L_1(\sigma(x),\infty)$ which satisfies the inequality (7). □

Remark 1. *The kernel function $K(x,t)$ has the following properties which are obtained from (11), (12) and (16)-(18):*

$$K(x,\sigma(x)) = \begin{cases} \frac{1}{2}\int_x^\infty q(t)dt, & x > a, \\ \frac{\theta^+}{2}\int_x^\infty \frac{1}{\sqrt{\rho(t)}}q(t)dt, & 0 < x < a. \end{cases} \tag{21}$$

$$K(x,\beta(a-x)+a+0)-K(x,\beta(a-x)+a-0)$$
$$=\frac{\theta^-}{2}\left\{\int_a^\infty q(u)du-\frac{1}{\beta}\int_x^a q(u)du\right\},\quad 0<x<a. \tag{22}$$

Moreover, if $q(x)$ is differentiable, then $K(x,t)$ satisfies the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2}-\rho(x)\frac{\partial^2 K(x,t)}{\partial t^2}=q(x)K(x,t),\quad x\in(0,a)\cup(a,\infty),\quad t>\sigma(x), \tag{23}$$

and the conditions

$$\frac{d}{dx}K(x,\sigma(x))=\begin{cases}-\frac{1}{2}q(x), & x>a,\\ -\frac{\theta^+}{2\beta}q(x), & 0<x<a,\end{cases} \tag{24}$$

$$\frac{d}{dx}\left\{K(x,\beta(a-x)+a+0)-K(x,\beta(a-x)+a-0)\right\}=\frac{\theta^-}{2\beta}q(x), \tag{25}$$

$$K(a-0,t)=\alpha K(a+0,t),\qquad K_x^{'}(a-0,t)=\alpha^{-1}K_x^{'}(a+0,t), \tag{26}$$

$$\lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial x}=\lim_{x+t\to\infty}\frac{\partial K(x,t)}{\partial t}=0, \tag{27}$$

which define it uniquely.

Thus, in order for $K(x,t)$ to be the kernel of representation (6), it is necessary and sufficient that it satisfies relations (21)-(27).

3. Scattering data

In this section, we give the scattering data of the problem (1)-(4) and investigate some properties of this scattering data.

The solution $e(x,\lambda)$ is an analytic function of λ in the upper half-plane $Im\lambda>0$ and continuous for $Im\lambda\geq 0$. For real $\lambda\neq 0$, the function $e(x,\lambda)$ and $e(x,-\lambda)$ form a fundamental system of solutions of equation (1) with discontinuity conditions (2) and their Wronskian is as follows:

$$W\left\{e(x,\lambda),e(x,-\lambda)\right\}=e'(x,\lambda)e(x,-\lambda)-e(x,\lambda)e'(x,-\lambda)=2i\lambda.$$

Lemma 1. *For all values of λ, the equation (1) with discontinuity conditions (2) has a solution $w(x,\lambda)$ satisfying the conditions*

$$w(x,\lambda)=x(1+o(1)),\qquad w_x'(x,\lambda)=1+o(1),\qquad x\to 0, \tag{28}$$

and the solution $w(x,\lambda)$ is an analytic function of λ.

Proof. It is obtained from (1), (2) and (28) that the function $w(x,\lambda)$ satisfies the integral equation

$$w(x,\lambda)=w_0(x,\lambda)+\int_0^x w_0(t,x,\lambda)q(t)w(t,\lambda)dt, \tag{29}$$

where

$$w_0(x,\lambda)=\begin{cases}\frac{\sin\lambda\beta x}{\lambda\beta}, & 0<x<a,\\ \frac{\theta^+\sin\lambda(x-a+\beta a)}{\lambda}+\frac{\theta^-\sin\lambda(x-a-\beta a)}{\lambda}, & a<x<\infty,\end{cases} \tag{30}$$

$$w_0(t,x,\lambda)=\begin{cases}\frac{\sin\lambda\beta(x-t)}{\lambda\beta}, & t<x<a,\\ \frac{\sin\lambda(x-t)}{\lambda}, & a<t<x,\\ \frac{\theta^+\sin\lambda(x-a+\beta(a-t))}{\lambda}+\frac{\theta^-\sin\lambda(x-a+\beta(t-a))}{\lambda}, & t<a<x.\end{cases} \tag{31}$$

We seek the solution of integral equation (29) for $Im\lambda\geq 0$ in the form $w(x,\lambda)=xe^{-i\lambda x}z(x,\lambda)$. Then we can write $z(x,\lambda)$ as follows:

$$z(x,\lambda)=\frac{w_0(x,\lambda)e^{i\lambda x}}{x}+\int_0^x\frac{w_0(t,x,\lambda)e^{i\lambda(x-t)}}{x}tq(t)z(t,\lambda)dt.$$

This equation can be solved by applying the successive approximation method. Set

$$z(x,\lambda)=\sum_{k=0}^{\infty}z_k(x,\lambda), \tag{32}$$

where

$$z_0(x,\lambda)=\frac{w_0(x,\lambda)e^{i\lambda x}}{x},\qquad z_k(x,\lambda)=\int_0^x\frac{w_0(t,x,\lambda)e^{i\lambda(x-t)}}{x}tq(t)z_{k-1}(t,\lambda)dt.$$

Using relations (30) and (31), for $Im\lambda\geq 0$ and $0\leq t<x$ we have

$$\left|\frac{w_0(x,\lambda)e^{i\lambda x}}{x}\right|\leq s,\qquad \left|\frac{w_0(t,x,\lambda)e^{i\lambda(x-t)}}{x}\right|\leq s,$$

where $s=(\theta^++|\theta^-|)+\beta(\theta^+-|\theta^-|)$. Then,

$$|z_0(x,\lambda)|\leq s,\qquad |z_k(x,\lambda)|\leq s\int_0^x t|q(t)||z_{k-1}(t,\lambda)|dt\leq\frac{s}{k!}\left(s\int_0^x t|q(t)|dt\right)^k.$$

It follows that series (32) converges uniformly in the domain $x\in[0,b]$, $Im\lambda\geq 0$ for any $b>0$ and its sum $z(x,\lambda)$ satisfies the inequality

$$|z(x,\lambda)|\leq s\exp\left\{s\int_0^x t|q(t)|dt\right\};$$

in addition, $z(x,\lambda)$ is an analytic function of λ for $Im\lambda>0$ and continuous in the half-plane $Im\lambda\geq 0$. Therefore, $w(x,\lambda)$ satisfies both equations (1) and (29) and the inequality

$$\left|w(x,\lambda)e^{i\lambda x}\right|\leq xs\exp\left\{s\int_0^x t|q(t)|dt\right\}; \tag{33}$$

moreover, $w(x,\lambda)$ is an analytic function of λ for $Im\lambda>0$ and continuous in the closed half-plane $Im\lambda\geq 0$. Similarly, it is proved that the equation (29) has a solution for $Im\lambda \le 0$ and its solution $w(x,\lambda)$ is analytic in λ in the half-plane $Im\lambda < 0$ and continuous for $Im\lambda \le 0$. Consequently, $w(x,\lambda)$ satisfies equation (1), vanishes at $x = 0$ and is an entire function of λ. It follows from the equation (29) and the inequality (33) that

$$|w(x,\lambda) - w_0(x,\lambda)| \le xs^2 \int_0^x t|q(t)|dt \exp\left\{|Im\lambda x| + s\int_0^x t|q(t)|dt\right\},$$
$$|w'(x,\lambda) - w_0'(x,\lambda)| \le s^2 \int_0^x t|q(t)|dt \exp\left\{|Im\lambda x| + s\int_0^x t|q(t)|dt\right\},$$

and these inequalities imply that the solution $w(x,\lambda)$ satisfies conditions (28). $\square$

Lemma 2. *The following identity holds for all real $\lambda \ne 0$*

$$\frac{-2i\lambda w(x,\lambda)}{e(0,\lambda)} = e(x,-\lambda) - S(\lambda)e(x,\lambda),$$

where

$$S(\lambda) = \frac{e(0,-\lambda)}{e(0,\lambda)} = \overline{S(-\lambda)} = [S(-\lambda)]^{-1}$$

and the function $S(\lambda)$ is the scattering function of the problem (1)-(3).

Lemma 3. *The function $e(0,\lambda)$ may have only a finite number of zeros in the half-plane $Im\lambda > 0$ and these zeros lie on the imaginary axis.*

Remark 2. *The proofs of Lemma 2 and Lemma 3 are obtained similarly in the work of V. A. Marchenko (Lemma 3.1.5 and Lemma 3.1.6 in [19]).*

Let $i\lambda_k, (0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n)$ be the zeros of the function $e(0,\lambda)$ and denote

$$m_k^{-2} = \int_0^\infty |e(x,i\lambda_k)|^2 \rho(x)dx = \frac{-e'(0,i\lambda_k)\dot{e}(0,i\lambda_k)}{2i\lambda_k}, \quad k = 1,2,\ldots,n,$$

where $\dot{e}(x,\lambda) = \frac{d}{d\lambda}e(x,\lambda)$. The numbers m_k is the *normalized numbers* of the boundary value problem (1)-(3).

The functions

$$u(x,\lambda) = e(x,-\lambda) - S(\lambda)e(x,\lambda), \quad (-\infty < \lambda < \infty),$$
$$u(x,i\lambda_k) = m_k e(x,i\lambda_k), \quad (k = 1,2,\ldots,n)$$

are bounded solutions of the boundary value problem (1)-(3); moreover, as $x \to \infty$, the asymptotic relations hold:

$$u(x,\lambda) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \quad (-\infty < \lambda < \infty)$$
$$u(x,i\lambda_k) = m_k e^{-\lambda_k x}(1 + o(1)), \quad (k = 1,2,\ldots,n).$$

Definition 1. *A collection of quantities*

$$\{S(\lambda)(-\infty < \lambda < \infty); \lambda_k, m_k \ (k = 1,2,\ldots,n)\}$$

that specify the behavior of the normalized eigenfunctions at infinity is called the scattering data of the boundary value problem (1)-(3) satisfying condition (4).

Now, we give a property relating to the scattering function $S(\lambda)$:

Lemma 4. *The function $S_0(\lambda)-S(\lambda)$ is the Fourier transform of a function $F_S(x)$ of the form*

$$F_S(x)=F_S^{(1)}(x)+F_S^{(2)}(x),$$

where

$$S_0(\lambda)=\frac{e_0(0,-\lambda)}{e_0(0,\lambda)}=\frac{\theta^+e^{-i\lambda a(1-\beta)}+\theta^-e^{-i\lambda a(1+\beta)}}{\theta^+e^{i\lambda a(1-\beta)}+\theta^-e^{i\lambda a(1+\beta)}},$$

$F_S^{(1)}(x)\in L_1(-\infty,\infty)$, $F_S^{(2)}(x)\in L_2(-\infty,\infty)$ *and* $\sup\limits_{-\infty<x<\infty}|F_S^{(2)}(x)|<\infty$.

Proof. Denoting $K(0,t)=K(t)$ for the simplicity, we can write

$$e(0,\lambda)=e_0(0,\lambda)+\int_0^\infty K(0,t)e^{i\lambda t}dt=\theta^+e^{i\lambda a(1-\beta)}+\theta^-e^{i\lambda a(1+\beta)}+\widetilde{K}(-\lambda)$$

and

$$S_0(\lambda)-S(\lambda)=\frac{e_0(0,-\lambda)\widetilde{K}(-\lambda)}{e_0(0,\lambda)\left[e_0(0,\lambda)+\widetilde{K}(-\lambda)\right]}-\frac{\widetilde{K}(\lambda)}{e_0(0,\lambda)+\widetilde{K}(-\lambda)}. \tag{34}$$

Now, we examine that $\frac{\widetilde{K}(-\lambda)}{e_0(0,\lambda)}$ is a Fourier transformation of some summable function. For this purpose, it is obtained that the series

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{\theta^-}{\theta^+}\right)^n K_+(t-\beta a(2n+1)+a)$$

with $K_+(t)=K(t)$ for $t>0$ and $K_+(t)=0$ for $t<0$ converges some function $\psi(.)\in L_1(-\infty,\infty)$ since

$$\left|\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\theta^-}{\theta^+}\right)^n K_+(t-\beta a(2n+1)+a)dt\right|$$
$$\leq\sum_{n=0}^{\infty}\left|\frac{\theta^-}{\theta^+}\right|^n\int_{-\infty}^{\infty}|K_+(t-\beta a(2n+1)+a)|dt=\sum_{n=0}^{\infty}\left|\frac{\theta^-}{\theta^+}\right|^n\int_{-\infty}^{\infty}|K(t)|dt.$$

Then, we have

$$\frac{\widetilde{K}(-\lambda)}{e_0(0,\lambda)}=\frac{1}{\theta^+}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\theta^-}{\theta^+}\right)^n\int_{\beta a(2n+1)-a}^{\infty}K(t-\beta a(2n+1)+a)e^{i\lambda t}dt$$
$$=\frac{1}{\theta^+}\int_{-\infty}^{\infty}\psi(t)e^{i\lambda t}dt=\frac{1}{\theta^+}\widetilde{\psi}(-\lambda):=\widetilde{\varphi}(-\lambda), \tag{35}$$

and also

$$\frac{\widetilde{K}(\lambda)}{e_0(0,\lambda)}=\frac{1}{\theta^+}\widetilde{\psi}(\lambda)=\widetilde{\varphi}(\lambda),\qquad \frac{\widetilde{K}(-\lambda)}{e_0(0,-\lambda)}=\frac{1}{\theta^+}\widetilde{\psi}(-\lambda)=\widetilde{\varphi}(-\lambda). \tag{36}$$

Then, using relations (35) and (36), two terms on the right-hand side of the equality (34) can be written as follows:

$$\frac{e_0(0,-\lambda)\widetilde{K}(-\lambda)}{e_0(0,\lambda)\left[e_0(0,\lambda)+\widetilde{K}(-\lambda)\right]}=\frac{\widetilde{\varphi}(-\lambda)}{1+\widetilde{\varphi}(-\lambda)},\qquad \frac{\widetilde{K}(\lambda)}{e_0(0,\lambda)+\widetilde{K}(-\lambda)}=\frac{\widetilde{\varphi}(\lambda)}{1+\widetilde{\varphi}(-\lambda)}.$$

Consequently, the equality (34) is in the form:

$$S_0(\lambda)-S(\lambda)=\frac{\widetilde{\varphi}(-\lambda)-\widetilde{\varphi}(\lambda)}{1+\widetilde{\varphi}(-\lambda)}. \tag{37}$$

Now, to complete the proof of this lemma, we proceed as in the works of V. A. Marchenko ([19], Lemma 3.1.7); namely, we note that

$$\widetilde{h}(\lambda)=\begin{cases}1, & |\lambda|<1,\\ 2-|\lambda|, & 1\le|\lambda|\le 2\\ 0, & 2<|\lambda|\end{cases}$$

is the Fourier transform of a function $h(x)\in L_1(-\infty,\infty)$; also, $\widetilde{h}(\lambda N^{-1})$ is the Fourier transform of the function $h_N(x)=Nh(xN)$ and $\lim\limits_{N\to\infty}\|f-h_N*f\|_{L_1}=0$ for all $f(x)\in L_1(-\infty,\infty)$. Since the Fourier transform of $f(x)-h_N*f(x)$ is equal to $\left\{1-\widetilde{h}(\lambda N^{-1})\right\}\widetilde{f}(\lambda)$, for N large enough, the function $\left[1+\left\{1-\widetilde{h}(\lambda N^{-1})\right\}\widetilde{f}(\lambda)\right]^{-1}$ -1 is the Fourier transform of a function from $L_1(-\infty,\infty)$. Then, we can write the equality (37) as follows:

$$\begin{aligned}S_0(\lambda)-S(\lambda)&=[\widetilde{\varphi}(-\lambda)-\widetilde{\varphi}(\lambda)]\\&\quad+[\widetilde{\varphi}(-\lambda)-\widetilde{\varphi}(\lambda)]\left\{\left[1+\left(1-\widetilde{h}(\lambda N^{-1})\right)\widetilde{\varphi}(-\lambda)\right]^{-1}-1\right\}\\&\quad-[\widetilde{\varphi}(-\lambda)-\widetilde{\varphi}(\lambda)]\left\{\frac{1}{1+\left(1-\widetilde{h}(\lambda N^{-1})\right)\widetilde{\varphi}(-\lambda)}-\frac{1}{1+\widetilde{\varphi}(-\lambda)}\right\}.\end{aligned}\tag{38}$$

Since for sufficiently large N the function $\left[1+\left\{1-\widetilde{h}(\lambda N^{-1})\right\}\widetilde{f}(\lambda)\right]^{-1}-1$ is the Fourier transform of a summable function, the sum of the first two terms on the right-hand side of the equality (38) is the Fourier transform of a summable function $F_S^{(1)}(x)\in L_1(-\infty,\infty)$ and since $\widetilde{h}(\lambda N^{-1})$ =0 for $|\lambda|>2N$, the third term on the right-hand side of the equality (38) equals zero for $|\lambda|>2N$ and is bounded, so it is the Fourier transform of a bounded function $F_S^{(2)}(x)\in L_2(-\infty,\infty)$. □

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References

[1] Z. S. Agranovich, V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Gordon and Breach Science Publishers, New York and London, 1963.

[2] E. Bairamov, Y. Aygar, B. Eren, *Scattering theory of impulsive Sturm-Liouville equations*, Filomat **31**(2017), 5401–5409.

[3] K. Chadan, P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, Berlin, 1977.

[4] A. Çöl, *Inverse spectral problem for Sturm-Liouville operator with discontinuous coefficient and cubic polynomials of spectral parameter in boundary condition*, Adv. Difference Equ. **132**(2015), DOI 10.1186/s13662-015-0478-7.

[5] A. A. Darwish, *The inverse problem for a singular boundary value problem*, New Zeland J. Math. **22**(1993), 37–66.

[6] Z. FA El-Reheem, A. H. Nasser, *On the spectral investigation of the scattering problem for some version of one-dimensional Schrödinger equation with turning point*, Bound. Value Probl. **97**(2014), DOI 10.1186/1687-2770-2014-97.

[7] L. D. Faddeev, A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin, 2007.

[8] M. G. Gasymov, *The direct and inverse problem of spectral analysis for a class of equations with a discontinuous coefficient*, in: *Non-Classical Methods in Geophysics*, (M. M. Lavrent'ev, Eds.), Novosibirsk Nauka, 1977, 37–44, in Russian.

[9] I. M. Guseinov, R. T. Pashaev, *On an inverse problem for a second-order differential equation*, Usp. Mat. Nauk **57**(2002), 147–148.

[10] H. M. Huseynov, J. A. Osmanova, *On Jost solution of Sturm-Liouville equation with discontinuity conditions*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **27**(2007), 63–70.

[11] H. M. Huseynov, J. A. Osmanli, *Uniqueness of the solution of the inverse scattering problem for discontinuous Sturm-Liouville operator*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **29**(2009) 43–50.

[12] H. M. Huseynov, L. I. Mammadova, *The inverse scattering problem for Sturm-Liouville operator with discontinuity conditions on the semi-axis*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **39**(2013), 63–68.

[13] M. Jaulent, C. Jean, *The inverse problem for the one dimensional Schrödinger equation with an energy-dependent potential I, II*, Ann. Inst. Henri Poincare Sec. A **25**(1976), 105–118, 119–137.

[14] B. Ya Levin, *Fourier- and Laplace-type transformations by means of solutions of the second-order differential equations*, Dokl. Akad. Nauk SSSR **106**(1956), 187–190, in Russian.

[15] K. R. Mamedov, *On an inverse scattering problem for a discontinuous Sturm-Liouville equation with a spectral parameter in the boundary condition*, Bound. Value Probl. **171967** (2010), DOI 10.1155/2010/171967.

[16] K. R. Mamedov, F. A. Cetinkaya, *Boundary value problem for a Sturm-Liouville operator with piecewise continuous coefficient*, Hacet. J. Math. Stat. **44**(2015), 867–874.

[17] K. R. Mamedov, N. Palamut, *On a direct problem of scattering theory for a class of Sturm-Liouville operator with discontinuous coefficient*, Proc. Jangjeon Math. Soc. **12**(2009), 243–251.

[18] M. D. Manafov, A. Kablan, *Inverse scattering problems for energy-dependent Sturm-Liouville equations with point δ-interaction and eigenparameter-dependent boundary condition*, Electron. J. Differential Equations **2013**(2013), 1–9.

[19] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, AMS Chelsea Publishing, Providence, Rhode Island, 2011.

[20] Ö. Mizrak, K. R. Mamedov, A. M. Akhtyamov, *Characteristic properties of scattering data of a boundary value problem*, Filomat **31**(2017), 3945–3951.