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A hybrid approximation scheme for 1-D singularly perturbed parabolic convection-diffusion problems

Mohammad Izadi[1,*] and Şuayip Yüzbaşı[2]

[1] *Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*
[2] *Department of Mathematics, Faculty of Science, Akdeniz University, TR 07058 Antalya, Turkey*

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Abstract. Our study is concerned with a hybrid spectral collocation approach to solving singularly perturbed 1-D parabolic convection-diffusion problems. In this approach, discretization in time is carried out with the help of Taylor series expansions before the spectral based on novel special polynomials is applied to the spatial operator in the time step. A detailed error analysis of the presented technique is conducted with regard to the space variable. The advantages of this attempt are presented through comparison of our results in the model problems obtained by this technique and other existing schemes.
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1. Introduction

A singular perturbation problem contains a (positive) small parameter whose solution cannot be obtained or approximated by setting the parameter value to zero [17]. Various important physical phenomena in science and engineering can be described mathematically as singularly perturbed convection-diffusion problems. They are found e.g. in semiconductor device modelling [22], electromagnetic field problems in moving media [7], and turbulent jet diffusion flames [8]. Other typical examples are Navier-Stokes flow with a high Reynolds number and heat transfer model problems with high Péclet numbers. In these models, the small parameter that premultiplies the diffusion term typically exhibits boundary layers, which often makes most of the proposed methods unsuccessful in practice [26].

Our main aim is to develop a hybrid approximation technique to solve the 1-D singularly perturbed second-order partial differential equations of convection-diffusion type of the form:

$$\frac{\partial u(x,t)}{\partial t}+\alpha(x)\,\frac{\partial u(x,t)}{\partial x}+\beta(x)\,u(x,t)=\varepsilon\,\frac{\partial^2 u(x,t)}{\partial x^2}+g(x,t),\ 0\le x\le x_f,\ 0\le t\le t_f, \tag{1}$$

*Corresponding author. *Email addresses:* izadi@uk.ac.ir (M. Izadi) syuzbasi@akdeniz.edu.tr; suayipyuzbasi@gmail.com (Ş. Yüzbaşı)

where $\alpha(x), \beta(x)$, $g(x,t)$ are some familiar real-valued functions. The constant $0 < \varepsilon < 1$ denotes the perturbation parameter and x_f, t_f are two given real numbers. The initial condition is given by

$$u(x, t = 0) = u_0(x), \quad 0 \le x \le x_f, \tag{2}$$

while the following boundary conditions are supplemented with the initial-value problem (1)-(2):

$$u(x = 0, t) = b_0(t), \qquad u(x = x_f, t) = b_f(t), \quad 0 \le t \le t_f, \tag{3}$$

where $b_0(t)$ and $b_f(t)$ are two prescribed functions.

So far, various approximate and numerical procedures have been developed for treating the model problem (1)-(3). In this respect, we mention the approach based on the classical implicit Euler and the simple upwind schemes [1], the piecewise-analytical method [24], the B-spline collocation scheme [18], the parameter-uniform finite difference schemes [21], the Bessel (of the first kind) collocation approach [30], the Sinc-Galerkin method combined with a standard finite difference scheme [25], the Galerkin weighted residual method [29], and a higher order finite difference scheme [2]. In addition, the combined finite difference approaches implemented on uniform and non-uniform meshes were developed in [3]-[6] and [23, 27]. Some other computational methods proposed for the models closely related to (1) can be found in [9, 10], [12, 13], and [19, 20].

The primary purpose of this study is to derive a new hybrid approximation algorithm to solve (1). Since the underlying equation is a time-dependent problem, developing an accurate algorithm for the time advancement is of interest. In this respect, the Taylor approach with second-order accuracy is utilized. After discretizing the time variable, we employ the novel special functions together with the collocation points to approximate the solution with respect to a space variable. Indeed, the special polynomial of order q is defined explicitly as [28, 14]:

$$\mathcal{S}_m(x) = \frac{1}{2^m} \sum_{q=0}^{m} \frac{(2m-q)!}{(q-k)!\, q!} (2x)^q, \quad m = 0, 1, \ldots. \tag{4}$$

We should emphasize that these polynomials all have positive integer coefficients. Note also that this class of polynomials is related to the Bessel functions recently utilized in some research papers [11, 15, 16].

2. Time marching technique

To get an accurate time discretization for the parabolic convection-diffusion equation (1), we employ the Taylor expansion series approach. In this respect, we divided the interval $[0, t_f]$ into K uniform subintervals with grid points $t_0 = 0 < t_1 = \Delta t < \ldots < t_K = K\Delta t = t_f$ and the time step given by $\Delta t = t_k - t_{k-1}$. Let us denote u^k as the approximate solution at time level t_k. First, note that after evaluating the original equation (1) at time level t_k, one gets

$$\frac{\partial u^k}{\partial t} = \varepsilon \frac{\partial^2 u^k}{\partial x^2} - \alpha(x) \frac{\partial u^k}{\partial x} - \beta(x)\, u^k + g(x, t_k). \tag{5}$$

Applying the Taylor formula to $\frac{\partial u^k}{\partial t}$ gives us

$$\frac{\partial u^k}{\partial t} = \frac{u^{k+1} - u^k}{\Delta t} - \frac{1}{2}\Delta t\,\frac{\partial^2 u^k}{\partial t^2} + \mathcal{O}(\Delta t^2). \tag{6}$$

Differentiating (1) with regard to t reveals

$$\frac{\partial^2 u^k}{\partial t^2} = \varepsilon\,\frac{\partial^3 u^k}{\partial x^2 \partial t} - \alpha(x)\,\frac{\partial^2 u^k}{\partial x \partial t} - \beta(x)\,\frac{\partial u^k}{\partial t} + \frac{\partial g(x,t_k)}{\partial t}.$$

The next aim is to substitute all first-order derivatives $\frac{\partial u^k}{\partial t}$ by an approximation $\frac{u^{k+1}-u^k}{\Delta t}$. Multiplying both sides by Δt, the resultant equation becomes

$$\Delta t\,\frac{\partial^2 u^k}{\partial t^2} = \varepsilon\left(\frac{\partial^2 u^{k+1}}{\partial x^2} - \frac{\partial^2 u^k}{\partial x^2}\right) - \alpha(x)\left(\frac{\partial u^{k+1}}{\partial x} - \frac{\partial u^k}{\partial x}\right) - \beta(x)\left(u^{k+1} - u^k\right) + g^{k+1} - g^k, \tag{7}$$

where $g^k := g(x, t_k)$. Hence, we place (7) into the right-hand side of (6) and then equate with (5). After multiplying by $2\Delta t$, we get the time-discretized equation for (1) with second-order accuracy in time. If we introduce $Y_{k+1}(x) := u^{k+1}$ and

$$A(x) := -\Delta t\,\varepsilon, \quad B(x) := \Delta t\,\alpha(x), \quad C(x) := 2 + \Delta t\,\beta(x),$$
$$F_k(x) := \left[2 - \Delta t\,\beta(x)\right]u^k + \Delta t\,\varepsilon\frac{\partial^2 u^k}{\partial x^2} - \Delta t\,\alpha(x)\,\frac{\partial u^k}{\partial x} + \Delta t(g^{k+1} + g^k),$$

the resulting second-order equation with regard to the space variable can be written as:

$$A(x)\,Y''_{k+1}(x) + B(x)\,Y'_{k+1}(x) + C(x)\,Y_{k+1}(x) = F_k(x), \quad 0 \le x \le x_f, \tag{8}$$

for $k = 0, 1, \ldots, K-1$. To solve linear equation (8), we need the given initial condition $u^0 = u_0(x)$ and its first- and second-order derivatives. This implies that when $k = 0$, we have $Y_0(x) = u_0(x)$ for $0 \le x \le x_f$. At each time level t_{k+1}, we exploit the boundary conditions obtained from (3) at $x = 0, x_f$ as follows:

$$Y_{k+1}(0) := b_0^{k+1} = b_0(t_{k+1}), \quad Y_{k+1}(x_f) := b_f^{k+1} = b_f(t_{k+1}), \quad k = 0, 1, \ldots, K-1. \tag{9}$$

3. Special functions: basic matrix relations

We discretized parabolic convection-diffusion equation (1) with regard to time by relation (8) along with boundary conditions (9). Now, the task is to solve the initial-boundary value problem given in (8)-(9) with respect to the space variable x. To this end, we approximate the solutions $Y_{k+1}(x)$ in each time level as a combination of special functions defined by (4). We first assume that $\mathcal{Y}_{k,M}(x)$ is known as the calculated special approximation to $Y_k(x)$ at the time step t_k. Obviously, we have $Y_0(x)$ from the given initial condition $u_0(x)$. In the next time step t_{k+1}, we look for the approximate solution $\mathcal{Y}_{k+1,M}(x)$ for $k = 0, 1, \ldots, K-1$ in the form:

$$\mathcal{Y}_{k+1,M}(x) = \sum_{m=0}^{M} q_m^{k+1}\,\mathcal{S}_m(x), \quad x \in [0, x_f = 1]. \tag{10}$$

Here, q_m^{k+1}, $m = 0, 1, \dots, M$ unknown coefficients have to be found. To continue, we introduce

$$\boldsymbol{Q}_M^{k+1} = \begin{bmatrix} q_0^{k+1} & q_1^{k+1} & \dots & q_M^{k+1} \end{bmatrix}^T, \qquad \boldsymbol{\Sigma}_M(x) = \begin{bmatrix} \mathcal{S}_0(x) & \mathcal{S}_1(x) & \dots & \mathcal{S}_M(x) \end{bmatrix}.$$

Using the above vectors, $(M+1)$-term finite series (10) can be represented in the following matrix form:

$$\mathcal{Y}_{k+1,M}(x) = \boldsymbol{\Sigma}_M(x)\, \boldsymbol{Q}_M^{k+1}. \tag{11}$$

Next, we constitute the vector of monomials

$$\boldsymbol{X}_M(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^M \end{bmatrix},$$

as well as the lower triangular matrix $\boldsymbol{E}_M$ as follows:

$$\boldsymbol{E}_M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 3 & 3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{2^{1-M}\,(2M-2)!}{(M-1)!\,0!} & \frac{2^{2-M}\,(2M-3)!}{(M-2)!\,1!} & \frac{2^{3-M}\,(2M-4)!}{(M-3)!\,2!} & \dots & 1 & 0 \\ \frac{2^{-M}\,(2M)!}{(M)!\,0!} & \frac{2^{1-M}\,(2M-1)!}{(M-1)!\,1!} & \frac{2^{2-M}\,(2M-2)!}{(M-2)!\,2!} & \dots & \frac{2^{-1}\,(2M-(M-1))!}{1!\,(M-1)!} & 1 \end{pmatrix}_{(M+1)\times(M+1)}.$$

One can easily verify that $\boldsymbol{\Sigma}_M(x)$ can be written in terms of $\boldsymbol{X}_M(x)$ and $\boldsymbol{E}_M$ as

$$\boldsymbol{\Sigma}_M(x) = \boldsymbol{X}_M(x)\, \boldsymbol{E}_M^T. \tag{12}$$

It can be further shown that the derivatives of $\boldsymbol{X}_M(x)$ can be represented in terms of $\boldsymbol{X}_M(x)$ and the differentiation matrix $\boldsymbol{D}$ as

$$\frac{d^\ell}{dx^\ell}\boldsymbol{X}_M(x) = \boldsymbol{X}_M(x)\,(\boldsymbol{D}^T)^\ell, \quad \boldsymbol{D}^T = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & M \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(M+1)\times(M+1)}, \tag{13}$$

for $\ell = 1, 2$. Ultimately, we need a set of collocation points on $[0,1]$ to acquire an approximate solution of discretized model problem (8) in the form (10). In this respect, we divided $[0,1]$ uniformly into the collocation points:

$$x_\tau = \frac{\tau}{M}, \quad \tau = 0, 1, \dots, M. \tag{14}$$

4. Hybrid methodology

To proceed, our aim is to find matrix forms of all unknown terms Y_{k+1} and the first and second derivatives in (8). We first place relation (12) into (11). Therefore, our approximate solution (10) in the matrix expression is rewritten as:

$$\mathcal{Y}_{k+1,M}(x) = \boldsymbol{X}_M(x)\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}. \tag{15}$$

With the help of collocation points (14) and by putting them into (15) we get

$$\boldsymbol{Y}_{k+1} = \boldsymbol{P}\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}, \qquad \boldsymbol{Y}_{k+1} = \begin{pmatrix} \mathcal{Y}_{k+1,M}(x_0) \\ \mathcal{Y}_{k+1,M}(x_1) \\ \vdots \\ \mathcal{Y}_{k+1,M}(x_M) \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} \boldsymbol{X}_M(x_0) \\ \boldsymbol{X}_M(x_1) \\ \vdots \\ \boldsymbol{X}_M(x_M) \end{pmatrix}. \tag{16}$$

Then we differentiate (15) twice with regard to x. Hence, we utilize relation (13) for $\ell = 1, 2$ to approximate the first- and second-orders derivatives in (8) in matrix representation forms:

$$\begin{cases} Y'_{k+1}(x) \approx \mathcal{Y}^{(1)}_{k+1,M}(x) = \boldsymbol{X}_M(x)\,\boldsymbol{D}^T\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}, \\ Y''_{k+1}(x) \approx \mathcal{Y}^{(2)}_{k+1,M}(x) = \boldsymbol{X}_M(x)\,(\boldsymbol{D}^T)^2\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}. \end{cases} \tag{17}$$

Now, it is sufficient to insert collocation points into the former relations. Thus, the first and second derivatives in (17) can be expressed in the matrix forms:

$$\boldsymbol{Y}^{(1)}_{k+1} = \boldsymbol{P}\,\boldsymbol{D}^T\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}, \qquad \boldsymbol{Y}^{(1)}_{k+1} = \begin{pmatrix} \mathcal{Y}^{(1)}_{k+1,M}(x_0) \\ \mathcal{Y}^{(1)}_{k+1,M}(x_1) \\ \vdots \\ \mathcal{Y}^{(1)}_{k+1,M}(x_M) \end{pmatrix}, \tag{18}$$

$$\boldsymbol{Y}^{(2)}_{k+1} = \boldsymbol{P}\,(\boldsymbol{D}^T)^2\,\boldsymbol{E}_M^T\,\boldsymbol{Q}_M^{k+1}, \qquad \boldsymbol{Y}^{(2)}_{k+1} = \begin{pmatrix} \mathcal{Y}^{(2)}_{k+1,M}(x_0) \\ \mathcal{Y}^{(2)}_{k+1,M}(x_1) \\ \vdots \\ \mathcal{Y}^{(2)}_{k+1,M}(x_M) \end{pmatrix}. \tag{19}$$

By exploiting the approximations $\mathcal{Y}_{k+1,M}(x), \mathcal{Y}^{(1)}_{k+1,M}(x), \mathcal{Y}^{(2)}_{k+1,M}(x)$, we may rewrite (8) as:

$$A(x)\,\mathcal{Y}^{(2)}_{k+1,M}(x) + B(x)\,\mathcal{Y}^{(1)}_{k+1,M}(x) + C(x)\,\mathcal{Y}_{k+1,M}(x) = F_k(x), \quad 0 \le x \le 1. \tag{20}$$

Next, we put collocation points into (20) to arrive at the following system of equations

$$\boldsymbol{A}\boldsymbol{Y}^{(2)}_{k+1} + \boldsymbol{B}\boldsymbol{Y}^{(1)}_{k+1} + \boldsymbol{C}\boldsymbol{Y}_{k+1} = \boldsymbol{G}_k, \quad k = 0, 1, \ldots, K-1. \tag{21}$$

Here, the coefficient matrices $\boldsymbol{A}/\boldsymbol{B}/\boldsymbol{C}$ of size $(M+1)\times(M+1)$, and the vector $\boldsymbol{G}_k$ are as follows:

$$\boldsymbol{A}/\boldsymbol{B}/\boldsymbol{C}=\begin{pmatrix} A/B/C(x_0) & 0 & \dots & 0\\ 0 & A/B/C(x_1) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & A/B/C(x_M)\end{pmatrix},\ \boldsymbol{G}_k=\begin{pmatrix}F_k(x_0)\\ F_k(x_1)\\ \vdots\\ F_k(x_M)\end{pmatrix}_{(M+1)\times 1}.$$

After placing relations (16), (18)-(19) into (21), we get the following fundamental matrix equation for $k=0,1,\dots,K-1$:

$$\boldsymbol{Z}_k\,\boldsymbol{Q}_M^{k+1}=\boldsymbol{G}_k,\qquad [\boldsymbol{Z}_k;\boldsymbol{G}_k],\tag{22}$$

where

$$\boldsymbol{Z}_k:=\left\{\boldsymbol{A}\,\boldsymbol{P}\,(\boldsymbol{D}^T)^2+\boldsymbol{B}\,\boldsymbol{P}\,\boldsymbol{D}^T+\boldsymbol{C}\,\boldsymbol{P}\right\}\,\boldsymbol{E}_M^T.$$

It can be clearly seen that matrix equation (22) is a set of $(M+1)$ linear equations in terms of $(M+1)$ unknown coefficients $q_0^{k+1},q_1^{k+1},\dots,q_M^{k+1}$ to be determined. In order to solve (22), any classical solver can be utilized.

We finally need to incorporate boundary conditions (9) into the fundamental matrix equation (22). With the help of representation (15), the boundary conditions $\mathcal{Y}_{k+1,M}(0)=b_0^{k+1}$ and $\mathcal{Y}_{k+1,M}(1)=b_f^{k+1}$ are written in the matrix notations:

$$\begin{aligned}\widehat{\boldsymbol{Z}}_k^0\,\boldsymbol{Q}_M^{k+1}&=b_0^{k+1}, & \widehat{\boldsymbol{Z}}_k^0&:=\boldsymbol{X}_M(0)\,\boldsymbol{E}_M^T=[\hat{z}_0^0\quad \hat{z}_1^0\quad \dots\quad \hat{z}_M^0],\\ \widehat{\boldsymbol{Z}}_k^1\,\boldsymbol{Q}_M^{k+1}&=b_f^{k+1}, & \widehat{\boldsymbol{Z}}_k^1&:=\boldsymbol{X}_M(1)\,\boldsymbol{E}_M^T=[\hat{z}_0^1\quad \hat{z}_1^1\quad \dots\quad \hat{z}_M^1].\end{aligned}$$

We finally replace the first and last rows of the augmented matrix $[\boldsymbol{Z}_k;\boldsymbol{F}_k]$ by the vectors $[\widehat{\boldsymbol{Z}}_k^0;b_0^{k+1}]$ and $[\widehat{\boldsymbol{Z}}_k^1;b_f^{k+1}]$. Therefore, the following modified linear system of equations is obtained:

$$\left[\widehat{\boldsymbol{Z}}_k;\widehat{\boldsymbol{G}}_k\right]=\begin{pmatrix} \hat{z}_0^0 & \hat{z}_1^0 & \hat{z}_2^0 & \hat{z}_3^0 & \dots & \hat{z}_M^0 & ; & b_0^{k+1}\\ z_0^1 & z_1^1 & z_2^1 & z_3^1 & \dots & z_M^1 & ; & F_k(x_1)\\ z_0^2 & z_1^2 & z_2^2 & z_3^2 & \dots & z_M^2 & ; & F_k(x_2)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots\\ z_0^{M-1} & z_1^{M-1} & z_2^{M-1} & z_3^{M-1} & \dots & z_M^{M-1} & ; & F_k(x_{M-1})\\ \hat{z}_0^1 & \hat{z}_1^1 & \hat{z}_1^1 & \hat{z}_3^1 & \dots & \hat{z}_M^1 & ; & b_f^{k+1}\end{pmatrix}.\tag{23}$$

Now, by solving the above linear system we are able to obtain the unknown special coefficients in (15).

5. Error analysis

In this part, we state two theorems regarding error analysis for the solutions obtained by the special function method.

Theorem 1. *We suppose that* $y_{k+1}(x)$ *and* $\mathcal{Y}_{k+1,M}(x) = \boldsymbol{\Sigma}_M(x)\boldsymbol{Q}_M^{k+1}$ *are the exact solution and the special polynomial solution by relation* (15) *of time-discretized equation* (8) *for problem* (1). *Let* $y_{k+1,M}^{Mac}(x) = \boldsymbol{X}_M(x)\widetilde{\boldsymbol{Q}_M^{k+1}}$ *be the expansion of the Maclaurin series with the* M*-th degree of* $y_{k+1}(x)$ *in* $[0,1]$. *Then, the errors* $e_{k+1,M}(x) := y_{k+1}(x) - \mathcal{Y}_{k+1,M}(x)$ *of the special polynomial solution* $\mathcal{Y}_{k+1,M}(x)$ *at* t_{k+1} *are bounded:*

$$\|e_{k+1,M}(x)\|_\infty \leq \frac{1}{(M+1)!}\|y_{k+1}^{(M+1)}(c_x)\|_\infty + \|\widetilde{\boldsymbol{Q}_M^{k+1}}\|_\infty + \|\boldsymbol{E}_M^T\|_\infty \|\boldsymbol{Q}_M^{k+1}\|_\infty, \quad (24)$$

where $\boldsymbol{X}_M(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^M \end{bmatrix}$, $\boldsymbol{Q}_M^{k+1}$ *shows the coefficient matrix of the special polynomial solution* $\mathcal{Y}_{k+1,M}(x)$, *and the matrix* $\boldsymbol{E}_M$ *is as defined in* (12).

Proof. With the help of the triangle inequality and the Maclaurin expansion $y_{k+1,M}^{Mac}(x)$ with the M-th degree, the error $\|e_{k+1,M}(x)\|_\infty$ can be written as follows:

$$\begin{aligned} \|e_{k+1,M}(x)\|_\infty &= \|y_{k+1}(x) - y_{k+1,M}^{Mac}(x) + y_{k+1,M}^{Mac}(x) - \mathcal{Y}_{k+1,M}(x)\|_\infty \\ &\leq \|y_{k+1}(x) - y_{k+1,M}^{Mac}(x)\|_\infty + \|y_{k+1,M}^{Mac}(x) - \mathcal{Y}_{k+1,M}(x)\|_\infty. \quad (25) \end{aligned}$$

From (10) we know that the special polynomial solution $\mathcal{Y}_{k+1,M}(x) = \boldsymbol{\Sigma}_M(x)\boldsymbol{Q}_M^{k+1}$ can be expressed by the matrix form $\mathcal{Y}_{k+1,M}(x) = \boldsymbol{X}_M(x)\boldsymbol{E}_M^T\boldsymbol{Q}_M^{k+1}$. Here, $\boldsymbol{\Sigma}_M(x)= [S_0(x) \quad S_1(x) \quad \dots \quad S_M(x)]$. It is also known that the expansion of the Maclaurin series by the M-th degree of $y_{k+1}(x)$ is $y_{k+1,M}^{Mac}(x) = \boldsymbol{X}_M(x)\widetilde{\boldsymbol{Q}_M^{k+1}}$. Hence, the following inequality can be written:

$$\begin{aligned} \|y_{k+1,M}^{Mac}(x) - \mathcal{Y}_{k+1,M}(x)\|_\infty &= \|\boldsymbol{X}_M(x)\,\widetilde{\boldsymbol{Q}_M^{k+1}} - \boldsymbol{X}_M(x)\boldsymbol{E}_M^T\boldsymbol{Q}_M^{k+1}\|_\infty \\ &\leq \|\boldsymbol{X}_M(x)\left(\widetilde{\boldsymbol{Q}_M^{k+1}} - \boldsymbol{E}_M^T\boldsymbol{Q}_M^{k+1}\right)\|_\infty, \quad 0 \leq x \leq 1. \end{aligned} \quad (26)$$

Since $\|\boldsymbol{X}_M(x)\|_\infty \leq 1$ on $[0,1]$, we can arrange the last inequality (26) as:

$$\|y_{k+1,M}^{Mac}(x) - \mathcal{Y}_{k+1,M}(x)\|_\infty \leq \|\widetilde{\boldsymbol{Q}_M^{k+1}} - \boldsymbol{E}_M^T\boldsymbol{Q}_M^{k+1}\|_\infty. \quad (27)$$

On the other hand, by using the fact that the remainder term of the Maclaurin series $y_{k+1,M}^{Mac}(x)$ by the M-th degree is:

$$\|\sum_{m=M+1}^{\infty} \frac{y_{k+1}^{(m)}(0)}{m!}x^k\|_\infty = \frac{x^{M+1}}{(M+1)!}y_{k+1}^{(M+1)}(c_x), \quad 0 \leq c_x \leq 1,$$

we get the inequality

$$\|y_{k+1}(x) - y_{k+1,M}^{Mac}(x)\|_\infty \leq \frac{1}{(M+1)!}\|y_{k+1}^{(M+1)}(c_x)\|_\infty. \quad (28)$$

Lastly, by combining the upper bounds (25), (27) and (28), we have the following inequality:

$$\|e_{k+1,M}(x)\|_\infty \le \frac{1}{(M+1)!}\|y_{k+1}^{(M+1)}(c_x)\|_\infty + \|\widetilde{\boldsymbol{Q}_M^{k+1}} - \boldsymbol{E}_M^T\boldsymbol{Q}_M^{k+1}\|_\infty, \quad 0 \le x \le 1. \tag{29}$$

The proof is completed by applying the triangle inequality to (29). □

The next theorem provides an upper bound for the difference between $e_{k+1,M}(x)$ and $e_{k+1,M+1}(x)$ in the infinty norm. To this end, let us define

$$E_{k+1,M} := \|e_{k+1,M}(x)\|_\infty - \|e_{k+1,M+1}(x)\|_\infty, \quad k = 0, 1, \dots, K-1.$$

Theorem 2. *Let $y_{k+1}(x)$, $\mathcal{Y}_{k+1,M}(x)$ and $\mathcal{Y}_{k+1,M+1}(x)$ be the exact solution, the M-th and the $(M+1)$-th degree special polynomial solutions of equation* (8)*, respectively. The difference of the error norms for the solutions $\mathcal{Y}_{k+1,M}(x)$ and $\mathcal{Y}_{k+1,M+1}(x)$ is bounded by*

$$|E_{k+1,M}| = C\|e_{k+1,*}(x)\|_\infty \le \|\boldsymbol{E}_{M+1}^T\|_\infty\|\widetilde{\boldsymbol{Q}_{M+1}^{k+1}} - \boldsymbol{Q}_{M+1}^{k+1}\|_\infty, \tag{30}$$

where C is a positive number in $(0,1)$, $\boldsymbol{Q}_{M+1}^{k+1}$ and $\boldsymbol{Q}_M^{k+1}$ show the coefficient matrices in a special polynomial solution in (11)*, $\widetilde{\boldsymbol{Q}_{M+1}^{k+1}} = \begin{bmatrix}\boldsymbol{Q}_M^{k+1}\\ 0\end{bmatrix}$ and $\|e_{k+1,*}(x)\|_\infty = \max\{\|e_{k+1,M}(x)\|_\infty, \|e_{k+1,M+1}(x)\|_\infty\}$.*

Proof. From equation (11), we know that the solutions $\mathcal{Y}_{k+1,M}(x)$ and $\mathcal{Y}_{k+1,M+1}(x)$ can be written in the form:

$$\mathcal{Y}_{k+1,M}(x) = \boldsymbol{\Sigma}_M(x)\boldsymbol{Q}_M^{k+1}, \text{ and } \mathcal{Y}_{k+1,M+1}(x) = \boldsymbol{\Sigma}_{M+1}(x)\boldsymbol{Q}_{M+1}^{k+1}, \text{ respectively.}$$

Let $\widetilde{\boldsymbol{Q}_{M+1}^{k+1}}$ be the resultant vector when the zero element is added to the end of vector $\boldsymbol{Q}_M^{k+1}$. Then we can write

$$\mathcal{Y}_{k+1,M}(x) = \boldsymbol{\Sigma}_M(x)\boldsymbol{Q}_M^{k+1} = \boldsymbol{\Sigma}_{M+1}(x)\widetilde{\boldsymbol{Q}_{M+1}^{k+1}}. \tag{31}$$

On the other hand, by using the triangle inequality, we can write

$$|E_{k+1,M}| \le \|\mathcal{Y}_{k+1,M}(x) - \mathcal{Y}_{k+1,M+1}(x)\|_\infty. \tag{32}$$

By utilizing equation (31), we have:

$$\|\mathcal{Y}_{k+1,M}(x) - \mathcal{Y}_{k+1,M+1}(x)\|_\infty = \|\boldsymbol{\Sigma}_{M+1}(x)\left(\widetilde{\boldsymbol{Q}_{M+1}^{k+1}} - \boldsymbol{Q}_{M+1}^{k+1}\right)\|_\infty. \tag{33}$$

We know that $\|\boldsymbol{X}_{M+1}(x)\|_\infty \le 1$ on $[0,1]$ and that $\boldsymbol{\Sigma}_{M+1}(x) = \boldsymbol{X}_{M+1}(x)\boldsymbol{E}_{M+1}^T$ from equation (12). Then, equation (33) can be written as:

$$\|\mathcal{Y}_{k+1,M}(x) - \mathcal{Y}_{k+1,M+1}(x)\|_\infty \le \|\boldsymbol{E}_{M+1}^T\|_\infty\|\widetilde{\boldsymbol{Q}_{M+1}^{k+1}} - \boldsymbol{Q}_{M+1}^{k+1}\|_\infty. \tag{34}$$

Due to $\|e_{k+1,*}(x)\|_\infty = \max\{\|e_{k+1,M}(x)\|_\infty, \|e_{k+1,M+1}(x)\|_\infty\}$; then there is a positive number C in $(0,1)$ such that

$$|E_{k+1,M}| = C\|e_{k+1,*}(x)\|_\infty. \tag{35}$$

Finally, the proof is completed when relations (32), (34), and (35) are combined. □

6. Simulations results

We demonstrate the efficiency and accuracy of the presented hybrid approach applied to the model problem (1)-(3) on various benchmark examples and compare them to computational procedures described earlier. Matlab plots have been presented to illustrate the main results derived in this paper. Our calculations were performed by using Matlab 2021a software on a personal laptop with a 2.2 GHz Intel Core i7-10870H CPU machine with 1 TB of memory.

Test case 1. *We first consider* (1) *using the following coefficients and the right-hand side of [29, 30]*

$$\alpha(x) = 2x + 1, \quad \beta(x) = x^2, \quad g(x,t) = (x^2 + 2x + 2 - \varepsilon)\, e^{t+x}/\varepsilon.$$

The initial and boundary conditions are:

$$u(x,0) = e^x/\varepsilon, \quad u(0,t) = e^t/\varepsilon, \quad u(1,t) = e^{1+t}/\varepsilon.$$

In this case, the exact analytical solution takes the form:

$$u(x,t) = e^{t+x}/\varepsilon.$$

We first consider $\Delta t = 0.01$, $\varepsilon = 0.1$ and $t_f = 1$. Utilizing (10) with $M = 5$, the following approximate solutions at three different time steps $t = \Delta t$, $t = t_f/2$, and $t = t_f$ for $0 \leq x \leq 1$ are obtained:

$$\begin{aligned}
\mathcal{Y}_{1,5}(x) =& 0.14152651\, x^5 + 0.34857723\, x^4 + 0.34857723\, x^3 + 5.0399108\, x^2 \\
&+ 10.101416\, x + 10.100502, \\
\mathcal{Y}_{50,5}(x) =& 0.26819065\, x^5 + 0.47789125\, x^4 + 2.897853\, x^3 + 8.1912749\, x^2 \\
&+ 16.494468\, x + 16.487213,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Y}_{100,5}(x) =& 0.43639831\, x^5 + 0.80116732\, x^4 + 4.7661671\, x^3 + 13.509883\, x^2 \\
&+ 27.194127\, x + 27.182818.
\end{aligned}$$

The profile of the approximate solution using these parameters is shown in Fig. 1. In Fig. 2, we show the achieved absolute errors

$$E_k(x) := |u(x,t_k) - \mathcal{Y}_{k,5}(x)|, \quad k = 1, \ldots, K,$$

for $x \in [0,1]$, $\Delta t = 0.01$, $t_f = 1$ evaluated to various time levels $t = s\Delta t$, $s = 1, 2, \ldots, 100$.

To proceed, some comparisons are made in Table 1 to validate our computational results. In this respect, the error norms in L_2 and L_∞ at the final time $t = t_f$ are calculated via

$$L_2 := \sqrt{\frac{1}{M+1}\int_0^1 [u(x,T) - \mathcal{Y}_{K,M}(x)]^2 dx}, \quad L_\infty := \max_{0 \leq x \leq 1} |u(x,T) - \mathcal{Y}_{K,M}(x)|.$$

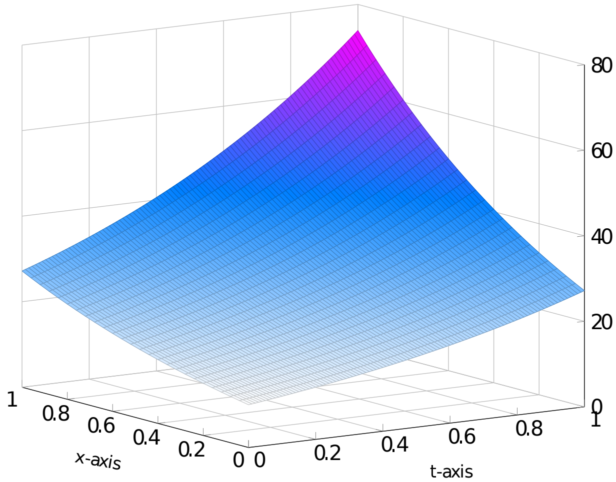

Figure 1: *Graph of the approximated solution in test case 1 using $\Delta t = 0.01$, $t_f = 1$, $\varepsilon = 0.1$, and $M = 5$*

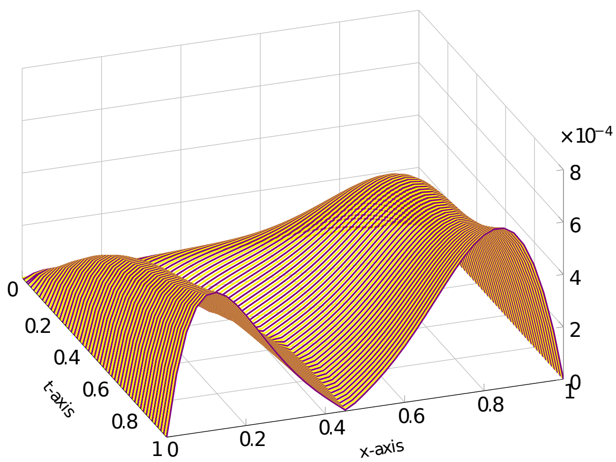

Figure 2: *Graph of the absolute error in test case 1 using $\Delta t = 0.01$, $t_f = 1$, $\varepsilon = 0.1$, and $M = 5$*

In Table 1, the achieved errors by the presented hybrid approach are evaluated at $t_f = 1$ and for $x \in [0, 1]$. We use $M = 5$ and the time steps are $\Delta t = 0.01$ and 0.001 in all computations. In addition, similar results obtained via the Bessel collocation method (BCM) [29] and the Galerkin weighted residual method (GWRM) [30] utilizing the same computational parameters are tabulated in Table 1. Obviously, the results show the ability of the proposed technique with less computational efforts and better performance than two existing well-established schemes. To confirm this fact, a comparison with GWRM with respect to required CPU time (in seconds) is presented in Table 2. Here, we utilize $\Delta t = 0.02$, $M = 5$, and diverse values of $\varepsilon = \frac{1}{10^i}$, $i = 1, 2, 3, 4$ are as in Table 1. It can be clearly observed that the performance of the proposed hybrid technique is better than GWRM [30]. Note further that our hybrid matrix method is easier to be implemented in practice.

Table 1: *Comparison results of L_∞ and L_2 error norms for test case 1 using $M = 5$, $\Delta t = 0.01, 0.001$, $t_f = 1$, and various ε*

	$\Delta t = 10^{-2}$		$\Delta t = 10^{-3}$		BCM [30]		GWRM [29]	
ε	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
10^{-1}	6.1125_{-4}	1.5199_{-4}	3.7094_{-4}	1.0841_{-4}	9.6181_{-4}	1.9982_{-4}	1.9640_{-3}	5.0046_{-4}
10^{-2}	9.7979_{-3}	2.3087_{-3}	4.2413_{-3}	1.5312_{-3}	6.0181_{-3}	1.7535_{-3}	4.3049_{-2}	7.0597_{-3}
10^{-3}	1.0256_{-1}	2.5898_{-2}	4.4547_{-2}	1.6011_{-2}	6.3998_{-2}	1.7367_{-2}	4.7793_{-1}	8.4708_{-2}
10^{-4}	1.0305_{+0}	2.6225_{-1}	4.5044_{-1}	1.6087_{-1}	6.5455_{-1}	1.7352_{-1}	4.8544_{+0}	8.6685_{-1}

Table 2: *Comparison results of L_2 error norms and required CPU time (in seconds) for test case 1 using $M = 5$, $\Delta t = 0.02$, $t_f = 1$, and various ε*

	Present ($\Delta t = 2 \times 10^{-2}$)		GWRM [29]	
ε	L_2	CPU [s]	L_2	CPU [s]
10^{-1}	6.8827539_{-4}	5.0025	5.0046_{-4}	7.8753
10^{-2}	6.3628758_{-3}	4.9949	7.0597_{-3}	7.8603
10^{-3}	6.9269659_{-2}	4.9920	8.4708_{-2}	7.9880
10^{-4}	7.0034490_{-1}	4.9989	8.6685_{-1}	8.0860

Next, we examine the effect of increasing M on the error norms. To this end, we fix $\Delta t = 0.01$, $t_f = 1$, and utilize two different $\varepsilon = 1$ and $\varepsilon = 10^{-2}$. The results of L_2/L_∞ error norms versus the number of bases are shown in Fig. 3. We use $M = 1, 2, 4, 8$ in these experiments. It can be obviously seen that for a fixed value of ε, the magnitude of errors is decreased. Additionally, note that the error norms increase as ε goes to zero. This is due to the fact that the relevant coefficients in the underlying equation approache zero and then the model problem becomes inherently singular. Therefore, the accuracy of the spectral method is usually be deteriorated. Indeed, this observation is also available in other methods for singular perturbed problems, see cf. [18, 24], and references therein.

Test case 2. *As the second test problem, we consider*

$$\alpha(x) = 2 - x^2, \quad \beta(x) = x, \quad g(x,t) = 10x(1-x)\, t^2 e^{-t}.$$

The initial and boundary conditions are taken as zero for $0 \le x, t \le 1$

$$u(x,0) = 0, \quad u(0,t) = 0, \quad u(1,t) = 0.$$

No exact solution is known for this model problem.

Let us first consider $\Delta t = 0.01$, $t_f = 1$, $\varepsilon = 2^{-2}$, and $M = 5$. The approximated

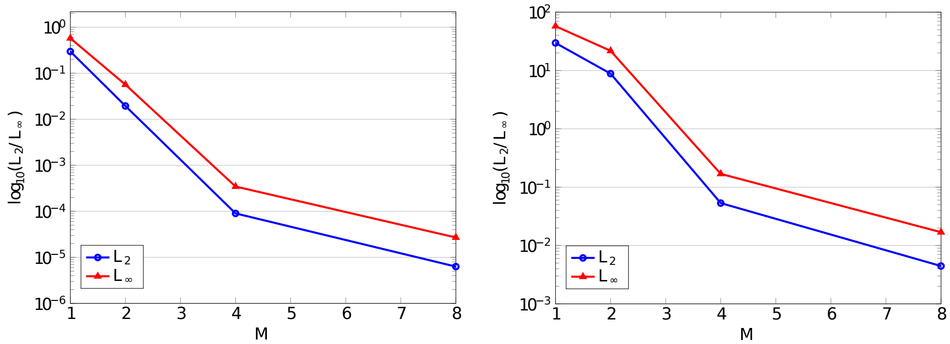

Figure 3: *Plot of L_2/L_∞ error norms using $\varepsilon = 1$ (left) and $\varepsilon = 10^{-2}$ (right) with $\Delta t = 0.001$, $t_f = 1$, and various M in test case 1*

solutions at $t = 2\Delta t$, $t = t_f/2$, and $t = t_f$ take the forms:

$$\begin{aligned}\mathcal{Y}_{1,5}(x) =& -6.3361 \times 10^{-6}\,x^5 + 1.8148 \times 10^{-5}\,x^4 - 2.0170 \times 10^{-5}\,x^3 \\ & - 9.1229 \times 10^{-6}\,x^2 + 0.1249951626\,x + 1.7480 \times 10^{-5}, \\ \mathcal{Y}_{10,5}(x) =& -0.06961376\,x^5 + 0.1806816\,x^4 - 0.39029857\,x^3 + 0.23039179\,x^2 \\ & + 0.048838945\,x + 4.3601509 \times 10^{-106},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y}_{10,5}(x) =& -0.088708544\,x^5 - 0.088708544\,x^4 - 0.59782978\,x^3 + 0.57230508\,x^2 \\ & + 0.048838945\,x + 0.15795965.\end{aligned}$$

We note that for $t = \Delta t$, the approximated solution is zero due to the given initial and boundary conditions. The snapshots of numerical solutions at all time steps $t = s\Delta t$ for $s = 1, \ldots, 100$ are visualized in Fig. 4. As previously mentioned that exact solution of this test case is not available. To measure the accuracy of the present hybrid method, we define the residual error function related to our discretized equation (8). To this end, we calculate the following error function at time level t_{k+1} as:

$$\mathcal{R}_{k+1,M}(x) := A(x)\,\mathcal{Y}^{(2)}_{k+1,M}(x) + B(x)\,\mathcal{Y}^{(1)}_{k+1,M}(x) + C(x)\,\mathcal{Y}_{k+1,M}(x) - F_k(x),\ 0 \le x \le 1. \tag{36}$$

Using the same parameters used in Fig. 4, we compute the corresponding error functions (36). The profile of the error is depicted in Fig. 5.

Let us fix $M = 4$ and see the behavior of error norms when ε varies. In addition, we fix $t_f = 1$ and compute L_2/L_∞ norms of the residual error function at the final time as:

$$R_2 := \|\mathcal{R}_{K,M}(x)\|_2, \quad R_\infty := \max_{x \in [0,1]} |\mathcal{R}_{K,M}(x)|.$$

The results of R_2/R_∞ error norms are reported in Table 3. For comparison, the outcomes of the previously avaliable computational schemes are also displayed in

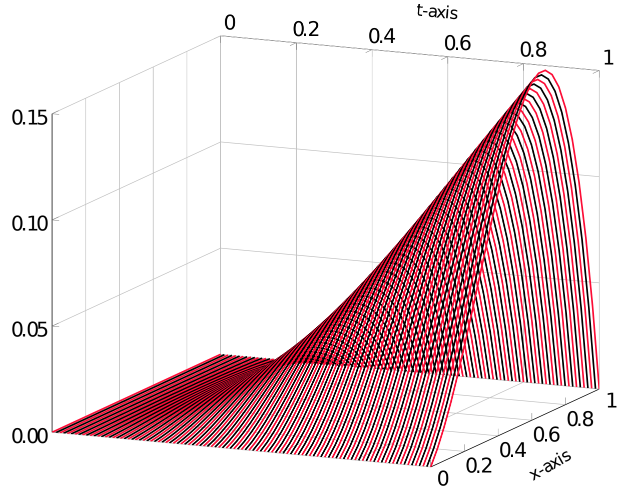

Figure 4: *Graph of numerical solutions in test case 2 at different time instants* $t = s\Delta t$, $s = 1, 2, \ldots, 100$ *for* $\Delta t = 0.01, M = 5$, $\varepsilon = 2^{-2}$, *and* $t_f = 1$

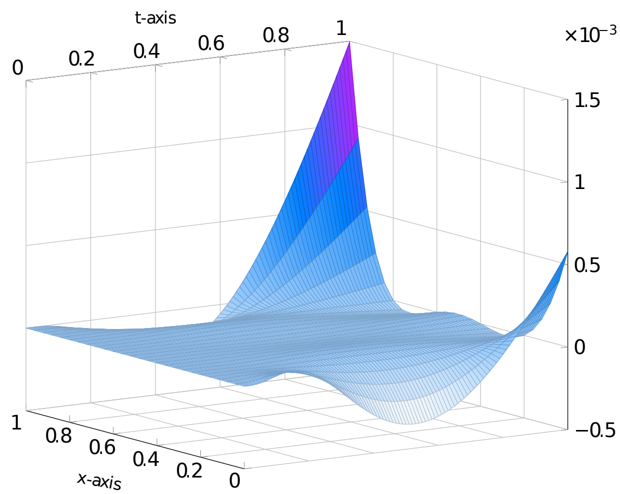

Figure 5: *Graph of residual error functions in test case 2 at different time instants* $t = s\Delta t$, $s = 1, 2, \ldots, 100$ *for* $\Delta t = 0.01, M = 5$, $\varepsilon = 2^{-2}$, *and* $t_f = 1$

Table 3. In this respect, we utilize the BCM [30], GWRM [29], the B-spline collocation method (BSCM) [18] and the piecewise analytical method (PAM) [24]. It can be seen that our numerical results are more accurate while employing a smaller number of bases.

Finally, for the second test case we depict numerical solutions for two other different smaller values of $\varepsilon = 2^{-4}$ and $\varepsilon = 2^{-8}$. For the simulations, we employ $M = 5$, $\Delta t = 0.01$, and $t_f = 1$. The profiles of numerical solutions are shown in Fig. 6. Related residual error functions are further visualized in Fig. 6.

Table 3: *Comparison results of L_∞ and L_2 error norms for test case 1 using $M = 4$, $\Delta t = 0.01$, $t_f = 1$, and various ε*

	Present		BCM [30]	GWRM [29]	BSCM [18]		PAM [24]	
ε	R_∞	R_2	$N=4$	$N=4$	$N=16$	$N=32$	$N=16$	$N=32$
2^{-2}	2.8048_{-3}	3.0067_{-4}	1.090_{-3}	2.723_{-3}	2.030_{-2}	1.113_{-2}	2.60_{-3}	9.92_{-4}
2^{-4}	3.7054_{-2}	4.8718_{-3}	1.141_{-2}	2.630_{-2}	2.810_{-2}	1.857_{-2}	1.15_{-2}	5.10_{-3}
2^{-6}	5.9778_{-2}	8.0477_{-3}	1.187_{-2}	6.464_{-2}	3.048_{-1}	1.275_{-1}	2.25_{-2}	1.67_{-2}
2^{-8}	6.7140_{-2}	9.0684_{-3}	2.909_{-2}	4.001_{-2}	8.395_{-1}	4.648_{-1}	1.52_{+2}	1.44_{-2}

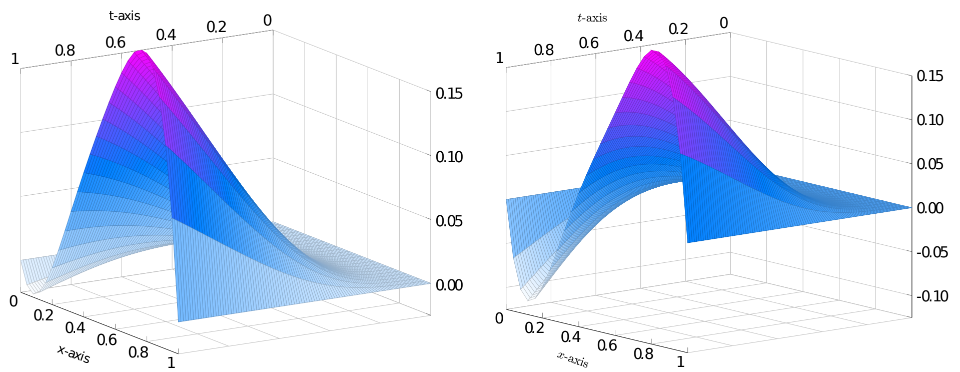

Figure 6: *Graphs of the numerical solution with $\varepsilon = 2^{-4}$ (left) and $\varepsilon = 2^{-8}$ (right) in test case 2 for $\Delta t = 0.01, M = 5$, and $t_f = 1$*

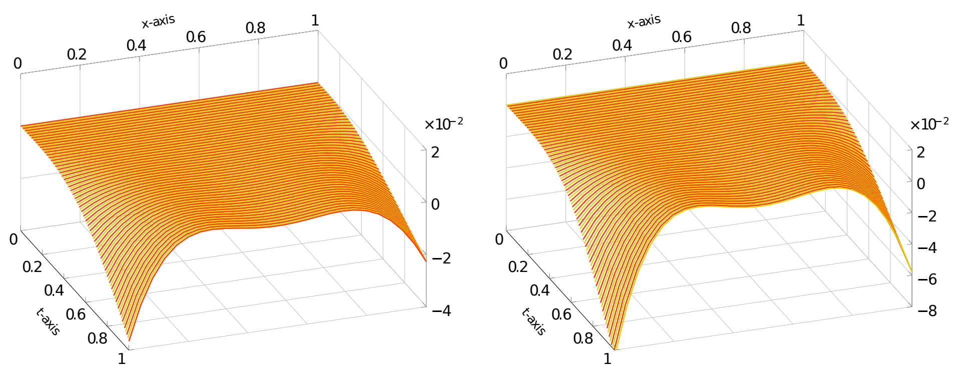

Figure 7: *Graphs of residual errors with $\varepsilon = 2^{-4}$ (left) and $\varepsilon = 2^{-8}$ (right) in test case 2 for $\Delta t = 0.01, M = 5$, and $t_f = 1$*

7. Conclusions

In this study, we have developed a collocation method based upon the hybrid of the novel special polynomials and the Taylor series formula for the numerical solution of convection-diffusion of parabolic type equations encountered in diverse disciplines of engineering science. The main feature of the presented work is that we need to solve an algebraic system of equations in each time step rather than a global system obtained in earlier spectral collocation methods. Error analysis of the hybrid technique is stated and proved. Numerical experiments are summarized in figures and tables show the accuracy and efficiency of the proposed approach in comparison with some available published schemes.

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