Math. Commun. **27**(2022), 63–75

A nonstandard construction of direct limit group actions*

Takuma Imamura†

Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

Received May 22, 2021; accepted February 2, 2022

Abstract. Manevitz and Weinberger (1996) proved that the existence of effective K-Lipschitz $\mathbb{Z}/n\mathbb{Z}$-actions implies the existence of effective K-Lipschitz $\mathbb{Q}/\mathbb{Z}$-actions for all compact connected manifolds with metrics, where K is a fixed Lipschitz constant. The $\mathbb{Q}/\mathbb{Z}$-actions were constructed from suitable actions of a sufficiently large hyperfinite cyclic group ${}^*\mathbb{Z}/\gamma\,{}^*\mathbb{Z}$ in the sense of nonstandard analysis. By modifying their construction, we prove that for every direct system $(\Lambda, G_\lambda, i_{\lambda\mu})$ of torsion groups with monomorphisms, the existence of effective K-Lipschitz G_λ-actions implies the existence of effective K-Lipschitz $\varinjlim G_\lambda$-actions. This generalises Manevitz and Weinberger's result.

AMS subject classifications: 54H15, 54J05, 18A30, 20A15

Key words: Group actions, direct limits of groups, locally finite groups, nonstandard analysis

1. Introduction

Let M be a compact connected manifold with a metric. Using nonstandard analysis, Manevitz and Weinberger [18] proved that if M admits an effective K-Lipschitz action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ for each $n \in \mathbb{Z}_+$, then M also admits an effective K-Lipschitz action of the rational circle group $\mathbb{Q}/\mathbb{Z}$. (The standard approach to this type of result with applications can be found in [22].) A sketch of the proof is as follows: let γ be an infinite hyperinteger that is divisible by all non-zero integers (e.g. the factorial $\omega!$ of an arbitrary positive infinite hyperinteger ω). Then $\mathbb{Q}/\mathbb{Z}$ can be embedded into the hyperfinite cyclic group ${}^*\mathbb{Z}/\gamma\,{}^*\mathbb{Z}$ by identifying $k/n+\mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ with $k\,(\gamma/n)+\gamma\,{}^*\mathbb{Z} \in {}^*\mathbb{Z}/\gamma\,{}^*\mathbb{Z}$. By the transfer principle, the nonstandard extension ${}^*M$ admits an internal effective K-Lipschitz action of ${}^*\mathbb{Z}/\gamma\,{}^*\mathbb{Z}$. By restricting the domain and taking its standard part, we obtain the desired $\mathbb{Q}/\mathbb{Z}$-action on M. The effectiveness of the resulting action follows from Newman's theorem in the version of [1, III.9.6 Corollary]. Their proof requires no advanced knowledge of transformation group theory. However, their proof contains an error involving the use of the downward transfer principle (see the footnote in the proof of Theorem 6). Fortunately, their proof can be corrected, as we shall see below.

Considering that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct limit of $\mathbb{Z}/n\mathbb{Z}$ $(n \in \mathbb{Z}_+)$, it is natural to attempt to generalise Manevitz and Weinberger's result to direct limits of

*This work was partly supported by the Morikazu Ishihara (Shikata) Research Encouragement Fund and by JST ERATO HASUO Metamathematics for Systems Design Project (No. JPMJER1603).
†Corresponding author. *Email address:* `timamura@kurims.kyoto-u.ac.jp` (T. Imamura)

a more general class of groups. In Section 2, we recall some results from nonstandard analysis and topology. In Section 3, we prove that for every direct system $\mathbf{G} := (\Lambda, G_\lambda, i_{\lambda\mu})$ of torsion groups with monomorphisms, if M admits an effective K-Lipschitz G_λ-action for each $\lambda \in \Lambda$, then M also admits an effective K-Lipschitz $\varinjlim \mathbf{G}$-action. The result on $\mathbb{Q}/\mathbb{Z}$-actions is an immediate corollary to our result. One can also obtain the following corollary on $\mathbb{Z}\left[1/p\right]/\mathbb{Z}$-actions: if the cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$ $(n \in \mathbb{N})$ effectively act on M by K-Lipschitz maps, then the p-Prüfer group $\mathbb{Z}\left[1/p\right]/\mathbb{Z}$ does as well. In Section 4, we conclude the paper by mentioning related works involving nonstandard approximations of direct and inverse limits.

2. Preliminaries

First of all, we recall the model-theoretic framework of nonstandard analysis (NSA). We refer to [20, 3, 16] for model-theoretic NSA and [12] for axiomatic NSA. The reader is assumed to be familiar with the rudiments of mathematical logic. NSA uses the following two universes:

1. The standard universe $(\mathbb{U}, \in)$. Assume the following:

   Transitivity The underlying set $\mathbb{U}$ is a transitive set, i.e. $x \in \mathbb{U}$ implies $x \subseteq \mathbb{U}$.

   Richness All standard mathematical objects we need (such as groups and manifolds that appear in this paper) belong to $\mathbb{U}$.

   Absoluteness All (but finitely many) set-theoretic formulae we need are absolute with respect to $\mathbb{U}$. In other words, given a set-theoretic formula $\varphi(\vec{x})$ that appears in this paper (such as "U is an open set of X" and "f is continuous at x") and parameters $\vec{a} \in \mathbb{U}$, the sentence $\varphi(\vec{a})$ is true in $\mathbb{U}$ (by interpreting $\forall$ and $\exists$ as quantifiers over $\mathbb{U}$) if and only if $\varphi(\vec{a})$ is actually true (by interpreting $\forall$ and $\exists$ as quantifiers over all mathematical objects). In particular, all axioms of ZFC we need are true in $\mathbb{U}$.

   While the existence of such a universe $\mathbb{U}$ is provable in ZFC by the reflection principle, the reader familiar with category theory may consider $\mathbb{U}$ as a Grothendieck universe. (The absoluteness holds for all bounded formulae in this case.) A more clever approach can be found in [6].

2. The nonstandard universe $({}^*\mathbb{U}, {}^*\!\in)$ with the embedding ${}^*(-) \colon \mathbb{U} \hookrightarrow {}^*\mathbb{U}$ satisfying the following principles:

   Transfer For any sentence $\varphi(\vec{a})$ with parameters $\vec{a}$ in $\mathbb{U}$, $\varphi(\vec{a})$ is true in $\mathbb{U}$ if and only if $\varphi({}^*\vec{a})$ is true in ${}^*\mathbb{U}$. The "only if" part is referred to as *upward transfer*. The "if" part is referred to as *downward transfer*.

   κ-saturation Let κ be a *fixed* infinite cardinal. Let $p(\vec{x})$ be a set of formulae with variables $\vec{x}$ and parameters in ${}^*\mathbb{U}$. Suppose that $|p(\vec{x})| < \kappa$. If every finite subset $q(\vec{x})$ of $p(\vec{x})$ has a solution in ${}^*\mathbb{U}$, the whole $p(\vec{x})$ has a solution in ${}^*\mathbb{U}$. The limitation of the cardinality of $p(\vec{x})$ cannot be relaxed, because the unlimited saturation principle leads to a contradiction. To prove the main results of this paper, we only need to assume the following weaker principle.

Weak saturation Let $p(\vec{x})$ be a set of formulae with variables $\vec{x}$ and parameters in ${}^*\mathbb{U}$. Suppose that each parameter of $p(\vec{x})$ belongs to the image of the embedding ${}^*(-)$. If every finite subset $q(\vec{x})$ of $p(\vec{x})$ has a solution in ${}^*\mathbb{U}$, the whole $p(\vec{x})$ has a solution in ${}^*\mathbb{U}$.

See [3] for the construction of ${}^*\mathbb{U}$.

A mathematical object is said to be *standard* (or $\mathbb{U}$-*small* in terminology of category theory) if it is an element of $\mathbb{U}$; *internal* if it is an element of ${}^*\mathbb{U}$; and *external* if it is not internal. Given a concept X on $\mathbb{U}$ defined by a formula $\varphi(\vec{x}, \vec{a})$, the concept on ${}^*\mathbb{U}$ defined by the associated formula $\varphi(\vec{x}, {}^*\vec{a})$ is called *internal* X, *hyper* X, and ${}^*X$. We drop the star ${}^*(-)$ unless there is a risk of confusion. In particular, we identify the *membership relation ${}^*\in$ with the genuine membership relation $\in$.

Example 1 (The ordered field of hyperreals)**.** *Let K be an ordered field. We may assume without loss of generality that K is an extension of $\mathbb{Q}$. An element of K is called an* infinite *(with respect to $\mathbb{Q}$) if its absolute value is an upper bound of $\mathbb{Q}$. An element of K is called an* infinitesimal *if its absolute value is a lower bound of $\mathbb{Q}_+$. The ordered field K is said to be* non-Archimedean *if one of the following equivalent conditions holds: (i) it has an infinite; (ii) it has a non-zero infinitesimal.*

The property that $\mathbb{R}$ is an ordered field can be described as a formula. By transfer, ${}^*\mathbb{R}$ is an ordered field. The field ${}^*\mathbb{R}$ and its elements are called by the above convention the hyperreal field *and* hyperreal numbers*. The restriction of $\mathbb{U} \hookrightarrow {}^*\mathbb{U}$ gives an embedding of $\mathbb{R}$ into ${}^*\mathbb{R}$. Consider the set $p(x) = \{\text{“}x \in {}^*\mathbb{R}\text{”}\} \cup \{\text{“}a < |x|\text{”} \mid a \in \mathbb{Q}\}$ of formulae with one variable x (and parameters ${}^*\mathbb{R}$ and $a \in \mathbb{Q}$). Every finite subset of $p(x)$ is solvable in ${}^*\mathbb{U}$, so $p(x)$ is solvable in ${}^*\mathbb{U}$ by weak saturation. The solutions of $p(x)$ are precisely hyperreal numbers whose absolute values are greater than all (standard) rational numbers, i.e. infinites. Similarly, one can obtain non-zero infinitesimals by considering the set $q(x) = \{\text{“}x \in {}^*\mathbb{R}\text{”}\} \cup \{\text{“}x \neq 0\text{”}\} \cup \{\text{“}|x| < a\text{”} \mid a \in \mathbb{Q}_+\}$. Hence ${}^*\mathbb{R}$ is a non-Archimedean ordered field.*

We recall some fundamental results from nonstandard topology.

Definition 1 (see [20])**.** *Let (X, τ_X) be a standard topological space. For $x \in X$, the set $\mu_X(x) := \bigcap_{x \in U \in \tau_X} {}^*U$ is called the* monad *of x.*

Definition 2 (see [20])**.** *Let X be a standard metric space. For $x, y \in {}^*X$, we say that x and y are* infinitely close *($x \approx_X y$) if the *distance ${}^*d_X(x, y)$ is an infinitesimal.*

For a standard metric space X, the monad $\mu_X(x)$ of $x \in X$ is precisely the set of all points of ${}^*X$ infinitely close to x.

Lemma 1 (see [20])**.** *Let X be a standard topological space and $x \in X$. There exists a *open set U (i.e. a member of ${}^*\tau_X$) such that $x \in U \subseteq \mu_X(x)$.*

Proof. Apply weak saturation to the set

$$p(U) := \{\text{“}x \in U\text{”}, \text{“}U \in {}^*\tau_X\text{”}\} \cup \{\text{“}U \subseteq {}^*V\text{”} \mid x \in V \in \tau_X\}. \qquad \square$$

Theorem 1 (see [20]). *Let X be a standard topological space and $x \in X$. A subset U of X is a neighbourhood of x if and only if $\mu_X(x) \subseteq {}^*U$.*

Proof. The "only if" part is trivial by the definition of μ_X. To prove the "if" part, suppose that $\mu_X(x) \subseteq {}^*U$. By Lemma 1, there exists a $V \in {}^*\tau_X$ such that $x \in V \subseteq \mu_X(x) \subseteq {}^*U$, i.e. ${}^*U$ is a *neighbourhood of x. By downward transfer, U is a neighbourhood of x. □

Corollary 1 (see [20]). *Let X be a standard topological space. A subset U of X is an open set if and only if for all $x \in U$, $\mu_X(x) \subseteq {}^*U$.*

Corollary 2 (see [20]). *Let X be a standard topological space. A subset F of X is a closed set if and only if $\mu_X(x) \cap {}^*F \neq \varnothing$ implies $x \in F$ for all $x \in X$.*

Theorem 2 (see [20]). *A standard topological space X is Hausdorff if and only if $\mu_X(x) \cap \mu_X(y) = \varnothing$ for all distinct $x, y \in X$.*

Proof. Let $x, y \in X$. It suffices to show that x and y are separable by neighbourhoods if and only if $\mu_X(x) \cap \mu_X(y) = \varnothing$. Suppose x and y are separable by neighbourhoods U_x and U_y. By Theorem 1, $\mu_X(x) \cap \mu_X(y) \subseteq {}^*U_x \cap {}^*U_y = {}^*(U_x \cap U_y) = {}^*\varnothing = \varnothing$. Conversely, suppose $\mu_X(x) \cap \mu_X(y) = \varnothing$. There exist $U_x, U_y \in {}^*\tau_X$ such that $x \in U_x \subseteq \mu_X(x)$ and $y \in U_y \subseteq \mu_X(x)$ by Lemma 1. Note that $U_x \cap U_y \subseteq \mu_X(x) \cap \mu_Y(y) = \varnothing$. By downward transfer, there exist (standard) $U_x, U_y \in \tau_X$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$. □

Theorem 3 (see [20]). *A standard topological space X is compact if and only if ${}^*X = \bigcup_{x \in X} \mu_X(x)$.*

Proof. Suppose X is compact. Let $x \in {}^*X$. Consider the family of closed sets of X defined by

$$\mathcal{F} := \{\, F \subseteq X \mid F \text{ is closed and } x \in {}^*F \,\}.$$

For each $F_1, \ldots, F_n \in \mathcal{F}$, since $x \in {}^*F_1 \cap \cdots \cap {}^*F_n \neq \varnothing$, $F_1 \cap \cdots \cap F_n \neq \varnothing$ by downward transfer. The intersection $\bigcap \mathcal{F}$ has an element y by the compactness of X. Let U be an arbitrary open neighbourhood of y. Then $X \setminus U \notin \mathcal{F}$, i.e. $x \notin {}^*(X \setminus U) = {}^*X \setminus {}^*U$. Hence $x \in \mu_X(x)$, because U was arbitrary.

Suppose ${}^*X = \bigcup_{x \in X} \mu_X(x)$. Let $\{\, F_i \,\}_{i \in I}$ be a family of closed subsets with the finite intersection property. The intersection $\bigcap_{i \in I} {}^*F_i$ has an element $x \in {}^*X$ by weak saturation. Choose a $y \in X$ such that $x \in \mu_X(y)$. For each $i \in I$, since $x \in \mu_X(y) \cap {}^*F_i \neq \varnothing$, $y \in F_i$ by Corollary 2. Therefore $\bigcap_{i \in I} F_i$ is non-empty. □

Corollary 3 (see [20]). *If X is a standard compact Hausdorff space, there exists a unique map ${}^\circ(-)\colon {}^*X \to X$ (called the* standard part map*) such that $x \in \mu_X({}^\circ x)$.*

Theorem 4 (see [20]). *A standard map $f\colon X \to Y$ between topological spaces is continuous at $x \in X$ if and only if ${}^*f[\mu_X(x)] \subseteq \mu_Y(f(x))$.*

Proof. Suppose f is continuous at x. Let U be a neighbourhood of $f(x)$. Then $f^{-1}[U]$ is a neighbourhood of x. By Theorem 1, ${}^*f[\mu_X(x)] \subseteq {}^*f\left[{}^*f^{-1}[{}^*U]\right] \subseteq {}^*U$. Since U was arbitrary, ${}^*f[\mu_X(x)] \subseteq \mu_Y(f(x))$.

Conversely, suppose ${}^*f\left[\mu_X(x)\right] \subseteq \mu_Y(f(x))$. Let U be a neighbourhood of $f(x)$. By Lemma 1, there exists a $V \in {}^*\tau_X$ such that $x \in V \subseteq \mu_X(x)$. Then ${}^*f[V] \subseteq {}^*f\left[\mu_X(x)\right] \subseteq \mu_Y(f(x)) \subseteq {}^*U$ by Theorem 1. By downward transfer, there exists a (standard) $V \in \tau_X$ such that $f[V] \subseteq U$. Hence f is continuous at x. $\square$

3. Main results

3.1. Nonstandard approximations of direct limits

Let $\mathbf{G} := (\Lambda, G_\lambda, i_{\lambda\mu})$ be a standard direct system of groups and homomorphisms. A *cocone* over $\mathbf{G}$ consists of a group G and a homomorphism $j_\lambda \colon G_\lambda \to G$ for each $\lambda \in \Lambda$ which makes the following diagram commutative:

$$\begin{tikzcd} & G & \\ G_\lambda \arrow[ur, "j_\lambda"] \arrow[rr, "i_{\lambda\mu}"] & & G_\mu \arrow[ul, "j_\mu"'] \end{tikzcd}$$

Given two cocones (G, j_λ) and (H, k_λ) over $\mathbf{G}$, a *morphism* between them is a homomorphism $f \colon G \to H$ (of groups) such that the diagram

$$\begin{tikzcd} G \arrow[rr, "f"] & & H \\ & G_\lambda \arrow[ul, "j_\lambda"] \arrow[ur, "k_\lambda"'] & \end{tikzcd}$$

is commutative for all $\lambda \in \Lambda$. The collection of cocones over $\mathbf{G}$ and their morphisms forms a category. The initial object of this category is called the *colimiting cocone* over $\mathbf{G}$ and is denoted by $i_\lambda \colon G_\lambda \to \varinjlim \mathbf{G}$. The group $\varinjlim \mathbf{G}$ is called the *direct limit* of $\mathbf{G}$. The direct limit $\varinjlim \mathbf{G}$ can be constructed as the quotient of the disjoint union $\bigsqcup_{\lambda \in \Lambda} G_\lambda := \bigcup_{\lambda \in \Lambda} (\{\lambda\} \times G_\lambda)$ modulo the equivalence relation $\equiv$ defined by $(\lambda, g) \equiv (\mu, h)$ if and only if $i_{\lambda\nu}(g) = i_{\mu\nu}(h)$ for some $\nu \geq \lambda, \mu$.

Lemma 2. *There exists an index $\gamma \in {}^*\Lambda$ such that $\Lambda \leq \gamma$.*

Proof. Consider the set $p(x) := \{\text{“}x \in \Lambda\text{”}\} \cup \{\text{“}\lambda \leq \gamma\text{”} \mid \lambda \in \Lambda\}$ and apply weak saturation. $\square$

Theorem 5. *Let γ be as in Lemma 2. There exists an embedding $j \colon \varinjlim \mathbf{G} \to {}^*G_\gamma$ such that the following diagram is commutative for every $\lambda \in \Lambda$:*

$$\begin{tikzcd} \varinjlim \mathbf{G} \arrow[rr, "j"] & & {}^*G_\gamma \\ & G_\lambda \arrow[ul, "i_\lambda"] \arrow[ur, "{}^*i_{\lambda\gamma}"'] & \end{tikzcd}$$

Proof. The homomorphisms $j_\lambda := {}^*i_{\lambda\gamma} \upharpoonright G_\lambda \colon G_\lambda \to {}^*G_\gamma$ $(\lambda \in \Lambda)$ form a cocone over $\mathbf{G}$: $j_\mu \circ i_{\lambda\mu} = {}^*i_{\mu\gamma} \upharpoonright G_\mu \circ {}^*i_{\lambda\mu} \upharpoonright G_\lambda = {}^*i_{\lambda\gamma} \upharpoonright G_\lambda = j_\lambda$. By the universal mapping property, there exists a (unique) homomorphism $j\colon \varinjlim \mathbf{G} \to {}^*G_\gamma$ such that the above diagram is commutative. More specifically, given $g \in \varinjlim \mathbf{G}$, find a $\lambda \in \Lambda$ and a $g_\lambda \in G_\lambda$ such that $g = i_\lambda(g_\lambda)$, and then define $j(g) := {}^*i_{\lambda\gamma}(g_\lambda)$.

To prove the injectivity, let $g \in \ker j$. Choose $g_\lambda \in G_\lambda$ such that $g = i_\lambda(g_\lambda)$. Then, ${}^*i_{\lambda\gamma}(g_\lambda) = j(g) = e$ by definition. Hence there exists a $\mu \in {}^*\Lambda$ such that ${}^*i_{\lambda\mu}(g_\lambda) = e$. By downward transfer, there exists a $\mu \in \Lambda$ such that $i_{\lambda\mu}(g_\lambda) = e$. Therefore $g = e$. $\square$

As a corollary, we obtain another construction of direct limits.

Corollary 4. $\varinjlim \mathbf{G} \cong j\left[\varinjlim \mathbf{G}\right] = \bigcup_{\lambda\in\Lambda} {}^*i_{\lambda\gamma}[G_\lambda]$.

The above argument also applies to any other algebraic system in the sense of universal algebra which includes rings, gyrogroups, lattices and Heyting algebras.

Example 2 ([18]). *Let $\Lambda := \mathbb{Z}_+$, $G_\lambda := \mathbb{Z}/\lambda\mathbb{Z}$ and $i_{\lambda\mu}(k+\lambda\mathbb{Z}) := k(\mu/\lambda) + \mu\mathbb{Z}$, where the index set $\mathbb{Z}_+$ is ordered by the divisibility relation. Its direct limit is isomorphic to the rational circle group $\mathbb{Q}/\mathbb{Z}$. The canonical homomorphism $i_\lambda\colon \mathbb{Z}/\lambda\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is given by $k + \lambda\mathbb{Z} \mapsto k/\lambda + \mathbb{Z}$. Let γ be a positive hyperinteger divisible by all non-zero (standard) integers, e.g. the factorial $\omega!$ of an infinite hypernatural number $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. The direct limit $\mathbb{Q}/\mathbb{Z}$ is then embedded into ${}^*\mathbb{Z}/\gamma\,{}^*\mathbb{Z}$ by $k/n + \mathbb{Z} \mapsto k(\gamma/n) + \gamma\,{}^*\mathbb{Z}$.*

Example 3. *Fix a prime number p. Let $\Lambda := \mathbb{N}$, $G_\lambda := \mathbb{Z}/p^\lambda\mathbb{Z}$ and $i_{\lambda\mu}\left(k + p^\lambda\mathbb{Z}\right) := p^{\mu-\lambda}k + p^\mu\mathbb{Z}$, where $\mathbb{N}$ is ordered by the usual linear order $\leq$. Its direct limit, called the p-*Prüfer group*, is isomorphic to $\mathbb{Z}[1/p]/\mathbb{Z}$. Let γ be an infinite hypernatural number. The direct limit $\mathbb{Z}[1/p]/\mathbb{Z}$ can be embedded into ${}^*\mathbb{Z}/p^\gamma\,{}^*\mathbb{Z}$ by $\sum_{i=0}^n a_i p^{-i} + \mathbb{Z} \mapsto \sum_{i=0}^n a_i p^{\gamma-i} + p^\gamma\,{}^*\mathbb{Z}$.*

Example 4. *Let k be a commutative field. Let $\Lambda := \mathbb{N}$ (with $\leq$), $G_\lambda := GL(\lambda, k)$ and*

$$i_{\lambda\mu}(A_\lambda) := \begin{pmatrix} A_\lambda & O_{\lambda,\mu-\lambda} \\ O_{\mu-\lambda,\lambda} & I_{\mu-\lambda} \end{pmatrix}.$$

The direct limit $GL(\infty, k)$ is the group of regular $\infty \times \infty$-matrices of the form:

$$\begin{pmatrix} A_\lambda & O_{\lambda,\infty} \\ O_{\infty,\lambda} & I_\infty \end{pmatrix},\ A_\lambda \in GL(\lambda, k).$$

Let $\gamma \in {}^*\mathbb{N}$ be an infinite hypernatural number. Then $GL(\infty, k)$ is embedded into ${}^*GL(\gamma, {}^*k)$ by

$$j\begin{pmatrix} A_\lambda & O_{\lambda,\infty} \\ O_{\infty,\lambda} & I_\infty \end{pmatrix} := \begin{pmatrix} A_\lambda & O_{\lambda,\gamma-\lambda} \\ O_{\gamma-\lambda,\lambda} & I_{\gamma-\lambda} \end{pmatrix}.$$

3.2. Construction of direct limit group actions

Definition 3. *Suppose that a group G acts on a metric space M. Let $K > 0$. The action is said to be* effective *if for each $g \in G \setminus \{ e_G \}$, $d_M(x, gx) > 0$ for some $x \in M$. The action is said to be* K-Lipschitz *if $d_M(gx, gy) \leq K d_M(x, y)$ for all $g \in G$ and all $x, y \in M$.*

Our main theorem is the following.

Theorem 6. *Let M be a compact connected manifold with a metric d_M. Let $\mathbf{G} := (\Lambda, G_\lambda, i_{\lambda\mu})$ be a direct system of torsion groups, where $i_{\lambda\mu}$ is a monomorphism for all $\lambda \leq \mu$. If there exists an effective K-Lipschitz G_λ-action on M for each $\lambda \in \Lambda$, then there exists an effective K-Lipschitz $\varinjlim \mathbf{G}$-action on M.*

Remark 1. *If for each $\lambda \in \Lambda$ there exists an effective K-Lipschitz G_λ-action Φ_λ on M such that $\Phi_\mu \circ i_{\lambda\mu} = \Phi_\lambda$ for all $\lambda \leq \mu$, then there exists an effective K-Lipschitz $\varinjlim \mathbf{G}$-action on M. In order to prove it, we just construct the action by gluing the given actions: $\Psi(i_\lambda(g), x) := \Phi_\lambda(g, x)$. We do not assume such a coherency condition in Theorem 6.*

Before proving the theorem, we consider some direct consequences (see Example 2 and Example 3).

Corollary 5 (see [18])**.** *Let M be a compact connected manifold with a metric. If there exists an effective K-Lipschitz $\mathbb{Z}/n\mathbb{Z}$-action on M for each $n \in \mathbb{Z}_+$, then there exists an effective K-Lipschitz $\mathbb{Q}/\mathbb{Z}$-action on M.*

Corollary 6. *Let M be a compact connected manifold with a metric and p a prime number. If there exists an effective K-Lipschitz $\mathbb{Z}/p^n\mathbb{Z}$-action on M for each $n \in \mathbb{N}$, then there exists an effective K-Lipschitz $\mathbb{Z}[1/p]/\mathbb{Z}$-action on M.*

Note that $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}[1/p]/\mathbb{Z}$ are locally finite, i.e. every finitely generated subgroup is finite. In fact, the above corollaries are a consequence of a more general corollary on locally finite group actions.

Corollary 7. *Let M be a compact connected manifold with a metric and let G be a locally finite group. If every finite subgroup H of G acts effectively on M by K-Lipschitz maps, then G acts effectively on M by K-Lipschitz maps.*

Proof. Consider the set Λ of all finite subgroups of G ordered by the inclusion relation $\subseteq$. We first verify that Λ is a directed set. Let $H_1, \ldots, H_n \in \Lambda$. Since G is locally finite, the group H' generated by $H_1 \cup \cdots \cup H_n$ is finite, i.e. $H' \in \Lambda$. The finite group H' is an upper bound of $\{ H_1, \ldots, H_n \}$. For $H, H' \in \Lambda$ with $H \subseteq H'$, let $i_{HH'} \colon H \to H'$ be the inclusion map. Then $\mathbf{G} := (\Lambda, H, i_{HH'})$ forms a direct system of torsion groups with monomorphisms.

By the local finiteness, the group $\langle g \rangle$ generated by g is finite for each $g \in G$, so $G = \bigcup_{g \in G} \langle g \rangle \subseteq \bigcup_{H \in \Lambda} H \subseteq G$. It is easy to see that the direct limit $\varinjlim \mathbf{G}$ is isomorphic to $\bigcup_{H \in \Lambda} H$, which is precisely G. The statement of the corollary now follows by Theorem 6. $\square$

In Theorem 6, to prove effectiveness, we employ the following version of Newman's theorem.

Theorem 7 (see [5, Theorem 2]). *Let M be a connected manifold with a metric d_M. There exists a constant $\varepsilon := \varepsilon(M, d_M) > 0$ such that for every effective action of a finite group G on M, there exist $g \in G$ and $x \in M$ such that $d_M(x, gx) \geq \varepsilon$.*

A group action $G \curvearrowright M$ is said to be *ε-effective* if for every $g \in G \setminus \{ e_G \}$ there exists an $x \in M$ such that the orbit Gx has a diameter of at least ε. In this terminology, Newman's theorem states every effective action of a finite group is ε-effective, where $\varepsilon > 0$ depends only on M.

Proof of Theorem 6. By the absoluteness of $\mathbb{U}$, we may assume without loss of generality that all the objects that appeared in the statement (such as M and $\mathbf{G}$) are standard. For simplicity, denote $G = \varinjlim \mathbf{G}$. Let $j\colon G \to {}^*G_\gamma$ be the embedding of Theorem 5, where $\gamma \in {}^*\Lambda$ is an upper bound of Λ. By upward transfer, there exists an internal effective K-Lipschitz action $\Phi\colon {}^*G_\gamma \times {}^*M \to {}^*M$. Since M is compact Hausdorff, each point $x \in {}^*M$ is infinitely close to a unique point ${}^\circ x \in M$ (see Corollary 3). Now define a map $\Psi\colon G \times M \to M$ by putting

$$\Psi(g, x) := {}^\circ(\Phi(j(g), x)).$$

Claim 1. Ψ *is an action.*

Proof. Let $g, h \in G$ and $x \in M$. Since $j\colon G \to {}^*G_\gamma$ and $\Phi\colon {}^*G_\gamma \to {}^*(\mathrm{Aut}(M))$ are homomorphisms, we have that

$$\begin{aligned}
\Psi(e, x) &= {}^\circ(\Phi(j(e), x)) \\
&= {}^\circ(\Phi(e, x)) \\
&= {}^\circ x \\
&= x,
\end{aligned}$$

and

$$\begin{aligned}
\Psi(gh, x) &= {}^\circ(\Phi(j(gh), x)) \\
&= {}^\circ(\Phi(j(g)\, j(h), x)) \\
&= {}^\circ(\Phi(j(g), \Phi(j(h), x))) \\
&= {}^\circ\big(\Phi\big(j(g), {}^\circ(\Phi(j(h), x))\big)\big) \\
&= \Psi(g, \Psi(h, x)).
\end{aligned}$$

Note that the fourth equality of the latter comes from the K-Lipschitz property of Φ:

$$\begin{aligned}
&{}^*d_M\big(\Phi(j(g), \Phi(j(h), x)), \Phi\big(j(g), {}^\circ(\Phi(j(h), x))\big)\big) \\
&\quad \leq K\, {}^*d_M\big(\Phi(j(h), x), {}^\circ(\Phi(j(h), x))\big) \\
&\quad = \text{finite} \times \text{infinitesimal} \\
&\quad = \text{infinitesimal}.
\end{aligned}$$

Hence $\Phi(j(g), \Phi(j(h), x))$ and $\Phi\left(j(g), {}^{\circ}(\Phi(j(h), x))\right)$ have the same standard part. $\square$

Claim 2. Ψ *is K-Lipschitz.*

Proof. Let $g \in G$ and $x, y \in M$. Since $\Psi(g, x) \approx_M \Phi(j(g), x)$ and $\Psi(g, y) \approx_M \Phi(j(g), y)$,

$$(\Psi(g, x), \Psi(g, y)) \approx_{M \times M} (\Phi(j(g), x), \Phi(j(g), y)).$$

By the the continuity of the metric function $d_M \colon M \times M \to \mathbb{R}$ (Theorem 4), we have

$$\begin{aligned} d_M(\Psi(g, x), \Psi(g, y)) &\approx_{\mathbb{R}} {}^{*}d_M(\Phi(j(g), x), \Phi(j(g), y)) \\ &\leq K d_M(x, y). \end{aligned}$$

It follows that $d_M(\Psi(g, x), \Psi(g, y)) \leq K d_M(x, y)$. $\square$

Claim 3. Ψ *is effective.*

Proof. Let $g \in G \setminus \{e\}$. Choose a $g_\lambda \in G_\lambda$ such that $g = i_\lambda(g_\lambda)$. Since i_λ is a homomorphism, g_λ is not a unit element. Since G_λ is a torsion group, the group $\langle g_\lambda \rangle$ generated by g_λ is a finite subgroup of ${}^{*}G_\lambda$. (Note that ${}^{*}A = A$ holds for all standard finite sets A by upward transfer.) Consider the internal action $\Phi_\lambda \colon \langle g_\lambda \rangle \times {}^{*}M \to {}^{*}M$ defined by

$$\Phi_\lambda(h, x) := \Phi\left({}^{*}i_{\lambda\gamma}(h), x\right).$$

Since ${}^{*}i_{\lambda\gamma}$ is injective by upward transfer, Φ_λ is effective. By Newman's theorem and upward transfer, there exists a *standard* constant $\varepsilon := \varepsilon(M, d_M) > 0$ such that "there exist an $h \in \langle g_\lambda \rangle$ and an $x \in {}^{*}M$ such that ${}^{*}d_M(x, \Phi_\lambda(h, x)) \geq \varepsilon$".[§] There exists a *standard* $n \in \mathbb{N}$ such that $h = g_\lambda^n$. Then ${}^{*}d_M(x, \Phi_\lambda(g_\lambda^n, x)) \geq \varepsilon$ holds. Since Φ is K-Lipschitz,

$$\begin{aligned} \Psi(g^n, {}^{\circ}x) &= {}^{\circ}(\Phi(j(g^n), {}^{\circ}x)) \\ &\approx_M \Phi(j(g^n), {}^{\circ}x) \\ &\approx_M \Phi(j(g^n), x) \\ &= \Phi\left({}^{*}i_{\lambda\gamma}(g_\lambda^n), x\right) \\ &= \Phi_\lambda(g_\lambda^n, x). \end{aligned}$$

By the continuity of d_M,

$$d_M({}^{\circ}x, \Psi(g^n, {}^{\circ}x)) \approx_{\mathbb{R}} {}^{*}d_M(x, \Phi_\lambda(g_\lambda^n, x)) \geq \varepsilon.$$

Hence $d_M({}^{\circ}x, \Psi(g^n, {}^{\circ}x)) \geq \varepsilon > 0$. Since $\Psi(g)^n = \Psi(g^n) \neq \mathrm{id}_M$, it follows that $\Psi(g) \neq \mathrm{id}_M$. $\square$

[§] The downward transfer principle cannot be applied to the quoted statement, because it contains a nonstandard object, namely Φ_λ. Manevitz and Weinberger [18, p. 152, ll. 2124] accidentally applied the downward transfer principle to the corresponding statement in the original proof. As you can see, this error can be avoided.

4. Conclusion

Our proof can be summarised as follows. For each index $\lambda \in \Lambda$, there exists an effective K-Lipschitz action $G_\lambda \curvearrowright M$. Fix an infinitely large index $\gamma \in {}^*\Lambda$. By transfer, there exists an effective K-Lipschitz action $\Phi\colon {}^*G_\gamma \curvearrowright {}^*M$ in ${}^*\mathbb{U}$. Since the direct limit $\varinjlim \mathbf{G}$ can be embedded into ${}^*G_\gamma$, the desired action is obtained as the restriction of the standard part ${}^\circ\Phi\colon {}^*G_\gamma \curvearrowright M$. The effectiveness of the resulting action follows from Newman's theorem.

The crux of this paper is the idea of approximating categorical limits by nonstandard objects rather than the results themselves. This enables us to study categorical limits with nonstandard analysis. Here are some examples of nonstandard approximations of direct and inverse limits.

4.1. Čech theory and McCord theory

First, recall the definition of Čech (co)homology groups following [4]. Let X be a topological space and G an abelian group. The family Cov_X of all open covers of X forms a (downward) directed set with respect to the refinement relation. If λ is a refinement of μ, there exists a (canonical) homomorphism $V(\lambda) \to V(\mu)$ of Vietoris complexes. Here the *Vietoris complex* $V(\lambda)$ is the simplicial set, where $a_0, \dots a_p \in X$ span a $p$simplex if $a_0, \dots a_p \in U$ for some $U \in \lambda$. The *Čech (co) homology groups* of X with coefficients in G are then defined as the limits:

$$\check{H}_\bullet(X; G) := \varprojlim_{\lambda \in \mathrm{Cov}_X} H_\bullet(V(\lambda); G).$$
$$\check{H}^\bullet(X; G) := \varinjlim_{\lambda \in \mathrm{Cov}_X} H^\bullet(V(\lambda); G).$$

By Lemma 1, $\check{H}^\bullet(X)$ can be embedded into ${}^*H^\bullet({}^*V(\lambda))$ for all infinitely fine $\lambda \in {}^*\mathrm{Cov}_X$. This gives a nonstandard construction of Čech cohomology.

McCord [17] has given a much deeper construction of Čech (co)homology. Let X be a standard topological space and G an internal abelian group. For $p \in \mathbb{N}$, a $p+1$tuple $(a_0, \dots, a_p)$ from ${}^*X$ are called a *pmicrosimplex* if $a_0, \dots, a_p \in \mu_X(x)$ for some $x \in X$. Denote the set of p-microsimplexes by Δ^p. A hyperfinite formal sum $\sum_{i=1}^n g_i\sigma_i$ of $p$microsimplexes σ_i with coefficients $g_i \in G$, where $\{g_i\}_{i=1}^n$ and $\{\sigma_i\}_{i=1}^n$ are both internal, is called a *pmicrochain.* (Formally, a p-microchain is an internal map $\sigma\colon {}^*X^{p+1} \to G$ whose support is a hyperfinite subset of Δ^p.) The set $M_p(X; G)$ of p-microchains forms an abelian group with respect to the usual addition. The boundary homomorphisms $\partial_p\colon M_p(X; G) \to M_{p-1}(X; G)$ are defined by

$$\partial_p \sum_{i=1}^n g_i\left(a_0^i, \dots, a_p^i\right) := \sum_{i=1}^n \sum_{j=0}^p (-1)^j g_i\left(a_0^i, \dots, a_{j-1}^i, a_{j+1}^i, \dots, a_p^i\right).$$

Thus $M_\bullet(X; G)$ forms a chain complex.

$$0 \longleftarrow M_0(X; G) \xleftarrow{\partial_1} M_1(X; G) \xleftarrow{\partial_2} M_2(X; G) \longleftarrow \cdots\cdots$$

The *McCord homology groups* of X with coefficients in G are defined by

$$H_\bullet^M(X;G) := H_\bullet(M_\bullet(X;G)),$$

where $H_\bullet$ is the homology functor of chain complexes. Roughly speaking, McCord's homology of X is the homology of the Vietoris complex of the monads $\{\, \mu_X(x) \mid x \in X \,\}$. Indeed, the following isomorphism results are known.

Theorem 8 (see [7]). *Assume that* ${}^*\mathbb{U}$ *is sufficiently saturated. Let X be a standard compact space and G a standard abelian group. Then* $H_\bullet^M(X;{}^*G) \cong \check{H}_\bullet(X;{}^*G)$.

Theorem 9 (see [14]). *Assume that* ${}^*\mathbb{U}$ *is sufficiently saturated. Let X be a standard completely regular space and G a standard abelian group. Then* $H_\bullet^M(X;{}^*G) \cong \varinjlim_K \check{H}_\bullet(K;{}^*G)$, *where the direct limit runs over all compact subspaces of X.*

As a result of taking inverse limits, Čech homology may violate the exactness axiom in the EilenbergSteenrod axioms depending on the choice of the coefficient group (see [19]). Garavaglia [7] proved that Čech homology is exact for all compact pairs if and only if the coefficient group is equationally compact. In contrast, McCord homology satisfies the exactness axiom for all coefficient groups ([17]). See also [13].

One can also consider a cohomological counterpart of McCord's theory. In contrast with McCord homology, there are at least two different definitions of McCord cohomology. One is the homology based on *external* cochains. Let X be a standard topological space and G an abelian group. Define the cochain complex $M^\bullet(X;G)$ by

$$M^p(X;G) := \hom(M_p(X;{}^*\mathbb{Z}), G),$$

where the coboundary homomorphisms $d_p\colon M^p(X;G) \to M^{p+1}(X;G)$ are defined as usual:

$$d_p\varphi(u) := \varphi(\partial_p u).$$

Note that $\varphi\colon M_p(X;{}^*\mathbb{Z}) \to G$ may not be determined by its values on Δ^p. The *McCord cohomology groups* of X with coefficients in G are defined as

$$H_M^\bullet(X;G) := H^\bullet(M^\bullet(X;G)),$$

where $H^\bullet$ is the cohomology functor of cochain complexes.

Theorem 10 (see [23]). *Assume that* ${}^*\mathbb{U}$ *is sufficiently saturated. Let X be a standard locally contractible paracompact space and G an abelian group. Then* $H_M^\bullet(X;{}^*G) \cong \check{H}^\bullet(X;\hom({}^*\mathbb{Z},G))$.

Assume that G is internal. We say that a p-cochain $\varphi \in M^p(X;G)$ is *essentially internal* if there exists an internal homomorphism $f\colon {}^*\big(\mathbb{Z}\langle X^{p+1}\rangle\big) \to G$ such that $\varphi \restriction M_p(X;{}^*\mathbb{Z}) = f \restriction M_p(X;{}^*\mathbb{Z})$, where $\mathbb{Z}\langle X^{p+1}\rangle$ denotes the free $\mathbb{Z}$-module generated by X^{p+1}. As a consequence of upward transfer, each essentially internal cochain $\varphi\colon M_p(X;{}^*\mathbb{Z}) \to G$ is completely determined by its values on Δ^p, so φ can be identified with a map $\varphi \restriction \Delta^p\colon \Delta^p \to G$ having an extension $f \restriction {}^*X^{p+1}\colon {}^*X^{p+1} \to G$ in ${}^*\mathbb{U}$. The set $M_\mu^\bullet(X;G)$ of essentially internal cochains forms a subcomplex of $M^\bullet(X;G)$. Finally, define

$$H_\mu^\bullet(X;G) := H^\bullet\big(M_\mu^\bullet(X;G)\big).$$

Theorem 11 (see [23]). *Assume that* $^*\mathbb{U}$ *is sufficiently saturated. Let* X *be a standard paracompact space and* G *an internal abelian group. Then* $H^\bullet_\mu(X;G) \cong \check{H}^\bullet(X;G)$.

The uniform versions of Čech and McCord theories are studied in [9, 11].

4.2. Shape theory

Wattenberg [21] introduced and studied the envelope functor of metric spaces, which is a nonstandard analogue of Borsuk's shape theory. Intuitively, the envelope of a metric space X is the strong homotopy type of the infinitesimal boldification of $^*X$ within an ambient normed linear space $^*Y$. Shape theory can be formulated in terms of inverse systems (see [19]). Wattenberg's theory is then considered as an example of nonstandard approximations of inverse limits.

4.3. Ends

Let (X,ξ) be a pointed metric space. For each $r > 0$, let E_r be the set of all unbounded connected components of $X \setminus B_r(\xi)$, where $B_r(\xi)$ denotes the open ball. If $r < s$, there exists a canonical surjection $E_s \to E_r$ which sends $Q \in E_s$ to $Q' \in E_r$ so that $Q \subseteq Q'$. The elements of the inverse limit

$$e(X,\xi) := \varprojlim E_r$$

in the category of sets are called *ends* (based at ξ). This notion plays a central role in geometric group theory (see e.g. [2, 15]). The nonstandard construction of ends can be found in [8, 10].

Acknowledgement

The author is grateful to the anonymous reviewer(s) for their invaluable comments which improved the manuscript.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York-London, 1972.

[2] M. R. Bridson, A. Häfliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin Heidelberg, 1999.

[3] C. C. Chang, H. J. Keisler, *Model Theory*, 3rd ed., North-Holland, Amsterdam-New York-Oxford-Tokyo, 1990.

[4] C. H. Dowker, *Homology Groups of Relations*, Ann. of Math. **56**(1952), 84–95.

[5] A. Dress, *Newman's theorems on transformation groups*, Topology **8**(1969), 203–207.

[6] S. FEFERMAN, *Set-theoretic foundations of category theory*, in: *Reports of the Midwest Category Seminar III*, (S. MacLane Ed.), Springer-Verlag, Berlin Heidelberg, 1969, 201–247.

[7] S. GARAVAGLIA, *Homology with equationally compact coefficients*, Fund. Math. **100**(1978), 89–95.

[8] I. GOLDBRING, *Ends of groups: a nonstandard perspective*, J. Log. Anal. **3**(2011), 1–28.

[9] T. IMAMURA, *Nonstandard homology theory for uniform spaces*, Topology Appl. **209**(2016), 22–29. Corrigendum in DOI: 10.13140/RG.2.2.36585.75368.

[10] T. IMAMURA, *A nonstandard invariant of coarse spaces*, Grad. J. Math. **5**(2020), 1–8.

[11] T. IMAMURA *Relationship among various Vietoris-type and microsimplicial homology theories*, Arch. Math. (Brno) **57**(2021), 131–150.

[12] V. KANOVEI, M. REEKEN, *Nonstandard Analysis, Axiomatically*, Springer-Verlag, Berlin Heidelberg, 2004.

[13] T. KORPPI, *Vanishing of derived limits of non-standard inverse systemss*, Topology Appl. **157**(2010), 2692–2703.

[14] T. KORPPI, *On the homology of compact spaces by using non-standard methods*, Topology Appl. **157**(2010), 2704–2714.

[15] C. LÖH, *Geometric Group Theory*, Springer International Publishing, 2017.

[16] P. A. LOEB, M. P. H. WOLFF, *Nonstandard Analysis for the Working Mathematician*, 2nd ed., Springer Netherlands, Dordrecht Heidelberg, New York-London, 2015.

[17] M. C. MCCORD, *Non-standard analysis and homology*, Fund. Math. **74**(1972), 21–28.

[18] L. M. MANEVITZ, S. WEINBERGER, *Discrete circle actions: A note using nonstandard analysis*, Israel J. Math. **94**(1996), 147–155.

[19] S. MARDEŠIĆ, J. SEGAL, *Shape Theory*, North-Holland, Amsterdam-New York-Oxford, 1982.

[20] A. ROBINSON. *Non-standard Analysis*, North-Holland, Amsterdam, 1966.

[21] F. WATTENBERG, *Nonstandard analysis and the theory of shape*, Fund. Math. **98**(1978), 41–60.

[22] S. WEINBERGER. *Free* $\mathbb{Q}/\mathbb{Z}$ *Actions*, Comment. Math. Helv. **62**(1987), 450–464.

[23] R T. ŽIVALJEVIĆ, *On a cohomology theory based on hyperfinite sums of microsimplexes*, Pacific J. Math. **128**(1987), 201–208.