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On the probability density function of the Hartman-Watson distribution*

Tibor K. Pogány[1,2,†] and Saralees Nadarajah[3]

[1] *Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Bécsi út 96/b, Hungary*
[2] *Faculty of Maritime Studies, University of Rijeka, Studentska 2, HR-51 000 Rijeka, Croatia*
[3] *Department of Mathematics, University of Manchester, Manchester M13 9PL, UK*

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Abstract. The Hartman-Watson distribution arises in many areas of applied probability. But its probability density function is difficult to compute. Recently, [14] gave the first two terms of its asymptotic expansion. Here, we derive the probability density function and a full asymptotic expansion in computable forms.

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1. Introduction

Let X denote a Hartman-Watson random variable. The probability density function of X is given by

$$f_r(t) = \frac{\theta(r,t)}{I_0(t)}, \qquad t \geq 0, \tag{1}$$

where [5, p. 5, Eq. (12)]

$$I_\nu(t) = \sum_{n\geq 0} \frac{\left(\frac{t}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)\, n!}$$

denotes the modified Bessel function of the first kind of the order ν and [14, Eq. (1)]

$$\theta(r,t) = \frac{r}{\sqrt{2\,\pi^3\,t}}\, \mathrm{e}^{\frac{\pi^2}{2t}} \int_0^\infty \mathrm{e}^{-\frac{x^2}{2t} - r\cosh(x)}\, \sinh(x)\, \sin\left(\frac{\pi x}{t}\right)\, \mathrm{d}x \tag{2}$$

for all $r > 0$, $t \geq 0$; consult the introductory article by Hartman and Watson [8].

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†Corresponding author. *Email addresses:* `pogany.tibor@uni-obuda.hu`, `tibor.poganj@uniri.hr` (T. K. Pogány), `saralees.nadarajah@manchester.ac.uk` (S. Nadarajah)

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The Hartman-Watson distribution arises in many problems. One important problem in mathematical finance is the pricing problem of Asian options. The pricing problem involves the computation of the expectation

$$\mathbb{E}\left[\left(A_t^{(\nu)} - K\right)^+\right],$$

where

$$A_t^{(\nu)} = \int_0^t \mathrm{e}^{2(W_h+\nu h)}\ \mathrm{d}h,$$

and W is a standard Brownian motion, whilst $K > 0$ is an absolute constant. Section 2.2 of [1] showed that the conditional density of $A_t^{(\nu)}$, given $W_t + \nu t = x$, can be expressed as

$$\Pr\left[A_t^{(\nu)} \in \mathrm{d}u \middle| W_t + \nu t = x\right] = \frac{\sqrt{2\pi t}}{u} \exp\left[\frac{x^2}{2t} - \frac{1+\mathrm{e}^{2x}}{2u}\right] I_0\left(\frac{\mathrm{e}^x}{u}\right)\ f_{\mathrm{e}^x/u}(t)\ \mathrm{d}u,$$

where the density

$$f_r(t) = \frac{r}{I_0(r)\sqrt{2\pi^3 t}} \exp\left(\frac{\pi^2}{2t}\right) \int_0^\infty \exp\left[-\frac{y^2}{2t} - r\cosh(y)\right] \sinh(y)\sin\left(\frac{\pi y}{t}\right) \mathrm{d}y,$$

which coincides with (1). Hence, the pricing problem of Asian options involves the Hartman-Watson distribution.

Many authors have tried to compute (2). We mention [1, 2, 3, 4, 6, 9, 15]. The articles [10, 11] present some new approaches to dealing with the Hartman-Watson distribution, that is the use of the special function $f_r(t)$. [14] gave the first two terms of its asymptotic expansion as $t \to 0^+$ under the constraint that r and t form a hyperbola $rt = \rho$ (ρ is a fixed absolute constant), that is, r and t are dependent in an inversely proportional way. In this note, we give a computable series representation for $f_r(t)$ (Section 2), *not* assuming that r and t are connected in any way. We also give its asymptotic behavior in full as $t \to 0^+$ (Section 3). A future work is to perform numerical studies to investigate accuracies of the derived representations.

2. Series expansion of $f_r(t)$ in terms of parabolic cylinder functions

Let us notice that the integrand of $\theta(r,t)$ consists of a sum of four functions. To show that we start by expanding the exponential term $\mathrm{e}^{-r\cosh(x)}$ into Taylor series, *viz*,

$$\mathrm{e}^{-r\cosh(x)} = \sum_{n\geq 0} \frac{(-r)^n}{n!}\ \cosh^n(x). \tag{3}$$

We have to find the power series

$$\cosh(x) = \sum_{k\geq 0} \frac{x^{2k}}{(2k)!}$$

raised to the power $n \in \mathbb{N}_0$. The coefficients $b_m = b_m(n)$ of the output power series

$$\left(\sum_{k\geq 0} a_k x^{2k}\right)^n = \sum_{m\geq 0} b_m\ x^{2m} \tag{4}$$

are given by the recurrence [7, p. 17, Sect. 0.314]

$$b_0 = a_0^n, \qquad b_m = \frac{1}{m\ a_0}\sum_{j=1}^{m}\left[j(n+1)-m\right] a_j\ b_{m-j}. \tag{5}$$

As in our case $a_k = [(2k)!]^{-1}$, $k \in \mathbb{N}_0$, the system for $b_m = b_m(n)$, $m \in \mathbb{N}_0$ becomes

$$b_0 = 1, \qquad b_m = \frac{1}{m}\sum_{j=1}^{m}\frac{j(n+1)-m}{(2j)!}\ b_{m-j}, \qquad m \in \mathbb{N}.$$

$b_m = b_m(n)$ are polynomials in n of degree $\deg(b_m) = m$. The first few polynomials are

$$b_0 = 1;\ b_1 = \frac{n}{2};\ b_2 = \frac{n}{24}(3n-2);\ b_3 = \frac{n}{720}\left(15n^2-30n+16\right);$$
$$b_4 = \frac{n}{40320}\left(420n^3-1680n^2+2352n-1091\right),$$

which suggests that the magnitude, for any fixed n and m, behaves like $b_m \sim \mathscr{O}(n^m)$. Rewriting recurrence (5) as

$$b_m = \frac{n}{2m}\ b_{m-1} - \frac{1-m}{2m}\ b_{m-1} + \frac{1}{m}\sum_{j=2}^{m}\frac{j(n+1)-m}{(2j)!}\ b_{m-j},$$

we see that the leading coefficient of the polynomial $b_m(n)$ equals

$$[n^m]\ b_m(n) = \frac{1}{2^{m-2}\cdot m!} \qquad m \geq 4, \tag{6}$$

where $[n^m]$ is the operator which extracts the coefficient of n^m in the polynomial $b_m = b_m(n)$. Relation (3) transforms into

$$\begin{aligned} \mathrm{e}^{-r\cosh(x)} &= \sum_{n\geq 0}\frac{(-r)^n}{n!}\left(1+\sum_{m\geq 1}\sum_{j=1}^{m}\frac{j(n+1)-m}{m\ (2j)!}\ b_{m-j}\ x^{2m}\right) \\ &= \mathrm{e}^{-r} + \sum_{n\geq 0}\sum_{m\geq 1}\frac{(-r)^n}{n!\ m}\ \sum_{j=1}^{m}\frac{j(n+1)-m}{(2j)!}\ b_{m-j}\ x^{2m}. \end{aligned}$$

Now, we insert this series into the integrand of $\theta(r,t)$. Firstly, the reversion of the order of integration and summation is legitimate since the radius of convergence of power series (4), by virtue of (6), equals

$$\rho^{-1} = \limsup_{m\to\infty}\ \sqrt[m]{b_m} = \limsup_{m\to\infty}\frac{\mathrm{e}}{2m} = 0.$$

In turn, this means that inside the interval of convergence term by term integration is enabled for the considered expansion.

Next, expressing the hyperbolic sine as the difference of exponentials, we obtain the following linear combination which contains four integrals, *viz.*

$$\theta(r,t) = \frac{r\ \mathrm{e}^{\frac{\pi^2}{2t}-r}}{\sqrt{(2\pi)^3\ t}} \left[\int_0^\infty \mathrm{e}^{-\frac{x^2}{2t}+x}\ \sin\left(\frac{\pi x}{t}\right)\ \mathrm{d}x - \int_0^\infty \mathrm{e}^{-\frac{x^2}{2t}-x} \sin\left(\frac{\pi x}{t}\right)\ \mathrm{d}x\right] \tag{7}$$

$$+ \frac{r\ \mathrm{e}^{\frac{\pi^2}{2t}}}{\sqrt{(2\ \pi)^3\ t}} \sum_{n\geq 0}\sum_{m\geq 1}\sum_{j=1}^{m} \frac{(-r)^n}{n!\ m}\ \frac{j(n+1)-m}{(2j)!}\ b_{m-j}$$
$$\times \left[\int_0^\infty x^{2m}\ \mathrm{e}^{-\frac{x^2}{2t}+x} \sin\left(\frac{\pi x}{t}\right)\mathrm{d}x - \int_0^\infty x^{2m}\mathrm{e}^{-\frac{x^2}{2t}-x}\sin\left(\frac{\pi x}{t}\right)\mathrm{d}x\right]. \tag{8}$$

To proceed, we express the integrals involved in terms of the parabolic cylinder functions $U(a,z)$, $V(a,z)$ which are the principal solutions of the Weber differential equation [16, p. 149]

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - \left(\tfrac{1}{4}\ z^2 + a\right)\ w = 0.$$

Whittaker's notation of the first standard solution $D_\nu(z) = U\left(-\nu - \frac{1}{2}, z\right)$ is also frequent in use. We remark that $U(a,-z)$, $V(a,-z)$ are also solutions of Weber's differential equation. The inter-connection formula between the standard solutions reads [16, p. 150,Eq. (11.1.5)]:

$$\sin(a\pi)\ U(a,z) + U(a,-z) = \frac{\pi}{\Gamma\left(a+\frac{1}{2}\right)}\ V(a,z). \tag{9}$$

Now, recall the Laplace transform [7, p. 498, Eq. **3.953.1**] or [5, p. 153, Eq. (25)]:

$$\int_0^\infty x^{\mu-1}\mathrm{e}^{-\beta x^2-\gamma x}\ \sin(ax)\ \mathrm{d}x = \frac{\mathrm{i}\ \Gamma(\mu)\ \mathrm{e}^{\frac{\gamma^2-a^2}{8\beta}}}{2\sqrt{(2\beta)^\mu}} \left[\mathrm{e}^{\frac{\mathrm{i}a\gamma}{4\beta}}\ D_{-\mu}\left(\frac{\gamma+\mathrm{i}a}{\sqrt{2\beta}}\right)\right.$$
$$\left. - \mathrm{e}^{-\frac{\mathrm{i}a\gamma}{4\beta}}\ D_{-\mu}\left(\frac{\gamma-\mathrm{i}a}{\sqrt{2\beta}}\right)\right],$$

where $\Re(\mu) > -1$, $\Re(\beta) > 0$, $a > 0$ and $\mathrm{i} = \sqrt{-1}$. Obviously, this integral covers all four integrals in $\theta(r,t)$, compare (7) and (8). Setting $\mu = 2m+1$, $m \in \mathbb{N}_0$, $\beta = (2t)^{-1}$, $\gamma = \mp 1$ and $a = \pi t^{-1}$, we infer that

$$\mathscr{I}_m^{\pm} := \int_0^\infty x^{2m}\mathrm{e}^{-\frac{x^2}{2t}\mp x}\sin\left(\frac{\pi x}{t}\right)\ \mathrm{d}x$$
$$= \mp\frac{1}{2}\ (2m)!\ t^{m+\frac{1}{2}}\ \mathrm{e}^{\frac{t^2-\pi^2}{4t}} \left[D_{-2m-1}\left(\frac{\pm t - \mathrm{i}\pi}{\sqrt{t}}\right) + D_{-2m-1}\left(\frac{\pm t+\mathrm{i}\pi}{\sqrt{t}}\right)\right].$$

Accordingly, introducing the shorthand

$$\tau = \sqrt{t} + \frac{\mathrm{i}\ \pi}{\sqrt{t}}, \qquad t > 0,$$

and adapting relation (9) to our case:

$$D_{-2m-1}(\tau)+D_{-2m-1}(-\tau)=\frac{\pi}{(2m)!}\;V\left(2m+\tfrac{1}{2},\tau\right),$$

we see that

$$\begin{aligned}\mathscr{I}_m^- - \mathscr{I}_m^+ &= \frac{(2m)!}{2\mathrm{e}^{\frac{\pi^2-t^2}{4t}}}\;t^{m+\frac{1}{2}}\;\Big[D_{-2m-1}(-\tau)+D_{-2m-1}\left(-\tau^*\right)\\ &\quad + D_{-2m-1}\left(\tau^*\right)+D_{-2m-1}(\tau)\Big]\\ &= \frac{\pi}{2}\;\mathrm{e}^{\frac{t^2-\pi^2}{4t}}\;t^{m+\frac{1}{2}}\;\left[V\left(2m+\tfrac{1}{2},\tau\right)+V\left(2m+\tfrac{1}{2},\tau^*\right)\right]\\ &= \pi\;\mathrm{e}^{\frac{t^2-\pi^2}{4t}}\;t^{m+\frac{1}{2}}\;\Re\left[V\left(2m+\tfrac{1}{2},\tau\right)\right], \qquad m\in\mathbb{N}_0,\end{aligned}$$

where the last equality holds by the mirror symmetry property of $U\left(a,z^*\right)=U^*(a,z)$ and $V\left(a,z^*\right)=V^*(a,z)$ using (9), considering real a, where ζ^* denotes the complex conjugate of ζ.

The case $m=0$ corresponds to (7), while (8) is covered by $\mathscr{I}_m^- - \mathscr{I}_m^+$ for positive integer $m\in\mathbb{N}$; therefore, the following expression is deduced:

$$\begin{aligned}\theta(r,t)=\frac{r\;\mathrm{e}^{\frac{\pi^2+t^2}{4t}-r}}{2\sqrt{2}\;\pi}\;\Re\Big[V\left(\tfrac{1}{2},\tau\right)+\mathrm{e}^r\;\sum_{n\geq 0}\sum_{m\geq 1}\sum_{j=1}^{m}\frac{(-r)^n}{n!}\;\frac{j(n+1)-m}{m\;(2j)!}\\ \times b_{m-j}\;t^m\;V\left(2m+\tfrac{1}{2},\tau\right)\Big].\end{aligned}$$

We have the following theorem.

Theorem 1. *For all $r>0$ and $t\geq 0$, the probability density function of a Hartman-Watson distributed random variable can be expressed as*

$$f_r(t)=\frac{r\;\mathrm{e}^{\frac{|\tau|^2}{4}}}{2\sqrt{2}\;\pi\;I_0(t)}\;\sum_{n\geq 0}\sum_{m\geq 0}\frac{(-r)^n}{n!}\;b_m\;\Re\left[V\left(2m+\tfrac{1}{2},\tau\right)\right]\;t^m, \tag{10}$$

where $\tau=(t+\mathrm{i}\pi)/\sqrt{t}$ and the coefficients $b_m=b_m(n)$ are solutions of the recursive system

$$b_0=1, \qquad b_m=\frac{1}{m}\sum_{j=1}^{m}\frac{j(n+1)-m}{(2j)!}\;b_{m-j}, \qquad m\in\mathbb{N}$$

for any fixed nonnegative integer n.

One important conclusion which follows from Theorem 1 is that the parameter r affects the behavior of $f_r(t)$ independently of t. We point out once again that Pirjol [14] and other authors take r, t to be inversely proportional, while considering the asymptotic behaviour of (2), that is, the Hartman-Watson probability density function $f_r(t)$. This has not been our approach.

3. Asymptotic of $f_r(t)$ for vanishing t

Consider the function $\theta(r,t)$ expressed in Theorem 1, equation (10) as a double series. Since τ is located in the first quadrant near to the imaginary axis as $t \to 0^+$, that is, $\tan[\arg(\tau)] = \frac{\pi}{t} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, we have to use the formula for $V(a,z)$ as $|z| \to \infty$ for the sector $-\frac{\pi}{4}+\delta \le \arg(z) \le \frac{3\pi}{4}+\delta$. That formula reads [13, p. 309, Eq. 12.9.4]:

$$V(a,z) \sim \sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\frac{z^2}{4}}\, z^{a-\frac12} \sum_{k\ge 0} \frac{\left(\frac12 - a\right)_{2k}}{k!\, \left(2z^2\right)^k} + \frac{\mathrm{i}\, z^{-(a+\frac12)}\, \mathrm{e}^{-\frac{z^2}{4}}}{\Gamma\left(\frac12 - a\right)} \sum_{k\ge 0} \frac{(-1)^k \left(\frac12 + a\right)_{2k}}{k!\, \left(2z^2\right)^k}. \tag{11}$$

For all $m \in \mathbb{N}_0$, by virtue of the Legendre duplication formula and as the second addend term in (11) vanishes by gamma function singularities at the non-positive integer values of $a = 2m + \frac12$ in the denominator,

$$V\left(2m+\tfrac12, \tau\right) \sim \mathrm{i}\sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\frac{t^2-\pi^2}{4t}} \sum_{k=0}^m 2^k \frac{(-m)_k \left(-m+\frac12\right)_k}{k!} \left(t - \frac{\pi^2}{t} + 2\pi\mathrm{i}\right)^{m-k},$$

which can be presented as a weighted ${}_2F_0$-hypergeometric polynomial, *viz.*

$$V\left(2m+\tfrac12, \tau\right) \sim \mathrm{i}\sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\frac{t^2-\pi^2}{4t}} \left(t - \frac{\pi^2}{t} + 2\pi\mathrm{i}\right)^m {}_2F_0\left(-m, -m+\tfrac12; -; \frac{2t}{(t+\mathrm{i}\pi)^2}\right).$$

Letting here $t \to 0^+$, keeping in mind that ${}_2F_0[0] = 1$, we obtain

$$\Re\Big[V\left(2m+\tfrac12, \tau\right)\Big]\Big|_{t\to 0^+} \sim \sqrt{\frac{2}{\pi}} \mathrm{e}^{\frac{-\pi^2}{4t}} \pi^m \sum_{k=0}^m \binom{m}{k} \left(-\frac{\pi}{t}\right)^{m-k} 2^k \cos\left[\frac{\pi}{2}(k+1)\right], \tag{12}$$

so we have the following result.

Theorem 2. *For all $r > 0$, as $t \to 0^+$, we have*

$$f_r(t) \sim \frac{r}{2\pi}\left\{\mathrm{e}^{-r} + \sum_{n\ge 0}\frac{(-r)^n}{n!}\sum_{m\ge 1}\left(-\pi^2\right)^m\, b_m \sum_{k=0}^m \binom{m}{k}\left(-\frac{2t}{\pi}\right)^k \cos\left[\frac{\pi}{2}(k+1)\right]\right\}.$$

Proof. Letting $t \to 0^+$ in (10) and using (12), we obtain

$$f_r(t) \sim \frac{r}{2\,\pi}\sum_{n\ge 0}\frac{(-r)^n}{n!}\sum_{m\ge 0} b_m \left(t^2-\pi^2\right)^m \sum_{k=0}^m \binom{m}{k}\left(-\frac{2t}{\pi}\right)^k \cos\left[\frac{\pi}{2}(k+1)\right], \tag{13}$$

as $I_0(0) = 1$. Since $b_0 = 1$, the right-hand side of (13) corresponding to $m = 0$ gives the exponential term e^{-r}. When $t \to 0^+$ in the remaining part of the right-hand side expression of (13), we arrive at the statement. □

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References

[1] P. Barrieu, A. Rouault, M. Yor, *A study of the Hartman-Watson distribution motivated by numerical problems related to the pricing of Asian options*, J. Appl. Probab. **41**(2004), 1049–1058.

[2] G. Bernhart, J. F. Mai, *A note on the numerical evaluation of the Hartman-Watson density and distribution function*, in: *Innovations in Quantitative Risk Management*, (K. Blau, M. Scherer and R. Zagst, Eds.), Springer Proceedings in Mathematics and Statistics, Springer, 2015, 337–345.

[3] A. Comtet, C. Monthus, M. Yor, *Exponential functionals of Brownian motion and disordered systems*, J. Appl. Probab. **35**(1998), 255–271.

[4] D. Dufresn, *The integral of the geometric Brownian motion*, Adv. Appl. Probab. **33**(2001), 223–241.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Volume II, Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company Inc., New York-Toronto-London, 1954.

[6] S. Gerhold, *The Hartman-Watson distribution revisited: Asymptotics for pricing Asian options*, J. Appl. Probab. **48**(2011), 892–899.

[7] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Sixth edition, Translation edited by Alan Jeffrey and Daniel Zwillinger, Academic Press, Inc., San Diego, California, 2000.

[8] P. Hartman, G. S. Watson, *"Normal" distribution functions on spheres and the modified Bessel functions*, Ann. Probab. **2**(1974), 593–607.

[9] K. Ishyiama, *Methods for evaluating density functions of exponential functionals represented as integrals of geometric Brownian motion*, Methodol. Comput. Appl. Probab. **7**(2005), 271–283.

[10] J. Jakubowski, M. Wiśniewolski, *Another look at the Hartman-Watson distribution*, Potential Anal. **53**(2020), 1269–1297.

[11] A. Lyasoff, *Another look at the integral of exponential Brownian motion and the pricing of Asian options*, Finance Stoch. **20**(2016), 1061–1096.

[12] F. W. J. Olver, *Uniform asymptotic expansions for Weber parabolic cylinder functions of large orders*, J. Res. Nat. Bur. Standards Sect. B: Mathematics and Mathematical Physics **63B**(1959), 131–169.

[13] F. Olver, D. Lozier, R. Boisvert, C. Clark (Eds.), *The* NIST *Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.

[14] D. Pirjol, *Small-t expansion for the Hartman-Watson distribution*, Methodol. Comput. Appl. Probab. **23**(2021), 1537–1549.

[15] M. Schröder, *On the integral of the geometric Brownian motion. In honor of Joseph Mecke*, Adv. Appl. Probab. **35**(2003),159–183.

[16] N. M. Temme, *Asymptotic Methods for Integrals.* Series in Analysis **6**, World Scientific Publishing Co. Pte. Ltd., Hackensack, New Jersey, 2015.