On the probability density function of the Hartman-Watson distribution∗

TIBOR K. POGÁNY1,2,† AND SARALEES NADARAJAH3

1 Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Bécsi út 96/b, Hungary
2 Faculty of Maritime Studies, University of Rijeka, Studentska 2, HR-51000 Rijeka, Croatia
3 Department of Mathematics, University of Manchester, Manchester M13 9PL, UK

Received September 22, 2021; accepted January 10, 2022

Abstract. The Hartman-Watson distribution arises in many areas of applied probability. But its probability density function is difficult to compute. Recently, [14] gave the first two terms of its asymptotic expansion. Here, we derive the probability density function and a full asymptotic expansion in computable forms.

AMS subject classifications: 33C15, 41A58, 62E17, 62E20

Key words: Computable series representation, Hartman-Watson distribution, Laplace-Mellin transform, parabolic cylinder function

1. Introduction

Let $X$ denote a Hartman-Watson random variable. The probability density function of $X$ is given by

$$f_r(t) = \frac{\theta(r, t)}{I_0(t)}, \quad t \geq 0,$$

where [5, p. 5, Eq. (12)]

$$I_\nu(t) = \sum_{n \geq 0} \frac{\left(\frac{1}{2}\right)^{2n+\nu}}{\Gamma(n + \nu + 1) \, n!}$$

denotes the modified Bessel function of the first kind of the order $\nu$ and [14, Eq. (1)]

$$\theta(r, t) = \frac{r}{\sqrt{2 \pi^3 t}} \int_0^\infty e^{-\frac{x^2}{2} - r \cosh(x)} \sinh(x) \sin\left(\frac{\pi x}{t}\right) \, dx$$

for all $r > 0$, $t \geq 0$; consult the introductory article by Hartman and Watson [8].

∗The research of the first author has been supported in part by the University of Rijeka, Croatia; project code uniri-pr-prirod-19-16.
†Corresponding author. Email addresses: pogany.tibor@uni-obuda.hu, tibor.poganj@uniri.hr (T.K. Pogány), saralees.nadarajah@manchester.ac.uk (S. Nadarajah)

http://www.mathos.hr/mc ©2022 Department of Mathematics, University of Osijek
The Hartman-Watson distribution arises in many problems. One important problem in mathematical finance is the pricing problem of Asian options. The pricing problem involves the computation of the expectation

$$\mathbb{E} \left[ \left( A_t^{(\nu)} - K \right)^+ \right],$$

where

$$A_t^{(\nu)} = \int_0^t e^{2(W_h + \nu h)} \, dh,$$

and $W$ is a standard Brownian motion, whilst $K > 0$ is an absolute constant. Section 2.2 of [1] showed that the conditional density of $A_t^{(\nu)}$, given $W_t = x$, can be expressed as

$$\Pr \left[ A_t^{(\nu)} \in du \mid W_t = x \right] = \frac{\sqrt{2\pi t}}{u} \exp \left[ \frac{x^2}{2t} - 1 + e^{2x} \right] I_0 \left( \frac{e^x}{u} \right) f_{\nu/u}(t) \, du,$$

where the density

$$f_r(t) = \frac{r}{I_0(r) \sqrt{2\pi t}} \exp \left( \frac{\pi^2}{2t} \right) \int_0^\infty \exp \left[ -\frac{y^2}{2t} - r \cosh(y) \right] \sinh(y) \sin \left( \frac{\pi y}{t} \right) \, dy,$$

which coincides with (1). Hence, the pricing problem of Asian options involves the Hartman-Watson distribution.

Many authors have tried to compute (2). We mention [1, 2, 3, 4, 6, 9, 15]. The articles [10, 11] present some new approaches to dealing with the Hartman-Watson distribution, that is the use of the special function $f_r(t)$. [14] gave the first two terms of its asymptotic expansion as $t \to 0^+$ under the constraint that $r$ and $t$ form a hyperbola $rt = \rho$ ($\rho$ is a fixed absolute constant), that is, $r$ and $t$ are dependent in an inversely proportional way. In this note, we give a computable series representation for $f_r(t)$ (Section 2), not assuming that $r$ and $t$ are connected in any way. We also give its asymptotic behavior in full as $t \to 0^+$ (Section 3). A future work is to perform numerical studies to investigate accuracies of the derived representations.

2. Series expansion of $f_r(t)$ in terms of parabolic cylinder functions

Let us notice that the integrand of $\theta(r, t)$ consists of a sum of four functions. To show that we start by expanding the exponential term $e^{-r \cosh(x)}$ into Taylor series, namely,

$$e^{-r \cosh(x)} = \sum_{n \geq 0} \frac{(-r)^n}{n!} \cosh^n(x). \quad (3)$$

We have to find the power series

$$\cosh(x) = \sum_{k \geq 0} \frac{x^{2k}}{(2k)!}$$
raised to the power $n \in \mathbb{N}_0$. The coefficients $b_m = b_m(n)$ of the output power series

$$\left( \sum_{k \geq 0} a_k x^{2k} \right)^n = \sum_{m \geq 0} b_m(x) 2^m$$

(4)

are given by the recurrence [7, p. 17, Sect. 0.314]

$$b_0 = a_0^n, \quad b_m = \frac{1}{m} a_0 \sum_{j=1}^{m} \left[ j(n+1) - m \right] a_j b_{m-j}. \quad (5)$$

As in our case $a_k = \frac{1}{(2k)!}$, $k \in \mathbb{N}_0$, the system for $b_m = b_m(n)$, $m \in \mathbb{N}_0$ becomes

$$b_0 = 1, \quad b_m = \frac{1}{m} \sum_{j=1}^{m} \frac{j(n+1) - m}{(2j)!} b_{m-j}, \quad m \in \mathbb{N}.$$  

$b_m = b_m(n)$ are polynomials in $n$ of degree $\deg(b_m) = m$. The first few polynomials are

$$b_0 = 1; \quad b_1 = \frac{n}{2}; \quad b_2 = \frac{n}{24} (3n-2); \quad b_3 = \frac{n}{720} (15n^2 - 30n + 16);$$

$$b_4 = \frac{n}{40320} (420n^3 - 1680n^2 + 2352n - 1091),$$

which suggests that the magnitude, for any fixed $n$ and $m$, behaves like $b_m \sim \Theta(n^m)$. Rewriting recurrence (5) as

$$b_m = \frac{n}{2m} b_{m-1} - \frac{1}{2m} \frac{1}{b_{m-1}} + \frac{1}{m} \sum_{j=2}^{m} \frac{j(n+1) - m}{(2j)!} b_{m-j},$$

we see that the leading coefficient of the polynomial $b_m(n)$ equals

$$\left[ n^n \right] b_m(n) = \frac{1}{2m-2} \cdot \frac{1}{m!}, \quad m \geq 4, \quad (6)$$

where $\left[ n^n \right]$ is the operator which extracts the coefficient of $n^m$ in the polynomial $b_m = b_m(n)$. Relation (3) transforms into

$$e^{-r \cosh(x)} = \sum_{n \geq 0} \frac{(-r)^n}{n!} \left( 1 + \sum_{m \geq 1} \sum_{j=1}^{m} \frac{j(n+1) - m}{m(2j)!} b_{m-j} 2^m \right)$$

$$= e^{-r} + \sum_{n \geq 0} \sum_{m \geq 1} \frac{(-r)^n}{n! m} \sum_{j=1}^{m} \frac{j(n+1) - m}{(2j)!} b_{m-j} 2^m$$

Now, we insert this series into the integrand of $\theta(r,t)$. Firstly, the reversion of the order of integration and summation is legitimate since the radius of convergence of power series (4), by virtue of (6), equals

$$\rho^{-1} = \lim_{m \to \infty} \sup_{n} \sqrt[n]{b_m} = \lim_{m \to \infty} \frac{e}{2m} = 0.$$
In turn, this means that inside the interval of convergence term by term integration is enabled for the considered expansion.

Next, expressing the hyperbolic sine as the difference of exponentials, we obtain the following linear combination which contains four integrals, viz.

\[
\theta(r, t) = \frac{r e^{\frac{x^2}{2} - \frac{x^2}{r}}}{\sqrt{(2\pi)^3 t}} \left[ \int_0^\infty e^{-\frac{x^2}{2}} \sin \left( \frac{\pi x}{t} \right) \, dx - \int_0^\infty e^{-\frac{x^2}{2} - x} \sin \left( \frac{\pi x}{t} \right) \, dx \right]
\]

+ \frac{r e^{\frac{x^2}{2}}}{\sqrt{(2\pi)^3 t}} \sum_{n \geq 0} \sum_{m \geq 1} e^{\frac{m^2}{2}} \sum_{j=1}^{m} \frac{(-1)^n}{n!} \frac{(n+1)-m}{(2j)!} \sum_{j=1}^{m} \sin \left( \frac{\pi x}{t} \right) \, dx - \int_0^\infty x e^{-\frac{x^2}{2} - x} \sin \left( \frac{\pi x}{t} \right) \, dx \right].
\]

To proceed, we express the integrals involved in terms of the parabolic cylinder functions \( U(a, z) \), \( V(a, z) \) which are the principal solutions of the Weber differential equation [16, p. 149]

\[
d^2w \quad dx^2 - \left( \frac{1}{4} z^2 + a \right) w = 0.
\]

Whittaker’s notation of the first standard solution \( D_v(z) = U(-\nu - \frac{1}{2}, z) \) is also frequent in use. We remark that \( U(a, -z), V(a, -z) \) are also solutions of Weber’s differential equation. The inter-connection formula between the standard solutions reads [16, p. 150, Eq. (11.1.5)]:

\[
\sin(a\pi) U(a, z) + U(a, -z) = \frac{\pi}{\Gamma(a + \frac{1}{2})} V(a, z).
\]

Now, recall the Laplace transform [7, p. 498, Eq. 3.953.1] or [5, p. 153, Eq. (25)]:

\[
\int_0^\infty x^{\mu-1} e^{-\beta x^2 - \gamma x} \sin(ax) \, dx = \frac{i \Gamma(\mu) e^{\frac{\gamma^2}{2\beta} - \frac{\gamma^2}{2\beta}}}{2(2\beta)^{\mu}} \left[ \frac{e^{\frac{i\alpha}{\beta}} D_{-\mu} \left( \frac{\gamma + ia}{\sqrt{2\beta}} \right)}{2\Gamma(\mu)} - e^{-\frac{i\alpha}{\beta}} D_{-\mu} \left( \frac{\gamma - ia}{\sqrt{2\beta}} \right) \right],
\]

where \( \Re(\mu) > -1, \Re(\beta) > 0, a > 0 \) and \( i = \sqrt{-1} \). Obviously, this integral covers all four integrals in \( \theta(r, t) \), compare (7) and (8). Setting \( \mu = 2m + 1, m \in \mathbb{N}_0, \beta = (2t)^{-1}, \gamma = \mp 1 \) and \( a = \pi t^{-1} \), we infer that

\[
\mathcal{J}_m^\pm := \int_0^\infty x^{2m} e^{-\frac{x^2}{2\pi} \pm x} \sin \left( \frac{\pi x}{t} \right) \, dx
\]

\[
= \mp \frac{1}{2} (2m)! \, t^{m+\frac{1}{2}} e^{-\frac{\pi^2}{2t}} \left[ D_{-2m-1} \left( \pm t - i\pi \sqrt{t} \right) + D_{-2m-1} \left( \pm t + i\pi \sqrt{t} \right) \right],
\]

Accordingly, introducing the shorthand

\[
\tau = \sqrt{t} + \frac{i\pi}{\sqrt{t}}, \quad t > 0,
\]
and adapting relation (9) to our case:

\[
D_{-2m-1}(\tau) + D_{-2m-1}(-\tau) = \frac{\pi}{(2m)!} V\left(2m + \frac{1}{2}, \tau\right),
\]

we see that

\[
I - m - I + m = (2m)! e^{\frac{\pi^2}{4} - \frac{t^2}{4}} \left[ \tau \sum_{n=0}^{m} \sum_{m=1}^{m \geq 0} \frac{(-r)^n j(n+1) - m}{m(2j)!} b_m \Re \left[ V\left(2m + \frac{1}{2}, \tau\right)\right] \right],
\]

where the last equality holds by the mirror symmetry property of \(U(a, z^*) = U^*(a, z)\) and \(V(a, z^*) = V^*(a, z)\) using (9), considering real \(a\), where \(\zeta^*\) denotes the complex conjugate of \(\zeta\).

The case \(m = 0\) corresponds to (7), while (8) is covered by \(J_m^- - J_m^+\) for positive integer \(m \in \mathbb{N}\); therefore, the following expression is deduced:

\[
\theta(r, t) = \frac{r e^{\frac{\pi^2}{8} - r^2}}{2\sqrt{2\pi}} \Re \left[ V\left(\frac{1}{2}, \tau\right) + e^r \sum_{n=0}^{m} \sum_{m=1}^{m \geq 0} \frac{(-r)^n j(n+1) - m}{m(2j)!} b_m \Re \left[ V\left(2m + \frac{1}{2}, \tau\right)\right] \right].
\]

We have the following theorem.

**Theorem 1.** For all \(r > 0\) and \(t \geq 0\), the probability density function of a Hartman-Watson distributed random variable can be expressed as

\[
f_r(t) = \frac{e^{\frac{\pi^2}{8} - r^2}}{2\sqrt{2\pi} I_0(t)} \sum_{n=0}^{m} \sum_{m=0}^{m \geq 0} \frac{(-r)^n}{n!} b_m \Re \left[ V\left(2m + \frac{1}{2}, \tau\right)\right] t^m,
\]

where \(\tau = (t + i\pi) / \sqrt{t}\) and the coefficients \(b_m = b_m(n)\) are solutions of the recursive system

\[
b_0 = 1, \quad b_m = \frac{1}{m} \sum_{j=1}^{m} \frac{j(n+1) - m}{(2j)!} b_{m-j}, \quad m \in \mathbb{N}
\]

for any fixed nonnegative integer \(n\).

One important conclusion which follows from Theorem 1 is that the parameter \(r\) affects the behavior of \(f_r(t)\) independently of \(t\). We point out once again that Pirjol [14] and other authors take \(r, t\) to be inversely proportional, while considering the asymptotic behaviour of (2), that is, the Hartman-Watson probability density function \(f_r(t)\). This has not been our approach.
3. Asymptotic of \( f_r(t) \) for vanishing \( t \)

Consider the function \( \theta(r, t) \) expressed in Theorem 1, equation (10) as a double series. Since \( \tau \) is located in the first quadrant near to the imaginary axis as \( t \to 0^+ \), that is, \( \tan \{\arg(\tau)\} = \frac{\pi}{\tau} \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \), we have to use the formula for \( V(a, z) \) as \( |z| \to \infty \) for the sector \( -\frac{\pi}{4} + \delta \leq \arg(z) \leq \frac{3\pi}{4} + \delta \). That formula reads [13, p. 309, Eq. 12.9.4]:

\[
V(a, z) \sim \sqrt{\frac{2}{\pi}} e^{\frac{z^2}{4}} z^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - a)_{2k}}{k! (2z^2)^k} + \frac{i}{\Gamma(\frac{1}{2} - a)} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2} + a)_{2k}}{k! (2z^2)^k} \cdot \frac{z^{-\frac{1}{2}}}{k!}.
\]

For all \( m \in \mathbb{N}_0 \), by virtue of the Legendre duplication formula and as the second addend term in (11) vanishes by gamma function singularities at the non-positive integer values of \( a = 2m + \frac{1}{2} \) in the denominator,

\[
V(2m + \frac{1}{2}, \tau) \sim i \sqrt{\frac{2}{\pi \tau}} e^{\frac{\pi^2}{4\tau^2}} \sum_{k=0}^{m} 2^k \frac{(-m)_k}{k!} \frac{(-m + \frac{1}{2})_k}{(t - \pi^2 t + 2\pi i)^{m-k}},
\]

which can be presented as a weighted \( 2F_0 \)-hypergeometric polynomial, viz.

\[
V(2m + \frac{1}{2}, \tau) \sim i \sqrt{\frac{2}{\pi \tau}} e^{\frac{\pi^2}{4\tau^2}} \left( t - \pi^2 t + 2\pi i \right)^m 2F_0 \left( -m, -m + \frac{1}{2}; \frac{2t}{(t + 1\pi)^2} \right).
\]

Letting here \( t \to 0^+ \), keeping in mind that \( 2F_0[0] = 1 \), we obtain

\[
\mathbb{R} \left[ V(2m + \frac{1}{2}, \tau) \right]_{t \to 0^+} \sim \sqrt{\frac{2}{\pi \tau^2}} e^{\pi\pi^2} \sum_{k=0}^{m} \frac{m!}{k!} \left( \frac{\pi}{t} \right)^{m-k} 2^k \cos \left[ \frac{\pi}{2} (k + 1) \right],
\]

so we have the following result.

**Theorem 2.** For all \( r > 0 \), as \( t \to 0^+ \), we have

\[
f_r(t) \sim \frac{r}{2\pi} \left\{ e^{-r} + \sum_{n \geq 0} \frac{(-r)^n}{n!} \sum_{m \geq 1} (-\pi^2)^m b_m \sum_{k=0}^{m} \binom{m}{k} \left( -\frac{2t}{\pi} \right)^k \cos \left[ \frac{\pi}{2} (k + 1) \right] \right\}.
\]

**Proof.** Letting \( t \to 0^+ \) in (10) and using (12), we obtain

\[
f_r(t) \sim \frac{r}{2\pi} \sum_{n \geq 0} \frac{(-r)^n}{n!} \sum_{m \geq 1} b_m \left( t^2 - \pi^2 \right)^m \sum_{k=0}^{m} \binom{m}{k} \left( -\frac{2t}{\pi} \right)^k \cos \left[ \frac{\pi}{2} (k + 1) \right],
\]

as \( I_0(0) = 1 \). Since \( b_0 = 1 \), the right-hand side of (13) corresponding to \( m = 0 \) gives the exponential term \( e^{-r} \). When \( t \to 0^+ \) in the remaining part of the right-hand side expression of (13), we arrive at the statement.

**Acknowledgement**

The authors are grateful to the reviewers for their constructive suggestions which finally encompassed the results achieved. The authors are also indebted to Professors Amparo Gil and Javier Segura, University of Cantabria, Spain, for critical reading and suggestions.
References


