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On rings with one middle class of injectivity domains

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Abstract. A module M is said to be modest if the injectivity domain of M is the class of all crumbling modules. In this paper, we investigate the basic properties of modest modules. We provide characterizations of some classes of rings using modest modules. In particular, we show that a ring having the class of crumbling modules as the only right middle class of injectivity domains is either a right V-ring or right Noetherian; and a commutative ring with this property is regular. We also give criteria for a ring having the class of crumbling modules as the only right middle class of injectivity domains.
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1. Introduction

Throughout this study, all rings are associative with an identity element and all modules are right and unital. Let R be a ring and M an R-module. We denote the socle and the radical of M by $\operatorname{Soc} M$ and $\operatorname{Rad} M$, respectively. $J(R)$ stands for the Jacobson radical of R. For terminology and notations used in this paper, we refer the reader to [6, 9, 13, 16].

Poor modules are introduced in [1] as modules that have their injectivity domain as minimal as possible, namely the class of all semisimple modules, where the injectivity domain of a module M is the class $\mathfrak{In}^{-1}(M) = \{N \in \mathcal{M}od\text{-}R : M$ is N-injective$\}$. This definition gives a natural opposite to injectivity of modules, since only injective modules have the class of all modules as their injectivity domain. Recently, many studies have been conducted concerning poor modules along with their generalizations and restrictions (see [2, 3, 4, 7, 10]).

As a proper generalization of poor modules, the notion of working-class modules is introduced in [8]. A module M is *working-class* if the injectivity domain $\mathfrak{In}^{-1}(M)$ of M is contained in the class of all modules having zero radical. Properties along

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with examples of these modules are given and rings over which every module is either injective or working-class are investigated in the same work.

Let R be an arbitrary ring and $\{S_\gamma : \gamma \in \Gamma\}$ a complete set of representatives of isomorphism classes of simple right R-modules and $S_1 = \bigoplus_{\gamma\in\Gamma} S_\gamma$. It is shown in [8, Example 1] that S_1 is working-class. It means that every ring has a semisimple working-class module. However, over a right SSI-ring, which is not semisimple Artinian, any semisimple module is not poor, where a ring R is called a right SSI-ring if every semisimple right R-module is injective. Put $S_2 = \bigoplus_{\gamma\in\Gamma} S_\gamma^{(R)}$. It follows from [10, Theorem 1] that $\mathfrak{In}^{-1}(S_2) = \{\, N \in \mathcal{M}od\text{-}R : N \text{ crumbles}\,\}$, and so S_2 is poor if and only if every module that crumbles is semisimple. Here a module M is said to *crumble* (or it is called a *crumbling* module) if $\mathrm{Soc}(M/N)$ is a direct summand of M/N for every submodule $N \leq M$. Recall that a module M is called a V-module if every simple module is M-injective. By [10, Corollary 2], a module M crumbles if and only if it is a locally Noetherian V-module. Using this fact, we give the following result playing a key role in our work. We denote the class of all crumbling right R-modules by $\mathcal{CRM}od\text{-}R$, namely $\mathcal{CRM}od\text{-}R = \{\, N \in \mathcal{M}od\text{-}R : N \text{ crumbles}\,\}$.

Lemma 1. *Let N be a module. Then N crumbles if and only if every semisimple module is N-injective. In particular, $\mathcal{CRM}od\text{-}R \subseteq \mathfrak{In}^{-1}(M)$ for every semisimple module M.*

Proof. ($\Rightarrow$): Let $M = \bigoplus_{i\in I} M_i$, where each M_i is simple. Assume that N is a crumbling module. Therefore N is a V-module and so M_i is N-injective for all $i \in I$. Since N is locally Noetherian, it follows from [16, 27.3] that M is N-injective.

($\Leftarrow$): Let $K \leq N$. By the hypothesis, $\mathrm{Soc}(N/K)$ is N-injective and then $\mathrm{Soc}(N/K)$ is N/K-injective. This means that $\mathrm{Soc}(N/K)$ is a direct summand of N/K. Hence N crumbles. □

The following is a direct consequence of Lemma 1. We denote the class of all semisimple modules by $\mathcal{SSM}od\text{-}R$.

Corollary 1. *Let R be a ring. Then we have*

$$\bigcap_{M\in\mathcal{SSM}od\text{-}R} \mathfrak{In}^{-1}(M) = \mathcal{CRM}od\text{-}R.$$

Proof. It is clear that $\mathcal{CRM}od\text{-}R \subseteq \bigcap_{M\in\mathcal{SSM}od\text{-}R} \mathfrak{In}^{-1}(M)$ by Lemma 1. Let $A \in \bigcap_{M\in\mathcal{SSM}od\text{-}R} \mathfrak{In}^{-1}(M)$ and $B \leq A$. Since $\mathrm{Soc}(A/B)$ is semisimple, we get that $\mathrm{Soc}(A/B)$ is A/B-injective and so $\mathrm{Soc}(A/B)$ is a direct summand of A/B. Hence A crumbles. □

Motivated by this fact, it is natural to consider modules whose injectivity domain is the class of all crumbling modules. With this idea in mind, we call a module M *modest* if the injectivity domain $\mathfrak{In}^{-1}(M)$ of M is the class of all crumbling modules.

The main purpose of this paper is to study the concept of modest modules and their application on the rings with exactly one right middle class of injectivity domains. In section 2, we show that every ring has a modest module. We prove that rings over which every injective module is modest are right SSI-rings. We also show that a right WV-ring is a right Noetherian ring if and only if $J(R)$ is modest. It follows that a WV-ring R is right Artinian if and only if $J(R)$ is poor.

In section 3, we introduce the notion of the crumbling submodule $C(M)$ of a module M and investigate the properties of this submodule. Using crumbling submodules, we generalize semi-artinian modules to weakly semi-artinian modules. In particular, we prove that a ring R is right weakly semi-artinian if and only if every R-module is an essential extension of its crumbling submodule.

Theorem 1 guarantees the existence of a modest module for every ring. It is well known that over every ring there are poor and injective modules, so there exist three injectivity domains for every ring R: $\mathcal{SSM}od\text{-}R$, $\mathcal{CRM}od\text{-}R$, and $\mathcal{M}od\text{-}R$. In section 4, we study the rings over which these three classes are different and there are no other injectivity domains. We call them right CMC-rings. We show that a right CMC-ring is either a right V-ring or right Noetherian, and, moreover, if it is commutative, then it is regular. We also give criteria for being a right CMC-ring.

Injectivity domains of an arbitrary ring has been investigated in [14]. In this work, we consider rings admitting the class $\mathcal{CRM}od\text{-}R$ as the only right middle class of injectivity domains, namely right CMC-rings. As right CMC-rings have one middle class of injectivity domains, results presented here give a partial answer to the problem of determining rings with one right middle class of injectivity domains in the following manner: If a ring R has one right middle class of injectivity domains, say $\mathcal{I}$, then we have three cases:

(1) If $\mathcal{I} = \mathcal{CRM}od\text{-}R$, then R is a right CMC-ring.

(2) The case when $\mathcal{SSM}od\text{-}R = \mathcal{CRM}od\text{-}R$ (without the assumption that R has one right middle class) is investigated in [10] and it is shown that this equality is equivalent to the existence of a semisimple poor right R-module (see [10, Theorem 1]).

(3) The equality $\mathcal{M}od\text{-}R = \mathcal{CRM}od\text{-}R$ (without the assumption that R has one right middle class) holds if and only if R is a right SSI-ring (see Theorem 3). As in the previous case, there are no studies on rings with one right middle class in this case. One can consider this case over a right SSI-ring R which is not a right QI-ring. Here, a ring A is called a right QI-ring if every self-injective right A-module is injective. Then there exists a self-injective right R-module M, which is not injective and so $M \in \mathfrak{In}^{-1}(M)$ and $\mathfrak{In}^{-1}(M) \neq \mathcal{M}od\text{-}R$. Since R is a right SSI-ring, M is not semisimple which implies that $\mathfrak{In}^{-1}(M) \neq \mathcal{SSM}od\text{-}R$. Therefore, we have $\mathfrak{In}^{-1}(M) = \mathcal{I}$. Hence, $\mathcal{I}$ contains the class of all self-injective modules which are not injective.

As we are dealing with rings having one right middle class of injectivity domains, the following results will be of use. For a ring X, we use $\mathfrak{In}_X^{-1}(Y)$ to denote the injectivity domain of an X-module Y.

Lemma 2. *Let R be a ring and I an ideal of R. If $\mathfrak{In}_{R/I}^{-1}(M) \neq \mathfrak{In}_{R/I}^{-1}(N)$ for R/I-modules M and N, then $\mathfrak{In}_R^{-1}(M) \neq \mathfrak{In}_R^{-1}(N)$.*

Proof. Since $\mathfrak{In}_{R/I}^{-1}(M) \neq \mathfrak{In}_{R/I}^{-1}(N)$, without loss of generality, we may assume that there is an R/I-module $L \in \mathfrak{In}_{R/I}^{-1}(M) \backslash \mathfrak{In}_{R/I}^{-1}(N)$. Then there is an R/I-submodule

U of L and an R/I-homomorphism $f : U \to N$ that cannot be extended to L. Since R/I-submodules are R-submodules and R/I-homomorphisms of R/I-modules are R-homomorphisms as well, we have $L \notin \mathfrak{In}_R^{-1}(N)$. On the other hand, every R-submodule U of L is an R/I-module since $UI \subseteq LI = 0$ and every R-homomorphism $f : U \to M$ is also an R/I-homomorphism, and can therefore be extended to a homomorphism $g : L \to M$. Hence $L \in \mathfrak{In}_R^{-1}(M)$ and so $\mathfrak{In}_R^{-1}(M) \neq \mathfrak{In}_R^{-1}(N)$. □

Corollary 2. *Let R be a ring and I an ideal of R. Then the number of different injectivity domains over R/I does not exceed the number of different injectivity domains over R.*

2. Modest modules

Definition 1. *Let R be a ring and M an R-module. M is called* modest *if the injectivity domain $\mathfrak{In}^{-1}(M)$ of M is the class of all crumbling modules.*

Firstly, we give the following lemma proof of which follows from [16, 16.2 §27.2].

Lemma 3. *The class of crumbling modules is closed under submodules, factor modules and direct sums.*

Corollary 3. *Let M be a module such that mR crumbles for every $mR \in \mathfrak{In}^{-1}(M)$. If $N \in \mathfrak{In}^{-1}(M)$, then N crumbles.*

Proof. Suppose that M is N-injective for some module N. We can write $N = \sum_{m\in N} mR$. Since M is N-injective, M is mR-injective and so mR crumbles as assumed. Let $A = \bigoplus_{m\in N} mR$. Then A crumbles by Lemma 3. Therefore, there exists an epimorphism $\psi : A \longrightarrow N$. Applying the lemma once more, we have that N crumbles. □

Theorem 1. *Every ring has a modest module.*

Proof. Let R be an arbitrary ring and $\{C_\gamma : \gamma \in \Gamma\}$ a complete set of representatives of isomorphism classes of non-crumbling cyclic right R-modules. If $\Gamma = \emptyset$, then every cyclic right R-module crumbles and so every right R-module by the proof of Corollary 3. Therefore, every injective right R-module is modest. Assume that $\Gamma \neq \emptyset$. Since each C_γ $(\gamma \in \Gamma)$ is non-crumbling, there exists a factor module F_γ of C_γ such that $\mathrm{Soc}(F_\gamma)$ is a proper essential submodule of F_γ. Let $M = \bigoplus_{\gamma\in\Gamma} \mathrm{Soc}(F_\gamma)$ and $mR \in \mathfrak{In}^{-1}(M)$. Suppose that mR is a non-crumbling module. Then, for some $\gamma \in \Gamma$, we can write $mR \cong C_\gamma$. Since M is mR-injective, M is F_γ-injective and so $\mathrm{Soc}(F_\gamma)$ is F_γ-injective. This is a contradiction. Thus $\mathfrak{In}^{-1}(M) \subseteq \mathcal{CRM}od\text{-}R$.

Let N be a crumbling module. By Lemma 1, M is N-injective since M is semisimple. It means that M is modest. □

Proposition 1. *Let R be an arbitrary ring and $\{S_\gamma : \gamma \in \Gamma\}$ a complete set of representatives of isomorphism classes of simple right R-modules. Put $S_2 = \bigoplus_{\gamma\in\Gamma} S_\gamma^{(R)}$. Then S_2 is modest.*

Proof. By the proof of [10, Theorem 1]. □

Lemma 4. *Let M be a crumbling module. Then* $\operatorname{Rad} M = 0$.

Proof. Since crumbling modules are V-modules, it follows from [16, 23.1] that $\operatorname{Rad} M = 0$. $\square$

Note that a module with zero radical need not crumble, in general. For instance, consider the $\mathbb{Z}$-module $M = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ is the set of all prime integers. Then $\operatorname{Rad} M = 0$. Since $\operatorname{Soc}(M) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$ is not a direct summand of M, we obtain that M does not crumble.

By Lemma 4, modest modules are working-class. However, a working-class module need not be modest, in general. An example showing the existence of a working-class module which is not modest is given in Example 1.

Lemma 5. *Let R be an arbitrary ring, $\{S_\gamma : \gamma \in \Gamma\}$ a complete set of representatives of isomorphism classes of simple right R-modules and $S_1 = \bigoplus_{\gamma\in\Gamma} S_\gamma$. Then we have $\mathcal{CRM}od\text{-}R \subseteq \mathfrak{In}^{-1}(S_1) \subseteq \{N \in \mathcal{M}od\text{-}R : N \text{ is a } V\text{-module}\}$.*

Proof. By Lemma 1, we get that $\mathcal{CRM}od\text{-}R \subseteq \mathfrak{In}^{-1}(S_1)$. Let S_1 be N-injective for some R-module N. Then S_γ is N-injective for all $\gamma \in \Gamma$. It means that N is a V-module. $\square$

Theorem 2. *Let R be an arbitrary ring, $\{S_\gamma : \gamma \in \Gamma\}$ a complete set of representatives of isomorphism classes of simple right R-modules and $M = \prod_{\gamma\in\Gamma} S_\gamma$. Then*

$$\mathfrak{In}^{-1}(M) = \{N \in \mathcal{M}od\text{-}R : N \text{ is a } V\text{-module}\}.$$

Moreover, if R is right Noetherian, then $S_1 = \bigoplus_{\gamma\in\Gamma} S_\gamma$ and M are modest.

Proof. By [16, 16.1 (1)], for a module N, M is N-injective if and only if every S_γ ($\gamma \in \Gamma$) is N-injective. So $\mathfrak{In}^{-1}(M) = \{N \in \mathcal{M}od\text{-}R : N \text{ is a } V\text{-module}\}$.

Let R be a right Noetherian ring. Then every V-module is locally Noetherian and so S_1 and M are modest by Lemma 5. $\square$

Recall that a module M is *semi-artinian* if every nonzero factor module of M has an essential socle and a ring R is right semi-artinian if R_R is a semi-artinian module.

Lemma 6. *Let R be a right semi-artinian ring and M a crumbling R-module. Then M is semisimple.*

Proof. It is assumed that $\operatorname{Soc}(M) \neq 0$. Since M crumbles, there exists a submodule K of M such that $M = \operatorname{Soc}(M) \oplus K$. If K is a nonzero submodule of M, it follows from the hypothesis that K has a simple submodule, say S. Then $S \subseteq \operatorname{Soc}(M) \cap K = 0$, a contradiction. Hence M is semisimple. $\square$

As a consequence of this fact we have the following result.

Corollary 4. *Let R be a right semi-artinian ring and M a right R-module. Then M is modest if and only if it is poor.*

Now we shall characterize the rings over which all modules crumble. Recall that a ring R is *a right SSI-ring* if every semisimple right R-module is injective. A ring over which self-injective modules are injective is an example of right SSI-rings. It is shown in [5, Proposition 1] that a ring R is a right SSI-ring if and only if R is a right Noetherian V-ring.

Theorem 3. *The following statements are equivalent for a ring R.*

(*1*) *R_R crumbles.*

(*2*) *R is a right SSI-ring.*

(*3*) *Every right R-module crumbles.*

(*4*) *Every injective right R-module crumbles.*

(*5*) 0 *is a modest R-module.*

(*6*) *There exists an injective modest R-module.*

(*7*) *Every injective right R-module is modest.*

(*8*) *Every modest right R-module is injective.*

(*9*) *Every direct summand of a modest R-module is modest.*

Proof. (3) $\Rightarrow$ (1), (3) $\Leftrightarrow$ (5), (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (3), (3) $\Rightarrow$ (8) and (3) $\Rightarrow$ (9) are clear.

(1) $\Leftrightarrow$ (2): Since crumbling modules are exactly locally Noetherian V-modules, it follows from [5, Proposition 1] that R is a right SSI-ring.

(2) $\Rightarrow$ (3): Let M be an R-module and N any submodule of M. It is assumed that $\mathrm{Soc}(M/N)$ is injective and so there exists a submodule K/N of M/N such that $M/N = \mathrm{Soc}(M/N) \oplus K/N$. Thus M crumbles.

(3) $\Leftrightarrow$ (4): It follows from Lemma 3.

(6) $\Rightarrow$ (7): Let M be an injective modest module. Then $\mathcal{M}od\text{-}R = \mathfrak{In}^{-1}(M) = \mathcal{CRM}od\text{-}R$. For every injective module I, we have $\mathfrak{In}^{-1}(I) = \mathcal{M}od\text{-}R = \mathcal{CRM}od\text{-}R$ so that I is modest.

(8) $\Rightarrow$ (6): By Theorem 1, R has a modest module, say N. It is assumed that N is injective.

(9) $\Rightarrow$ (5): By Theorem 1, R has a modest module, say N. Then, it is assumed that 0 is a modest module as a direct summand of N. □

Using Theorem 3, it is easy to characterize the rings over which every module is modest.

Corollary 5. *Let R be a ring. If every right R-module is modest, then R is semisimple Artinian.*

Example 1 (see [11, Example 3.2])**.** *Let K be a field, A an infinite set, $Q = \prod_{i\in A} K_i$, where $K_i = K$ for all $i \in A$, $L = \bigoplus_{i\in A} K_i$, and let R be the subring generated by L*

and 1_Q. *Then* R *is a* V*-ring which is not Noetherian and so* R *is not an* SSI*-ring. By [8, Theorem 1] and Theorem 3, there exists a working-class* R*-module which is not modest.*

As a generalization of right V-rings, a ring R is called *a right weakly-*V*-ring* (a right WV-ring for short) if every simple R-module is R/I-injective for any right ideal I of R such that R/I is proper. A right WV-ring need not be right Noetherian, in general. In [12], it is shown that a right WV-ring R is Noetherian if and only if every cyclic right R-module M can be written as $M = A \oplus B$, where A is either a CS-module or a Noetherian module and B is a projective module. Here we obtain a similar relationship using modest modules.

Theorem 4. *Let* R *be a right* WV*-ring. Then* R *is a right Noetherian ring if and only if* $J(R)$ *is modest.*

Proof. ($\Rightarrow$): Let R be a right Noetherian ring and $J(R)$ N-injective for some R-module N. Suppose that $a \in N$. Clearly, $J(R)$ is aR-injective. Then $R/\operatorname{ann}_r(a) \cong aR$. If $\operatorname{ann}_r(a) = 0$, then $J(R)$ is injective and so $J(R) = 0$. Since R is a right WV-ring, it follows from [13, Corollary 6.8] that R is a right V-ring. Thus R is a right SSI-ring. Applying Theorem 3, we obtain that N crumbles.

Let $\operatorname{ann}_r(a) \neq 0$. Since R is right Noetherian and a right WV-ring, $R/\operatorname{ann}_r(a) \cong aR$ is a Noetherian V-module. Therefore, N crumbles by Lemma 3. Hence, $J(R)$ is modest by Lemma 1.

($\Leftarrow$): If $J(R) = 0$, R is a right SSI-ring by Theorem 3. Thus R is a right Noetherian ring. Suppose that $J(R) \neq 0$. It follows from [13, Lemma 6.12] that $J(R)$ is simple. Since R is a right WV-ring, $J(R)$ is $R/J(R)$-injective. Then, it is assumed that $R/J(R)$ is a right Noetherian R-module. Hence R is a right Noetherian ring. □

In general, a semisimple module need not be poor. Rings over which there is a semisimple poor right module are studied in [10]. Now, we give a characterization of right WV-rings which are not right V-rings such that every semisimple right module is either modest or injective.

Theorem 5. *Let* R *be a right* WV*-ring which is not a right* V*-ring. Then the following statements are equivalent.*

(*1*) *R is a right Noetherian ring.*

(*2*) *Every non-injective semisimple right R-module is modest.*

(*3*) *$J(R)$ is modest.*

Proof. (1) $\Rightarrow$ (2): Let M be a non-injective semisimple right R-module and M mR-injective for some cyclic right R-module mR. Since M is not injective, $R/\operatorname{ann}_r(m) \cong mR$ is proper. By (1), mR is Noetherian and so it crumbles since R is a right WV-ring. Therefore, $\mathfrak{In}^{-1}(M) \subseteq \mathcal{CRM}od\text{-}R$ by Corollary 3.

Let N be a crumbling module. By Lemma 1, M is N-injective. Thus M is modest.

(2) $\Rightarrow$ (3): It is clear since $J(R)$ is a non-injective simple right R-module.

(3) $\Rightarrow$ (1): It follows from Theorem 4. □

Corollary 6. *A right WV-ring is right Artinian if and only if $J(R)$ is poor.*

Proof. If $J(R) = 0$, it is clear by Theorem 4. Suppose that $J(R) \neq 0$.

($\Rightarrow$): R is a right WV-ring which is not a right V-ring. R is also right Noetherian. It follows from Theorem 5 and Corollary 4 that $J(R)$ is poor.

($\Leftarrow$): Since R is a right WV-ring and $J(R) \neq 0$, $J(R)$ is simple and $R/J(R)$-injective. It is assumed that $R/J(R)$ is semisimple. Thus, R is right Artinian. $\square$

A ring R is said to be *local* if $R/J(R)$ is simple, and it is said to be *semilocal* if $R/J(R)$ is semisimple. It is well known that over a semilocal ring a module with zero radical is semisimple. Using this fact we have the following result.

Corollary 7. *Let R be a semilocal ring and M a right R-module. Then the following statements are equivalent.*

(*1*) *M is modest.*

(*2*) *M is poor.*

(*3*) *M is working-class.*

Remark 1. *The structure of poor abelian groups is completely determined in [2, Theorem 1]. Let R be a commutative hereditary Noetherian ring and M an R-module. Assume that M is modest. By [8, Theorem 5], we obtain that M is poor. In particular, if R is the ring of integers, then M is a poor abelian group.*

3. Weakly semi-artinian modules and rings

Let M be a module. We define *the crumbling submodule* of M as the sum of all submodules of M that crumble and denote it by $C(M)$. It follows from Lemma 3 that $C(M)$ is the largest submodule of M that crumbles. It is clear that $\mathrm{Soc}(M) \subseteq C(M)$.

Proposition 2. *Let R be a ring and M an R-module. Then the following statements hold.*

(*1*) *If $f : M \to N$ is a homomorphism of R-modules, then $f(C(M)) \subseteq C(N)$.*

(*2*) *For every submodule K of M, $C(K) = K \cap C(M)$.*

(*3*) *$C(M) \trianglelefteq M$ if and only if $C(K) \neq 0$ for every $0 \neq K \leq M$.*

(*4*) *$C(M)$ is a fully invariant submodule of M.*

(*5*) *$C(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} C(M_i)$.*

(*6*) *$C(M)$ is coatomic.*

(*7*) *If $C(M) \subseteq \mathrm{Rad}(M)$, then $C(M)$ is a small submodule of M.*

Proof. (1): It follows from Lemma 3.
(2): It is clear that $C(K) \subseteq K \cap C(M)$. Let $a \in K \cap C(M)$. Then aR is a (locally) Noetherian V-module and so $a \in aR \subseteq C(K)$. Therefore, $K \cap C(M) \subseteq C(K)$ and $C(K) = K \cap C(M)$.
(3): ($\Rightarrow$) By (2), we have $C(K) = K \cap C(M) \neq 0$.
($\Leftarrow$) Let $C(M) \cap L = 0$ for some submodule L of M. It follows from (2) that $C(L) = C(M) \cap L = 0$, which contradicts with the assumption. Thus, $L = 0$ and $C(M) \trianglelefteq M$.
(4): It is a consequence of (1).
(5): Let $a \in C(\bigoplus_{i \in I} M_i)$. Then there exist elements $a_{i_j} \in M_{i_j}$ $(1 \leq j \leq n)$ such that $a = a_{i_1} + a_{i_2} + \ldots + a_{i_n}$. Since aR is a (locally) Noetherian V-module, $a_{i_j}R \subseteq aR$ is a (locally) Noetherian V-module and so $a_{i_j} \in a_{i_j}R \subseteq C(M_{i_j})$ for every j in $\{1, 2, \ldots, n\}$. Therefore, $a = a_{i_1} + a_{i_2} + \ldots + a_{i_n} \in C(M_{i_1}) \oplus C(M_{i_2}) \oplus \cdots \oplus C(M_{i_n}) \subseteq \bigoplus_{i \in I} C(M_i)$. By (1), we have $\bigoplus_{i \in I} C(M_i) \subseteq C(\bigoplus_{i \in I} M_i)$.
(6): Let $\mathrm{Rad}(C(M)/X) = C(M)/X$ for some submodule X of $C(M)$. Then $X + \mathrm{Rad}(C(M)) = C(M)$. Since $C(M)$ crumbles, $\mathrm{Rad}(C(M)) = 0$ by Lemma 4, and so $X = C(M)$.
(7): Let $C(M) + N = M$ for some submodule N of M. Then $\mathrm{Rad}(M) + N = M$, which implies that $\mathrm{Rad}(M/N) = M/N$. Note that $M/N \cong C(M)/[C(M) \cap N]$. By Lemma 3 and Lemma 4, we have $M/N = \mathrm{Rad}(M/N) = 0$. $\square$

Following Proposition 2, let us note that for a ring R, $C(R_R)$ is an ideal of R.

A module M is called *locally projective* whenever $g : A \longrightarrow B$ is an epimorphism and $f : M \longrightarrow B$ is a homomorphism of R-modules A and B, then for every finitely generated submodule M_0 of M there exists a homomorphism $h : M \longrightarrow A$ such that $gh|_{M_0} = f|_{M_0}$. It is clear that every projective module is locally projective.

Proposition 3. *Let R be a ring and M a locally projective R-module. Then $C(M) = MC(R_R)$.*

Proof. Clearly, we have $MC(R_R) \subseteq C(M)$ by Proposition 2-(1). Let $m \in C(M)$. Since M is a locally projective module, there are a finite number of homomorphisms $f_i : M \longrightarrow R$ and elements $m_i \in M$ $(1 \leq i \leq n)$ such that $m_1 f_1(m) + m_2 f_2(m) + \ldots + m_n f_n(m) = m$. Applying Proposition 2-(1) once more, we obtain that $f_i(m) \in C(R_R)$ and so $C(M) \subseteq MC(R_R)$. Hence $C(M) = MC(R_R)$. $\square$

Corollary 8. *For a projective right R-module M, $C(M) = MC(R_R)$.*

We call a module M *weakly semi-artinian* if $C(M/N) \trianglelefteq M/N$ for every nonzero factor module M/N of M, and a ring R right *weakly semi-artinian* if R_R is a weakly semi-artinian module. Semi-artinian modules and crumbling modules are weakly semi-artinian.

Proposition 4. *Let M be a weakly semi-artinian module and N a submodule of M. Then N and M/N are weakly semi-artinian.*

Proof. Let $K \leq N$ with $N/K \neq 0$. If $L/K \cap C(N/K) = 0$, it follows from Proposition 2 (2) that

$$\begin{aligned} 0 &= L/K \cap C(N/K) \\ &= L/K \cap [C(M/K) \cap N/K] \\ &= C(M/K) \cap L/K \end{aligned}$$

and so $L/K = 0$. It means that N is weakly semi-artinian.
Let $N \leq A \lneqq M$. Then we can write

$$\frac{\frac{M}{N}}{\frac{A}{N}} \cong \frac{M}{A}.$$

It follows from the assumption that M/N is weakly semi-artinian. □

Let c denote the left exact preradical which assigns to each module its crumbling submodule. The *crumbling series* of M is the ascending chain of submodules

$$0 = c_0(M) \subseteq c_1(M) \subseteq c_2(M) \subseteq \ldots \subseteq c_\alpha(M) \subseteq c_{\alpha+1}(M) \subseteq \ldots,$$

where $c_0(M) = 0, c_1(M) = C(M), c_{\alpha+1}(M)/c_\alpha(M) = C(M/c_\alpha(M))$ for every ordinal $\alpha \geq 0$ and $c_\alpha(M) = \bigcup_{0 \leq \beta < \alpha} c_\beta(M)$ if α is a limit ordinal. The corresponding radical $\overline{c}$ is obtained in the following way:

$$\overline{c}(M) = c_\alpha(M), \text{ where } \alpha \text{ is the first ordinal for which } c_\alpha(M) = c_{\alpha+1}(M).$$

Proposition 5. *Let M be a module. Then the following hold.*

(*1*) *Every submodule $c_\alpha(M)$ is fully invariant.*

(*2*) *If M is self-projective, then so is $M/c_\alpha(M)$ for all α.*

Proof. (1): Suppose that the statement is false. Then there is a least ordinal α such that $c_\alpha(M)$ is not fully invariant. Clearly, α is not zero, nor is it a limit ordinal. Let f be an endomorphism of M. Since $f(c_{\alpha-1}(M)) \subseteq c_{\alpha-1}(M)$ under the assumption on α, f induces an endomorphism f' of $M' = M/c_{\alpha-1}(M)$. Then we have $f'(c_1(M')) \subseteq c_1(M')$ by Proposition 2 so that $f(c_\alpha(M)) \subseteq c_\alpha(M)$. Hence $c_\alpha(M)$ is fully invariant, a contradiction.
(2) follows from (1) and the fact that factor modules of self-projective modules by fully invariant submodules are again self-projective (see [9, 3.1]). □

Theorem 6. *The following are equivalent for a module M.*

(*1*) *M is weakly semi-artinian.*

(*2*) *Every nonzero homomorphic image of M has a nonzero crumbling submodule.*

(*3*) *$c_\rho(M) = M$ for some ordinal $\rho \geq 0$.*

(*4*) *There exists an ascending chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset M_{\alpha+1} \subset \cdots \subset M_\tau = M$$

such that $M_{\alpha+1}/M_\alpha$ *crumbles for each* $0 \leq \alpha < \tau$ *and* $M_\alpha = \bigcup_{0 \leq \beta < \alpha} M_\beta$ *if* α *is a limit ordinal.*

(*5*) *There exists an ascending chain of submodules*

$$0 = L_0 \subset L_1 \subset \cdots \subset L_\sigma \subset L_{\sigma+1} \subset \cdots \subset L_\tau = M$$

such that $L_{\sigma+1}/L_\sigma$ *is weakly semi-artinian for all ordinals* $0 \leq \sigma < \tau$*, and, for all limit ordinals* σ*,* $L_\sigma = \bigcup_{0 \leq \rho < \sigma} L_\rho$*.*

Proof. (1) ⇒ (2) and (3) ⇒ (4) ⇒ (5) are clear.

(2) ⇒ (3): Considering the crumbling series of M, there exists an ordinal $\rho \geq 0$ such that $c_{\rho+1}(M) = c_\rho(M)$ so that $C(M/c_\rho(M)) = 0$. It follows from (2) that $M = c_\rho(M)$.

(5) ⇒ (1): Suppose that there is an ascending chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_\sigma \subset L_{\sigma+1} \subset \cdots \subset L_\tau = M$$

of submodules of M with the properties given in (5). Let K and N be submodules of M with $K \subsetneqq N$. We will show that $C(N/K) \neq 0$.

We have $N \cap L_\tau = N \neq K = K \cap L_\tau$. Let σ be the least ordinal such that $N \cap L_\sigma \neq K \cap L_\sigma$. It is clear that $0 < \sigma \leq \tau$ and σ is not a limit ordinal. Then $N \cap L_{\sigma-1} = K \cap L_{\sigma-1}$ so that

$$(N + L_{\sigma-1})/(K + L_{\sigma-1}) \cong N/[K + (N \cap L_{\sigma-1})] = N/K.$$

Let us show that $N \cap L_\sigma \nsubseteq K + L_{\sigma-1}$. If this is not true, then we have

$$N \cap L_\sigma \subseteq K + (N \cap L_{\sigma-1}) = K + (K \cap L_{\sigma-1}) = K$$

so that $N \cap L_\sigma = K \cap L_\sigma$, which is a contradiction.

Then the module

$$[N \cap L_\sigma + K + L_{\sigma-1}]/(K + L_{\sigma-1})$$

is a nonzero subfactor of the weakly semi-artinian module $L_\sigma/L_{\sigma-1}$ and so it has a nonzero crumbling submodule. Hence $C(N/K) \neq 0$. □

As a consequence, we see that a sum of weakly semi-artinian modules is weakly semi-artinian and every module contains a maximal weakly semi-artinian submodule which is the union of the crumbling series of the module. Using these facts, we can characterize right weakly semi-artinian rings in terms of their right modules.

Proposition 6. *The following are equivalent for a ring* R*.*

(*1*) R *is right weakly semi-artinian.*

(*2*) *Every right R-module is weakly semi-artinian.*

(*3*) *Every nonzero right R-module has a nonzero crumbling submodule.*

(*4*) *Every R-module is an essential extension of its crumbling submodule.*

It turns out that the notion weakly semi-artinian modules are related to T-nilpotency of $J(R)$. Let R be a ring and M a right R-module. Recall that $J(R)$ is T-nilpotent on M if for every $m \in M$ and every sequence $a_1, a_2, \ldots \in J(R)$ there exists an integer n such that $ma_1a_2 \cdots a_n = 0$. $J(R)$ is called left T-nilpotent if it is T-nilpotent on R_R, i.e. for every sequence $a_1, a_2, \ldots \in J(R)$ there exists an integer n such that $a_1a_2 \cdots a_n = 0$

Proposition 7. *$J(R)$ is T-nilpotent on every weakly semi-artinian right R-module.*

Proof. Let M be a weakly semi-artinian right R-module. For each $m \in M$ we can define $o(m)$ to be the smallest ordinal β for which $m \in c_\beta(M)$. Then $o(m)$ cannot be a limit ordinal, because $m \in \sum_{\alpha<\beta} c_\alpha(M)$ implies $m \in c_\alpha(M)$ for some $\alpha < \beta$. Hence we can write $o(m) = \alpha + 1$ for some α (unless $m = 0$). However, $c_{\alpha+1}(M)/c_\alpha(M)$ crumbles and so $\mathrm{Rad}(c_{\alpha+1}(M)/c_\alpha(M)) = 0$, and therefore $c_{\alpha+1}(M)J(R) \subseteq c_\alpha(M)$. It follows that $o(ma) < o(m)$ for every nonzero $m \in M$ and $a \in J(R)$. If $a_1, a_2, \ldots$ is a sequence of elements of $J(R)$ such that $ma_1a_2 \cdots a_n \neq 0$ for every n, then $o(m) > o(ma_1) > o(ma_1a_2) > \ldots > o(ma_1a_2 \cdots a_n)$, but since every strictly descending chain of ordinals is finite, this is impossible. □

Here are some consequences regarding right weakly semi-artinian rings. The proofs of some of the following results are straightforward and therefore omitted.

Corollary 9. *Let R be a right weakly semi-artinian ring. Then $J(R)$ is left T-nilpotent.*

Corollary 10. *A ring R is right perfect if and only if it is semilocal and right weakly semi-artinian.*

Corollary 11. *A ring R is right weakly semi-artinian if and only if $J(R)$ is left T-nilpotent and $R/J(R)$ is right weakly semi-artinian.*

Proof. ($\Rightarrow$): It is clear.

($\Leftarrow$): Let M be a nonzero right R-module. Since $J(R)$ is left T-nilpotent, there is a submodule L of M such that $LJ(R) = 0$ by [15, Lemma 2.9]. Then $L_{R/J(R)}$ is a right weakly semi-artinian module and so $C(L_{R/J(R)}) \neq 0$, which implies that $C(L_R) \neq 0$. This shows that M is weakly semi-artinian. By Proposition 6, R is right weakly semi-artinian. □

4. On rings with one right middle class

In this section, we consider the rings with exactly one right middle class of injectivity domains, say $\mathcal{I}$, in the case when $\mathcal{I} = \mathcal{CRM}od\text{-}R$.

Definition 2. *We call a ring R a right CMC-ring if it has $\mathcal{CRM}od\text{-}R$ as the only right middle class of injectivity domains.*

Lemma 7. *A factor ring of a right CMC-ring is either a right CMC-ring or it has no right middle class.*

Proof. It follows from Corollary 2. $\square$

Lemma 8. *Let R be a right CMC-ring. Suppose that there exists a singular right R-module M such that injective hull $E(M)$ of M does not crumble. Then the following conditions hold.*

(*1*) *Every nonsingular right R-module is injective (hence semisimple).*

(*2*) *The second singular submodule splits in any right R-module.*

(*3*) *There exists a ring direct sum $R = S \oplus T$, where S is semisimple and $C(T)$ is essential in T_T with $Z(T_T) \subseteq C(T_T)$.*

(*4*) $C(R_R) \trianglelefteq R_R$.

Proof. (1): Since every nonsingular module is $E(M)$-injective and $E(M)$ does not crumble, every nonsingular module is injective, hence semisimple.

(2): It follows from the proof of [10, Lemma 2 (ii)].

(3): By (2), we have $R_R = A \oplus Z_2(R_R)$ for some right ideal A. A_R is semisimple by (1). It follows from the proof of [10, Lemma 2 (iii)] that A is a two-sided ideal. Assume that $Z(R_R)$ does not crumble. Then $Z(E(R_R))$ does not crumble by Lemma 3. Since $Z(E(R_R))$ is a fully invariant submodule of $E(R_R)$, it is quasi-injective. Then $Z(E(R_R))$ is neither poor nor modest so that it is injective. $Z(R_R) = Z(E(R_R)) \cap R_R$ is a closed submodule of R, which implies that $Z(R_R) = Z_2(R_R)$ and $Z(R_R)$ splits in R_R. Then $Z(R_R)$ must be zero contradicting our assumption. Therefore, $Z(R_R)$ crumbles. Letting $S = A$ and $T = Z_2(R_R)$, we have a ring decomposition into a semisimple ring S and a ring T such that $C(T)$ is essential in T_T with $Z(T_T) \subseteq C(T_T)$.

(4): It follows from (3), since $\mathrm{Soc}(R_R) \subseteq C(R_R)$. $\square$

Corollary 12. *Let R be a right CMC-ring. Suppose that either R is a non-right GV-ring or $R/\mathrm{Soc}(R_R)$ is a non-right SSI-ring. Then the following conditions hold.*

(*1*) *Every nonsingular right R-module is injective (hence semisimple).*

(*2*) *The second singular submodule splits into any right R-module.*

(*3*) *There exists a ring direct sum $R = S \oplus T$, where S is semisimple and $C(T)$ is essential in T_T with $Z(T_T) \subseteq C(T_T)$.*

(*4*) $C(R_R) \trianglelefteq R_R$.

Proof. If R is not a right GV-ring, then there is a singular simple right R-module M which is not injective. Then $\mathrm{Rad}(E(M)) \neq 0$ for the injective hull $E(M)$ of M so that $E(M)$ does not crumble.

If $R/\operatorname{Soc}(R_R)$ is not a right SSI-ring, then there is a singular right R-module N which does not crumble by [9, 16.2]. By Lemma 3, $E(N)$ does not crumble.
In both cases, Lemma 8 completes the proof. $\square$

Lemma 9. *Let R be a right CMC-ring. Then R is either right weakly semi-artinian or right Noetherian.*

Proof. Assume that R is not right weakly semi-artinian. Let I be the union of the right crumbling series of R. Then R/I either has no right middle class or is a (nonzero) right CMC-ring with $C(R/I_{R/I}) = 0$ by Lemma 7 and the assumption. If R/I has no right middle class, it follows from the proof for [10, Lemma 5] that R is right Noetherian.

If R is a (nonzero) right CMC-ring with $C(R/I_{R/I}) = 0$, since $\operatorname{Soc}(R/I_{R/I}) \subseteq C(R/I_{R/I}) = 0$, Corollary 12 implies that R is a right SSI-ring, a contradiction. Therefore, R is right weakly semi-artinian in this case. $\square$

Lemma 10. *Let R be a right CMC-ring. If R has a locally Noetherian right module which does not crumble, then R is right Noetherian.*

Proof. Let M be a locally Noetherian right R-module which does not crumble and let $\{E_i | i \in I\}$ be any family of injective right R-modules. Then, by [9, 2.5 p. 10], $\bigoplus_{i\in I} E_i$ is M-injective. Since R is a right CMC-ring and M does not crumble, $\bigoplus_{i\in I} E_i$ is injective. Hence R is right Noetherian. $\square$

The following result shows where the class of right CMC-rings lies.

Theorem 7. *Let R be a right CMC-ring. Then R is either a right V-ring or right Noetherian.*

Proof. Assume that R is not a right V-ring. Then there exists a simple right R-module S, which is properly contained in $E(S)$. By Lemma 9, R is either right weakly semi-artinian or right Noetherian. If R is right weakly semi-artinian, then $E(S)/S$ has a nonzero submodule K/S that crumbles. Since S is Noetherian and K/S is locally Noetherian, K is also locally Noetherian by [16, 27.2 (iii)]. Since $S \ll K$, we have $0 \neq S \subseteq \operatorname{Rad}(K)$ which implies that K does not crumble. By Lemma 10, R is right Noetherian. $\square$

Remark 2. *Let R be a right CMC-ring which is not right Noetherian. It follows from Theorem 7 and [10, Example 3] that R is weakly semi-artinian, but not semi-artinian.*

Here we give an example of a right Noetherian ring which is not a right CMC-ring.

Example 2. *Let R be a right Noetherian and right WV-ring which is not a right V-ring.*

Let X be a non-injective right R-module and $aR \subseteq \mathfrak{In}^{-1}(X)$. Since X is not injective and $aR \cong R/\operatorname{ann}_r(a)$, we have $\operatorname{ann}_r(a) \neq 0$. Therefore, by [13, Corollary 6.13], $J(R) = \operatorname{Soc} R \subseteq \operatorname{ann}_r(a)$. It follows from [13, Corollary 6.8] *and assumption that $R/\operatorname{ann}_r(a) \cong aR$ crumbles. Thus, according to Corollary 3, $\mathfrak{In}^{-1}(X) \subseteq \mathcal{CRM}od\text{-}R$.*

Next, we show that $\mathcal{CRM}od\text{-}R = \{A \in \mathcal{M}od\text{-}R : A$ is singular$\}$. Let $N \in \mathcal{CRM}od\text{-}R$ and $0 \neq a \in N$. Since R is not a right V-ring and $aR \cong R/\operatorname{ann}_r(a)$, we have $\operatorname{ann}_r(a) \neq 0$ and so $a \in Z(N)$. It means that N is singular. The converse is clear.

Therefore, there exists a non-injective non-singular right R-module M such that $\mathfrak{In}^{-1}(M) = \mathcal{CRM}od\text{-}R = \{A \in \mathcal{M}od\text{-}R : A$ is singular$\}$. Following Corollary 12(1), R is not a right CMC-ring.

Using Theorem 7, commutative CMC-rings are easy to determine. Recall that a ring R is said to have no simple middle class if every simple R-module is either poor or injective.

Corollary 13. *Let R be a commutative CMC-ring. Then R is regular.*

Proof. Let R be a commutative CMC-ring. By Theorem 7, R is either a V-ring or Noetherian. If R is a V-ring, then it is regular. Suppose that R is not a V-ring. Then there is a simple R-module S properly contained in its injective hull $E(S)$. Since $S \ll E(S)$, $\operatorname{Rad}(E(S)) \neq 0$ so that $E(S)$ does not crumble. If there is a simple right R-module T with $T \ncong S$, then T is $E(S)$-injective and under the assumption on R, it is injective. Since R is Noetherian, S is poor by [3, Corollary 4.3]. Then every crumbling module is semisimple by [10, Theorem 1], a contradiction. □

Corollary 14. *Let R be a commutative CMC-ring. Then R has no simple middle class.*

The following two examples show that regular rings need not be CMC-rings. Indeed, there are infinitely many injectivity domains over the rings at hand.

Example 3. *We consider the ring in Example 1 once again. Let K be a field, A an infinite set, $Q = \prod_{i\in A} K_i$, where $K_i = K$ for all $i \in A$, $L = \bigoplus_{i\in A} K_i$, and let R be the subring generated by L and 1_Q.*

Then R is an SI-ring, $\operatorname{Soc}(R_R) = L \trianglelefteq R$ and L is a maximal ideal. For every subset I of A, let $M_I = \bigoplus_{i\in I} K_i$, $N_I = R_R/M_I$. For each $i \in A$, let e_i be the identity of K_i.

Claim 1: Let $U = uR$ be a cyclic R-module and $\bigoplus_{i\in I} V_i \leq U$, where I is infinite and $V_i \neq 0$ for all $i \in I$. Then $\bigoplus_{i\in I} V_i$ cannot be a direct summand of U.

Suppose that $\bigoplus_{i\in I} V_i$ is a direct summand in U. Then there is an epimorphism $f : U \to \bigoplus_{i\in I} V_i$. But $\operatorname{Im} f = f(uR) = f(u)R$ is contained in $\bigoplus_{i\in F} V_i$ for some finite subset F of I, so f cannot be an epimorphism, a contradiction.

Claim 2: Let $I, J \subseteq A$. If $I \subseteq J$; then M_I is N_J-injective.

Let $K \leq N_J$ and $f : K \to M_I$ be a homomorphism. Take any $r + M_J \in K$ and let $f(r + M_J) = \sum_{i\in I} m_i$ with $m_i \in K_i$. For each $i \in I$, $m_i = f(r + M_J)e_i =$

$f(re_i + M_J) = f(0) = 0$ *since* $re_i \in K_i \subseteq M_I \subseteq M_J$*; therefore* $f(r + M_J) = 0$*. So* $\mathrm{Hom}_R(K, M_I) = 0$ *for every* $K \leq N_J$*. Hence* M_I *is* N_J*-injective.*

Claim 3: If $I, J \subseteq A$ *and* $J \backslash I$ *is infinite, then* M_J *is not* N_I*-injective.*

Suppose that M_J *is* N_I*-injective. Since* $M_{J\backslash I}$ *is a direct summand of* M_J*, it is also* N_I*-injective. Let* $\pi : R_R \to N_I$ *be the canonical epimorphism. Since* $M_{J\backslash I} \cap \mathrm{Ker}\,\pi = M_{J\backslash I} \cap M_I = 0$*,* N_I *contains a submodule* $\bigoplus_{i\in J\backslash I} X_i$ *isomorphic to* $M_{J\backslash I}$*. Then* $\bigoplus_{i\in J\backslash I} X_i$ *is* N_I*-injective, but* $\bigoplus_{i\in J\backslash I} X_i$ *cannot be a direct summand of* N_I *by Claim 1. A contradiction.*

Claim 4: There are infinitely many different injectivity domains over R*.*

Take any family $\{I_k\}_{k=1,2,\dots}$ *of subsets of* A *such that* $I_1 \supseteq I_2 \supseteq \cdots$ *where* $I_k \backslash I_{k+1}$ *is infinite for every* k *(for example, for* $A = \mathbb{Z}^+$*,* $I_k = \{n \in \mathbb{Z} : 2^k | n\}$*). Then injectivity domains* $\mathfrak{In}^{-1}(M_{I_k})$ *are different. Indeed, for* $k < n$*,* M_{I_k} *is not* N_{I_n}*-injective, whereas* M_{I_n} *is* N_{I_n}*-injective, so* $\mathfrak{In}^{-1}(M_{I_k}) \neq \mathfrak{In}^{-1}(M_{I_n})$*.*

Example 4. *Let* $Q = \prod_{i=1}^{\infty} K_i$*, where* $K_i = K$ *is a field. Take any two infinite disjoint subsets* I *and* J *of* $\mathbb{Z}^+$*. Using arguments similar to those in the previous example, it is easy to show that* $M_I = \bigoplus_{i\in I} K_i$ *is* $\prod_{j\in J} K_j$*-injective, but not* $\prod_{i\in I} K_i$*-injective, so* $\mathfrak{In}^{-1}(M_I) \neq \mathfrak{In}^{-1}(M_J)$*.*

Taking infinitely many disjoint infinite subsets of $\mathbb{Z}^+$ *(for example,* $I_n = \{p_n^i | p_n$ *is the* n*th prime,* $i = 1, 2, \ldots\}$*), we see that there are infinitely many different injectivity domains.*

Remark 3. *By the proof in the example above, for such a ring* R*, every non-injective non-singular right* R*-module is modest.*

The following result gives a criteria for being a right CMC-ring. Let M be a module. If every M-injective module is injective, M is called a test module for injectivity.

Theorem 8. *A ring* R *is a right* CMC*-ring if and only if the following conditions are satisfied:*

(1) *There is a crumbling* R*-module which is not semisimple.*

(2) *There is an* R*-module* K *such that* $\mathrm{Soc}\,K$ *is not a direct summand of* K *and every such module is a test module for injectivity.*

(3) *For every* R*-module* A*, if there is a crumbling* R*-module* $B \notin \mathfrak{In}^{-1}(A)$*, then* A *is poor.*

Proof. $(\Rightarrow)$: Let R be a right CMC-ring. Then there are three modules M, N and L such that $\mathfrak{In}^{-1}(M) = \mathcal{SS}\mathcal{M}od\text{-}R \neq \mathfrak{In}^{-1}(N) = \mathcal{CR}\mathcal{M}od\text{-}R \neq \mathfrak{In}^{-1}(L) = \mathcal{M}od\text{-}R$, that is, M is poor, L is injective and N is modest, but not poor and not injective. So there is a crumbling module which is not semisimple and there is a non-crumbling module, therefore, there is an R-module K such that $\mathrm{Soc}\,K$ is not a direct summand of K.

To show that every such module K is a test module for injectivity, let T be a module which is K-injective. Then $K \in \mathfrak{In}^{-1}(T) \backslash \mathfrak{In}^{-1}(N)$, which implies that $\mathfrak{In}^{-1}(T) = \mathcal{M}od\text{-}R$ and so T is injective.

Let A be a module for which we have a crumbling R-module $B \notin \mathfrak{In}^{-1}(A)$. Then A is not injective and not modest. Since R is a right CMC-ring, $\mathfrak{In}^{-1}(A) = \mathcal{SSM}od\text{-}R$, that is, A is poor.

($\Leftarrow$): Let M be a poor module, N a modest module that is not poor (the existence of such a module follows from part (1) and Theorem 1) and L an injective module. We prove that there are only three different injectivity domains: $\mathfrak{In}^{-1}(M), \mathfrak{In}^{-1}(N)$ and $\mathfrak{In}^{-1}(L)$.

Suppose that there is a module T such that $\mathfrak{In}^{-1}(T)$ is different from these three classes. Then either $\mathfrak{In}^{-1}(T) \not\subseteq \mathfrak{In}^{-1}(N)$ or $\mathfrak{In}^{-1}(T) \subsetneqq \mathfrak{In}^{-1}(N)$.

a) $\mathfrak{In}^{-1}(T) \not\subseteq \mathfrak{In}^{-1}(N) = \mathcal{CRM}od\text{-}R$. In this case, there is a non-crumbling module U such that T is U-injective. Then there is a factor module V of U such that $\operatorname{Soc} V$ is not a direct summand of V. Since T is V-injective as well, T is injective by (2), so $\mathfrak{In}^{-1}(T) = \mathcal{M}od\text{-}R = \mathfrak{In}^{-1}(L)$, a contradiction.

b) $\mathfrak{In}^{-1}(T) \subsetneqq \mathfrak{In}^{-1}(N) = \mathcal{CRM}od\text{-}R$. In this case, there is a crumbling module C such T is not C-injective. Condition (3) implies that T is poor, so $\mathfrak{In}^{-1}(T) = \mathfrak{In}^{-1}(M) = \mathcal{SSM}od\text{-}R$, a contradiction. Hence R is a right CMC-ring. □

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