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JACOBSON'S LEMMA FOR THE GENERALIZED n-STRONG DRAZIN INVERSES IN RINGS AND IN OPERATOR ALGEBRAS

Yanxun Ren and Lining Jiang
Beijing Institute of Technology, China

Abstract. In this paper, we extend Jacobson's lemma for Drazin inverses to the generalized n-strong Drazin inverses in a ring, and prove that $1-ac$ is generalized n-strong Drazin invertible if and only if $1-ba$ is generalized n-strong Drazin invertible, provided that $a(ba)^2 = abaca = acaba = (ac)^2a$. In addition, Jacobson's lemma for the left and right Fredholm operators, and furthermore, for consistent in invertibility spectral property and consistent in Fredholm and index spectral property are investigated.

1. Introduction

Throughout the paper, let R be an associative ring with an identity 1. By R^{inv} we denote the set of all invertible elements in R. For $a \in R$, by $\{a\}'$ we denote the commutant of a, namely, $\{a\}' = \{b \in R : ab = ba\}$. Accordingly, by $\{a\}''$ we denote the set $\{\{a\}'\}'$ and call it the double commutant of a.

Recall that an element $a \in R$ is Drazin invertible ([17]) if there exists an element $x \in \{a\}'$ such that

$$xax = x \text{ and } a^kxa = a^k \text{ for some integer } k \geq 0.$$

The preceding x is unique if it exists, and denote it by a^D. The concept of the Drazin inverse was firstly generalized by Koliha for bounded linear operators between Banach spaces and for elements in Banach algebras ([20]) and then by Koliha and Patrício ([21]) for elements in a ring: An element $a \in R$ is

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called generalized Drazin invertible if there exists an element $x \in \{a\}''$ such that

$$xax = x \text{ and } a - a^2x \in R^{qnil},$$

where $R^{qnil} = \{a \in R : 1 + ab \in R^{inv} \text{ for all } b \in \{a\}'\}$. The preceding x is unique if it exists, and we denote it by a^d. In [26], Mosić generalized the concept of generalized Drazin inverse by replacing $a - a^2x \in R^{qnil}$ with $a^n - ax \in R^{qnil}$ $(n \in \mathbb{N})$.

DEFINITION 1.1 ([26]). *Let $n \in \mathbb{N}$. An element $a \in R$ is called generalized n-strong Drazin invertible if there exists an element $x \in \{a\}''$ such that*

$$xax = x \text{ and } a^n - ax \in R^{qnil}. \tag{1.1}$$

The preceding x is unique if it exists, and we denote it by a^{gnsd}. From [26, Lemma 2.1] we know that if a is generalized n-strong Drazin invertible, then a is generalized Drazin invertible and $a^{gnsd} = a^d$.

As we have known, Drazin inverse is a useful tool in analyzing Markov chains, difference equations, differential equations, iterative numerical methods, and so on. Taking into account its importance, properties of the classical Drazin inverse, and furthermore, variations of the inverse in rings and in Banach algebras are studied by many mathematicians. In particular, the topic for the generalized Drazin inverse and for the generalized n-strong Drazin inverse was studied extensively. In detail, the generalized n-strong Drazin inverse for $n = 1$, namely, the generalized strong Drazin inverse, was studied by Gürgün et al. in [19, 24]. The generalized n-strong Drazin inverse for $n = 2$ (so-called the generalized Hirano inverses) was investigated by Chen et al. in [8, 11]. The first purpose of the paper is to illustrate the generalized n-strong Drazin inverse for $n \in \mathbb{N}$ in terms of Jacobson's lemma.

Recall that Jacobson's lemma, which states that $1 - ba \in R^{inv}$ if and only if $1 - ab \in R^{inv}$, was initially a statement for classical inverse in a ring. It was generalized to inner inverses, Drazin inverses and generalized Drazin inverses in a ring in [10, 14, 15, 25, 6, 23, 29, 34]. More recently, in [26], the author generalized Jacobson's lemma to generalized strong Drazin inverses. In [31], Zeng and Yan considered Jacobson's lemma for the generalized Hirano inverses. In [9], Chen and Sheibani investigated Jacobson's lemma for the generalized n-strongly Drazin inverses. Motivated by these papers, the paper considers Jacobson's lemma for the generalized n-strong Drazin inverses, and prove that $1 - ac$ is generalized n-strong Drazin invertible if and only if $1 - ba$ is generalized n-strong Drazin invertible, provided that $a(ba)^2 = abaca = acaba = (ac)^2a$. Moreover, the paper illustrates Cline formula for the generalized n-strong Drazin inverses, and prove that ba is generalized n-strong Drazin invertible if and only if ac is generalized n-strong Drazin invertible under the preceding assumption.

Jacobson's lemma is useful in Banach algebras and in operator algebras (see [1, 3, 30, 32, 33]). Another aim of the paper is to study Jacobson's lemma for the left and right Fredholm operators on a Banach space. We prove in Section 3 that the range space (the null space) of $I-AC$ is complement if and only if the range space (the null space) of $I-BA$ is complement under the condition $A(BA)^2 = ABACA = ACABA = (AC)^2A$ for bounded linear operators A, B, C on a Banach space $\mathcal{X}$, and show that $I-AC$ is a left (right) Fredholm operator if and only if $I-BA$ is a left (right) Fredholm operator under the same assumption. In addition, the consistent in invertibility spectrum property for operator and consistent in Fredholm and index spectrum property for operator on a Hilbert space are investigated, respectively.

2. Jacobson's lemma for generalized n-strong Drazin inverses

In [10], Chen and Abdolyousefi investigated Jacobson's lemma for Drazin inverses in a Banach algebra, and gave the following result.

Theorem 2.1 ([10, Theorem 2.2]). *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$ satisfy*

$$(ac)^2a = abaca = acaba = a(ba)^2.$$

Then $\alpha = 1 - ba \in \mathcal{A}^d$ if and only if $\beta = 1 - ac \in \mathcal{A}^d$.

The first aim of this section is to extended the result to the generalized n-strong Drazin inverses in a ring, and furthermore, to illustrate Cline's formula for the inverses. We begin with the following lemma.

Lemma 2.2. [7, Lemma 2.1] *Let $a, b, c \in R$. If a, b, c satisfy*

$$a(ba)^2 = abaca = acaba = (ac)^2a, \tag{2.1}$$

then $ac \in R^{qnil}$ if and only if $ba \in R^{qnil}$.

Theorem 2.3. *Let $a, b, c \in R$ and $n \in \mathbb{N}$. If a, b, c satisfy the relation* (2.1)*, then $\beta = 1 - ac$ is generalized n-strong Drazin invertible if and only if $\alpha = 1 - ba$ is generalized n-strong Drazin invertible. In this case, we have*

$$(1-ba)^{gnsd} = \left[1 - bac\beta^{\pi}(1-\beta^{\pi}\beta\sum_{i=0}^{2}(ac)^i)^{-1}aca\right]\sum_{i=0}^{2}(ba)^i + bac\beta^{gnsd}aca,$$

and

$$(1-ac)^{gnsd} = \left[1 - aba\alpha^{\pi}(1-\alpha^{\pi}\alpha\sum_{i=0}^{2}(ba)^i)^{-1}bac\right]\sum_{i=0}^{2}(ac)^i + aba\alpha^{gnsd}bac$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gnsd}$ and $\beta^{\pi} = 1 - \beta\beta^{gnsd}$.

Proof. Suppose that $\alpha = 1 - ba$ is generalized n-strong Drazin invertible. Let $p = \alpha^{\pi}$, $x = \alpha^{gnsd}$. Since $\alpha^{gnsd} = \alpha^d$, this implies $p\alpha \in R^{qnil}$. Moreover, the element $1 - p\alpha(1 + ba + (ba)^2) \in R^{inv}$, and ba commutes with $v = [1 - p\alpha(1 + ba + (ba)^2)]^{-1}$ since $1 + ba + (ba)^2$ commutes with $p\alpha$. Let

$$y = (1 - abapvbac)(1 + ac + (ac)^2) + abaxbac.$$

In order to prove $y = \beta^{gnsd}$, it suffices to show that the following three conditions hold: $y \in \{\beta\}''$, $y\beta y = y$ and $\beta^n - \beta y \in R^{qnil}$.

Firstly, we need to verify $y \in \{\beta\}''$. Set $z \in R$ and $z\beta = \beta z$. Since

$$\begin{aligned}(baczaba)\alpha &= baczaba - baczababa \\ &= baczaba - babaczaba \\ &= \alpha(baczaba),\end{aligned}$$

it follows that $baczaba$ commutes with p, x and v. Due to the equation

$$p = pv(ba)^3, \tag{2.2}$$

one can get

$$\begin{aligned}z(abapbac) &= zaba(ba)^3pvbac \\ &= ababaczabapvbac \\ &= abapvbaczababac \\ &= abapvbabababacz \\ &= abapbacz,\end{aligned}$$

which implies

$$aba\alpha xbacz = zaba\alpha xbac. \tag{2.3}$$

Multiplying (2.3) by ac on the left, one has

$$acaba\alpha xbacz = aczaba\alpha xbac,$$

and hence

$$ababa\alpha xbacz = zababa\alpha xbac. \tag{2.4}$$

Applying (2.3) and (2.4), one can get $aba(1+ba)\alpha xbacz = zaba(1+ba)\alpha xbac$, and

$$abaxbacz - aba(ba)^2xbacz = zabaxbac - zaba(ba)^2xbac.$$

Moreover,

$$\begin{aligned}aba(ba)^2xbacz &= axbaczababac \\ &= abaczabaxbac \\ &= zabababaxbac \\ &= zaba(ba)^2xbac,\end{aligned}$$

and so

$$zabaxbac = abaxbacz. \tag{2.5}$$

Put $f = abapvbac(1 + ac + (ac)^2)$. Using (2.2), one has

$$\begin{aligned}zf &= zabapvbac(1 + ac + (ac)^2) \\ &= zaba(ba)^3pv^2bac(1 + ac + (ac)^2)\end{aligned}$$

$$\begin{aligned} &= ababaczabapv^2bac(1+ac+(ac)^2) \\ &= abapv^2baczababac(1+ac+(ac)^2) \end{aligned}$$

and

$$\begin{aligned} fz &= abapvbac(1+ac+(ac)^2)z \\ &= abapv^2(ba)^3bacz(1+ac+(ac)^2) \\ &= abapv^2baczacacac(1+ac+(ac)^2) \\ &= abapv^2baczababac(1+ac+(ac)^2). \end{aligned}$$

These equations imply that $zf = fz$, that is

$$zabapvbac(1+ac+(ac)^2) = abapvbac(1+ac+(ac)^2)z. \tag{2.6}$$

Therefore, it follows from (2.5), (2.6) and $(ac)z = z(ac)$ that $yz = zy$, which implies that $y \in \{\beta\}''$.

The next thing is to show $y\beta y = y$. Indeed,

$$\begin{aligned} y\beta &= 1-(ac)^3 - abapvbac(1-(ac)^3) + abaxbac(1-ac) \\ &= 1 - ababac + abaxbac - ababaxbac - abapv(1-(ba)^3)bac \\ &= 1 - abapbac - abapv\alpha(1+ba+(ba)^2)bac \\ &= 1 - abap(1+vp\alpha(1+ba+(ba)^2))bac \\ &= 1 - abapvbac. \end{aligned}$$

Since ba commutes with p, x, v and $px = 0$, one has

$$abapvbacabaxbac = abapvbababapxbac = 0.$$

Therefore,

$$\begin{aligned} y\beta y &= y - abapvbac(1+ac+(ac)^2) + abapvbacabapvbac(1+ac+(ac)^2) \\ &= y - abapv(1+ba+(ba)^2)bac + abapv^2(ba)^3(1+ba+(ba)^2)bac \\ &= y - abapv^2(p - p\alpha(1+ba+(ba)^2) - p(ba)^3)(1+ba+(ba)^2)bac \\ &= y. \end{aligned}$$

Finally we have to show $\beta^n - \beta y \in R^{qnil}$. Define

$$b' = pv(ba)^2b - \sum_{i=1}^{n}(-1)^{i-1}\begin{pmatrix} n \\ i \end{pmatrix}(ba)^{i-1}b$$

and

$$c' = pv(ba)^2b - \sum_{i=1}^{n}(-1)^{i-1}\begin{pmatrix} n \\ i \end{pmatrix}(ca)^{i-1}c.$$

Using (2.2), one has

$$\begin{aligned}\alpha^n - \alpha\alpha^{gnsd} &= (1-ba)^n - 1 + 1 - \alpha\alpha^{gnsd} \\ &= (pv(ba)^2 b - \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^{i-1} b)a \\ &= b'a \in R^{qnil},\end{aligned}$$

and

$$\begin{aligned}\beta^n - \beta y &= (1-ac)^n - 1 + abapvbac \\ &= a(pv(ba)^2 b - \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ca)^{i-1} c) \\ &= ac'.\end{aligned}$$

Moreover, since $pba = pv(ba)^3 ba = pca$ and $abap = aba(ba)^3 pv = acap$, by direct calculation one has

$$\begin{aligned}ab'ab'a = a &\left[pv(ba)^3 - \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \right] \\ &\cdot \left[pv(ba)^3 - \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \right] \\ &= ap - ap \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i - a \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i p \\ &\quad + a \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \\ &= ap - ap \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i - a \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ca)^i p \\ &\quad + a \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ca)^i \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \\ &= \left[ap - a \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ca)^i \right] \left[p - \sum_{i=1}^{n} (-1)^{i-1} \left(\begin{array}{c} n \\ i \end{array} \right) (ba)^i \right] \\ &= ac'ab'a.\end{aligned}$$

Using the same trick one has $a(b'a)^2 = ab'ac'a = ac'ab'a = (ac')^2 a$. Thus $\beta^n - \beta y = ac' \in R^{qnil}$ owing to Lemma 2.2,

At last, an argument similar to the first part can show the case of $\beta = 1 - ac$, and we omit it here. $\square$

Corollary 2.4. *Let $a, b, c \in R$ and $n \in \mathbb{N}$. If a, b, c satisfy $aba = aca$, then $1 - ac$ is generalized n-strong Drazin invertible if and only if $1 - ba$ is generalized n-strong Drazin invertible.*

From Theorem 2.3, one can get Jacobson's lemma for the generalized n-strong inverses which can also be seen in [9, Theorem 2.3].

Corollary 2.5. *Let $a, b \in R$ and $n \in \mathbb{N}$. Then $1 - ab$ is generalized n-strong Drazin invertible if and only if $1 - ba$ is generalized n-strong Drazin invertible.*

Now, we consider Cline's formula for the generalized n-strong inverses. In 1965, Cline ([12]) proved that if ba is Drazin invertible, then ab is Drazin invertible, where $a, b \in R$. In this case, $(ab)^D = a((ba)^D)^2 b$. This was known as Cline's formula for Drazin inverses. Cline's formula for generalized Drazin inverse were found in [22]. Chen et al. ([7]) gave an analogue of Cline's formula for the generalized Drazin inverse under the condition $a(ba)^2 = abaca = acaba = (ac)^2 a$. For the generalized n-strong Drazin inverse, we have the following result.

Theorem 2.6. *Let $a, b, c \in R$ and $n \in \mathbb{N}$. If a, b, c satisfy* (2.1)*, then ac is generalized n-strong Drazin invertible if and only if ba is generalized n-strong Drazin invertible. In this case, $(ac)^{gnsd} = a((ba)^{gnsd})^2 c$ and $(ba)^{gnsd} = b((ac)^{gnsd})^2 a$.*

Proof. Suppose that ac has the generalized n-strong Drazin inverse and $x = (ac)^{gnsd}$. Let $y = bx^2 a$. In light of [7, Theorem 2.2], we know that $y \in \{ba\}''$ and $y(ba)y = y$. It is easy to verify that $abx = acx$ and $abay = acay$. In order to show that ba has a generalized n-strong Drazin inverse, it is enough to prove that $(ba)^n - (ba)y \in R^{qnil}$. Set $b' = (ba)^{n-1}b - yb$ and $c' = c(ac)^{n-1} - cx$. Then

$$ac' = (ac)^n - acx \in R^{qnil}$$

and

$$b'a = (ba)^n - yba.$$

Moreover, by direct calculation one has

$$\begin{aligned}
ab'ab'a &= a[(ba)^{n-1}b - yb]a[(ba)^{n-1}b - yb]a \\
&= a(ba)^{2n} - a(ba)^n yba - ayba(ba)^n + aybayba \\
&= a(ca)^n (ba)^n - a(ca)^n bay^2 ba - abx^2 aba(ba)^n + abx^2 acayba \\
&= (ac)^n a(ba)^n - (ac)^n ayba - acxa(ba)^n + acxayba \\
&= [(ac)^n - acx]a[(ba)^n - yba] \\
&= a[c(ac)^{n-1} - cx]a[(ba)^{n-1}b - yb]a \\
&= ac'ab'a.
\end{aligned}$$

The same calculation shows that

$$ab'ab'a = ac'ab'a = ab'ac'a = ac'ac'a.$$

Therefore $(ba)^n - bay \in R^{qnil}$ in view of Lemma 2.2.

The same technique can be applied to the opposite implication, and we omit it here. ∎

Corollary 2.7. *Let $a, b, c \in R$ and $n \in \mathbb{N}$. If a, b, c satisfy $aba = aca$, then ac is generalized n-strong Drazin invertible if and only if ba is generalized n-strong Drazin invertible.*

The following result is a direct result of Theorem 2.6, which gives Cline formula for the generalized n-strong inverses.

Corollary 2.8. *Let $a, b \in R$ and $n \in \mathbb{N}$. Then ab is generalized n-strong Drazin invertible if and only if ba is generalized n-strong Drazin invertible.*

Using Theorems 2.3 and 2.6 again, one can get the following corollary.

Corollary 2.9. *Let R be an unitary Banach algebra, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. If $a, b, c \in R$ satisfy the relation* (2.1)*, then $\lambda - ac$ is generalized n-strong Drazin invertible if and only if $\lambda - ba$ is generalized n-strong Drazin invertible.*

3. Jacobson's lemma for left and right Fredholm operators

Jacobson's lemma is a powerful in the research of Banach algebra and operator algebra spectral theory. We devote this section to Jacobson's lemma for the complementarity of range space (null space) of operators acting on a Banach space, and further to the Jacobson's lemma for the left and right Fredholm operators acting on a Banach space. We start by explaining the relevant terminology. Let $L(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. For $A \in L(\mathcal{X})$, we use $n(A)$ and $d(A)$ to denote the dimension of null space $\ker A$ and the codimension of range space $\mathrm{ran}A$, respectively. Set $\Phi_-(\mathcal{X}) = \{A \in L(\mathcal{X}) : d(A) < +\infty\}$ and $\Phi_+(\mathcal{A}) = \{A \in L(\mathcal{X}) : n(A) < +\infty \text{ and } \mathrm{ran}A \text{ is closed}\}$. We say A a lower-semi Fredholm operator if $A \in \Phi_-(\mathcal{X})$, and an upper-semi Fredholm operator if $A \in \Phi_+(\mathcal{X})$. The index of A is denoted by $\mathrm{ind}(A) = n(A) - d(A)$, where $A \in \Phi_-(\mathcal{X}) \cup \Phi_+(\mathcal{X})$. An operator A is called a Fredholm operator if $A \in \Phi(\mathcal{X})$, where $\Phi(\mathcal{X}) = \Phi_-(\mathcal{X}) \cap \Phi_+(\mathcal{X})$. One can refer to [2] for details on Fredholm operators.

Recall that a closed subspace M is complemented in a Banach space $\mathcal{X}$ if there exists a closed subspace $N \subseteq \mathcal{X}$ such that $\mathcal{X} = M \bigoplus N$. In light of [13, Theorem 13.2] we know that M is complemented in $\mathcal{X}$ if and only if there exists a $P \in L(\mathcal{X})$ such that $P^2 = P$ and $\mathrm{ran}P = M$.

Sets of left and right Fredholm operators (see [35]) are defined by

$$\Phi_l(\mathcal{X}) = \{A \in \Phi_+(\mathcal{X}) : \mathrm{ran}A \text{ is complemented}\}$$

and

$$\Phi_r(\mathcal{X}) = \{A \in \Phi_-(\mathcal{X}) : \ker A \text{ is complemented}\},$$

respectively. Accordingly the left Fredholm spectrum $\sigma_{\Phi_l}(T)$ of T is defined by

$$\sigma_{\Phi_l}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_l(\mathcal{X})\},$$

and the right Fredholm spectrum of T is defined by

$$\sigma_{\Phi_r}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_r(\mathcal{X})\}.$$

LEMMA 3.1 ([33, Lemma 3.3 and 3.5]). *Let $A, B, C \in L(\mathcal{X})$ and $m \in \mathbb{N}$. If $A, B, C \in L(\mathcal{X})$ satisfy the condition*

$$A(BA)^2 = ABACA = ACABA = (AC)^2A, \tag{3.1}$$

then the following statements hold:

1. *$I - AC \in \Phi_-(\mathcal{X})$ if and only if $I - BA \in \Phi_-(\mathcal{X})$,*
2. *$I - AC \in \Phi_+(\mathcal{X})$ if and only if $I - BA \in \Phi_+(\mathcal{X})$,*
3. *$I - AC \in \Phi(\mathcal{X})$ with $\mathrm{ind}(I - AC) = m$ if and only if $I - BA \in \Phi(\mathcal{X})$ with the same index as $\mathrm{ind}(I - AC)$.*

LEMMA 3.2. *Suppose that $A, B, C \in L(\mathcal{X})$ satisfy the relation (3.1). Then the following statements hold:*

1. *$\mathrm{ran}(I - AC)$ is complemented if and only if $\mathrm{ran}(I - BA)$ is complemented,*
2. *$\ker(I - AC)$ is complemented if and only if $\ker(I - BA)$ is complemented.*

PROOF. (1) It is enough to prove that $\mathrm{ran}(I - BA)$ is complemented provided that $\mathrm{ran}(I - AC)$ is complemented. Assume that there exists a projection $P \in L(\mathcal{X})$ such that $\mathrm{ran}P = \mathrm{ran}(I - AC)$. It is easy to verify that $(I - P)AC = I - P$. Let

$$Q = I - (BA)^2C(I - P)A.$$

In order to prove that $\mathrm{ran}(I - BA)$ is complemented, it suffices to prove that $Q = Q^2$ and $\mathrm{ran}Q = \mathrm{ran}(I - BA)$.

Firstly we need to verify the relation $Q^2 = Q$. By direct calculation one has

$$\begin{aligned} Q^2 &= I - 2(BA)^2C(I - P)A + BABAC(I - P)ABABAC(I - P)A \\ &= I - 2(BA)^2C(I - P)A + BABAC(I - P)ACACAC(I - P)A \\ &= I - 2(BA)^2C(I - P)A + BABAC(I - P)A \\ &= Q. \end{aligned}$$

The next thing is to check the equality $\mathrm{ran}Q = \mathrm{ran}(I - BA)$. We claim that

$$\mathrm{ran}(BABACPA) \subseteq \mathrm{ran}(I - BA).$$

Indeed, set $y \in \operatorname{ran}(BABACPA)$. Then there exists some $x \in \mathcal{X}$ such that

$$y = BABACPAx \in BABAC\operatorname{ran}P = BABAC\operatorname{ran}(I - AC).$$

One can choose $z \in \mathcal{X}$ such that

$$\begin{aligned} y &= BABAC(I - AC)z \\ &= (BABAC - BABACAC)z \\ &= (I - BA)BABACz. \end{aligned}$$

This leads to the inclusion of $\operatorname{ran}(BABACPA) \subseteq \operatorname{ran}(I - BA)$. Since

$$\begin{aligned} Q &= I - BABAC(I - P)A \\ &= I - BABABA + BABACPA \\ &= (I - BA)(I + BA + BABA) + BABACPA, \end{aligned}$$

it follows that $\operatorname{ran}Q \subseteq \operatorname{ran}(I - BA)$.

For the other inclusion, set $y \in \operatorname{ran}(I - BA)$. Then there exists some $x \in \mathcal{X}$ such that $(I - BA)x = y$. Since

$$\begin{aligned} (I - P)Ay &= (I - P)ACA(I - BA)x \\ &= (I - P)(ACA - ACABA)x \\ &= (I - P)(I - AC)ACAx \\ &= 0, \end{aligned}$$

one has

$$Qy = [I - BABAC(I - P)A]y = y.$$

So $\operatorname{ran}Q = \operatorname{ran}(I - BA)$.

The converse implication is also true, and we omit it here.

(2) It is enough to assume that the space $\ker(I - AC)$ is complemented. Then there exists a projection $P \in L(\mathcal{X})$ such that $\operatorname{ran}P = \ker(I - AC)$. This implies that $(I - AC)P = 0$, that is, $P = ACP$. Set

$$Q = 2BABPABA - BABPACA.$$

In order to prove that $\ker(I - BA)$ is complemented, it suffices to show that $Q^2 = Q$ and $\operatorname{ran}Q = \ker(I - BA)$.

The first step is to check the relation $Q^2 = Q$. It is not hard to verify that

$$ABABABP = P, \text{ and } ACABABP = P.$$

Thus,

$$\begin{aligned} Q^2 &= 4BABPABABABPABA - 2BABPABABABPACA \\ &\quad - 2BABPACABABPABA + BABPACABABPACA \\ &= 4BABPABA - 2BABPACA - 2BABPABA + BABPACA \\ &= 2BABPABA - BABPACA \\ &= Q. \end{aligned}$$

The second step is to check the relation $\mathrm{ran}Q = \ker(I - BA)$. Indeed,

$$\begin{aligned}
(I - BA)Q &= (I - BA)(2BABPABA - BABPACA)\\
&= 2BABPABA - BABPACA - 2BABABPABA\\
&\quad + BABABPACA\\
&= 2BABPABA - BABPACA - 2BABABACPABA\\
&\quad + BABABACPACA\\
&= 2BABPABA - BABPACA - 2BABACACPACA\\
&\quad + BABACACPACA\\
&= 2BABPABA - BABPACA - 2BABPABA + BABPACA\\
&= 0.
\end{aligned}$$

This implies that $\mathrm{ran}Q \subseteq \ker(I - BA)$.

Now we show the other inclusion relation. Let $x \in \ker(I - BA)$. It is easy to verify that

$$(I - AC)ABAx = 0, \text{ and } (I - AC)ACAx = 0,$$

that is, $ABAx,\ ACAx \in \ker(I - AC) = \mathrm{ran}P$. Therefore,

$$PABAx = ABAx,\ PACAx = ACAx.$$

Moreover, since

$$\begin{aligned}
Qx &= (2BABPABA - BABPACA)x\\
&= 2BABABAx - BABACAx\\
&= 2BABABAx - BABACAx\\
&= (BA)^3x\\
&= x,
\end{aligned}$$

one has $\ker(I - BA) \subseteq \mathrm{ran}Q$. In conclusion one has $\mathrm{ran}Q = \ker(I - BA)$, and the space $\ker(I - BA)$ is complemented. $\square$

It is time to give the main theorem of the section, which follows immediately from Lemmas 3.1 and 3.2.

Theorem 3.3. *Let $A, B, C \in L(\mathcal{X})$ satisfy the relation* (3.1)*, then the following statements hold:*

(1) $I - AC \in \Phi_l(\mathcal{X})$ *if and only if* $I - BA \in \Phi_l(\mathcal{X})$,
(2) $I - AC \in \Phi_r(\mathcal{X})$ *if and only if* $I - BA \in \Phi_r(\mathcal{X})$.

Corollary 3.4. *Let $A, B, C \in L(\mathcal{X})$ satisfy the relation* (3.1)*. Then the following statements hold:*

(1) $\sigma_{\Phi_l}(AC)\backslash\{0\} = \sigma_{\Phi_l}(BA)\backslash\{0\}$,
(2) $\sigma_{\Phi_r}(AC)\backslash\{0\} = \sigma_{\Phi_r}(BA)\backslash\{0\}$.

Proof. (1) Let $\lambda \neq 0$ and $A' = \frac{A}{\lambda}$. It is not hard to check that

$$(A'C)^2 A' = A'BA'CA' = A'CA'BA' = A'(BA')^2.$$

In light of Theorem 3.3, one can obtain $I - A'C \in \Phi_l(\mathcal{X})$ if and only if $I - BA' \in \Phi_l(\mathcal{X})$. Considering the fact that ran$T$ is a linear space for any $T \in L(\mathcal{X})$, one has

$$\mathrm{ran}(\lambda I - AC) = \mathrm{ran}(I - A'C), \mathrm{ran}(\lambda I - BA) = \mathrm{ran}(I - BA').$$

Thus, $\lambda I - AC \in \Phi_l(\mathcal{X})$ if and only if $I - A'C \in \Phi_l(\mathcal{X})$, and $\lambda I - BA \in \Phi_l(\mathcal{X})$ if and only if $I - BA' \in \Phi_l(\mathcal{X})$. This means that $\sigma_{\Phi_l}(AC)\backslash\{0\} = \sigma_{\Phi_l}(BA)\backslash\{0\}$.

(2) With the same argument as the proof of (1). $\square$

4. Jacobson's lemma for CI and CFI operators

The purpose of this section is to investigate consistent in invertibility spectral property and consistent in Fredholm and index spectral property for operators on a Hilbert space $\mathcal{H}$, respectively. Recall that an operator $A \in L(\mathcal{H})$ is consistent in invertibility (CI for short) if for all $B \in L(\mathcal{H})$, $AB \in L(\mathcal{H})^{-1}$ if and only if $BA \in L(\mathcal{H})^{-1}$. The CI spectrum $\sigma_{CI}(A)$ of $A \in L(\mathcal{H})$ is defined by

$$\sigma_{CI}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not CI}\}.$$

Using Jacobson's lemma one can see that for $A, B \in L(\mathcal{H})$, the nonzero element of $\sigma(AB)$ and $\sigma(BA)$ are the same. This leads to a question whether $A \in L(\mathcal{H})$ satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ for all $B \in L(\mathcal{H})$. Applying Jacobson's lemma again we know that $\sigma(AB) = \sigma(BA)$ if and only if $0 \notin \sigma_{CI}(A)$ for all $B \in L(\mathcal{H})$. In [18], Gong and Han gave a complete characterization of $A \in L(\mathcal{H})$ which satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ for every $B \in L(\mathcal{H})$. Djordjevic ([16]) further described the CI operators on a Banach space. In [5], Cao et al. investigated the CI spectrum on a Banach space, and applied the CI spectrum to characterize those operators T satisfying Weyl type theorems. The relation between CI spectrum and Weyl type theorems also can be seen in [27, 28]. Illuminated by results above we answer the question affirmably that Jacobson's lemma still holds for CI spectrum. Let $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$ and $\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$. From [33] we have that $\sigma_*(AC)\backslash\{0\} = \sigma_*(BA)\backslash\{0\}$ under the condition (3.1), where $\sigma_* \in \{\sigma_l, \sigma_r\}$.

Lemma 4.1 ([5, Theorem 1]). *If $A \in L(\mathcal{H})$, then there is an equality*

$$\sigma_{CI}(A) = \sigma(A)\backslash(\sigma_l(A) \cap \sigma_r(A)).$$

Theorem 4.2. *Let $A, B, C \in L(\mathcal{H})$ satisfy the relation* (3.1). *Then $\sigma_{CI}(AC)\backslash\{0\} = \sigma_{CI}(BA)\backslash\{0\}$.*

Proof. Let $\lambda \in \sigma_{CI}(AC)\backslash\{0\}$. In light of Lemma 4.1, $\lambda \in \sigma_l(AC)\backslash \sigma_r(AC)$ or $\lambda \in \sigma_r(AC)\backslash\sigma_l(AC)$. One has that either $\lambda \in \sigma_l(BA)\backslash\sigma_r(BA)$ or $\lambda \in \sigma_r(BA)\backslash\sigma_l(BA)$. This means that $\lambda \in \sigma_{CI}(BA)\backslash\{0\}$.

Using the same manner one can get the converse inclusion. ∎

As a continuation of [16, 18], Cao in [4] gave the notion of consistent in Fredholm and index on a Hilbert space $\mathcal{H}$. Recall that an operator $A \in L(\mathcal{H})$ is consistent in Fredholm and index (abbrev. CFI), if for arbitrary $B \in L(\mathcal{H})$, AB and BA are Fredholm together and $\mathrm{ind}(AB) = \mathrm{ind}(AB) = \mathrm{ind}(B)$, or not Fredholm together. Given an operator $A \in L(\mathcal{H})$, its CFI spectrum $\sigma_{CFI}(A)$ is defined by

$$\sigma_{CFI}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not CFI}\}.$$

It was proved in [4, Corollary 3.3] that an operator $A \in L(\mathcal{H})$ is not CFI if and only if A is semi-Fredholm with $\mathrm{ind}(A) \neq 0$. The result below indicates that Jacobson's lemma is also suitable for CFI spectrum.

Theorem 4.3. *Let $A, B, C \in L(\mathcal{H})$ satisfy the relation* (3.1). *Then* $\sigma_{CFI}(AC)\backslash\{0\} = \sigma_{CFI}(BA)\backslash\{0\}$.

Proof. Let $\lambda \in \sigma_{CFI}(AC)\backslash\{0\}$. Set $A' = \frac{A}{\lambda}$. Then $I - A'C$ is a semi-Fredholm operator with $\mathrm{ind}(I - A'C) \neq 0$. It is not hard to verify that

$$(A'C)^2A' = A'BA'CA' = A'CA'BA' = A'(BA')^2.$$

In light of Lemma 3.1, $I - BA'$ is also a semi-Fredholm operator with $\mathrm{ind}(I - BA') \neq 0$. Hence $\lambda \in \sigma_{CFI}(BA)\backslash\{0\}$.

The same manner can be applied to show the converse inclusion. ∎

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Y. Ren
School of Mathematics and Statistics
Beijing Institute of Technology
100081 Beijing
China
E-mail: renyanxun110@126.com

L. Jiang
School of Mathematics and Statistics
Beijing Institute of Technology
100081 Beijing
China
E-mail: jianglining@bit.edu.cn

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