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FINITE W-ALGEBRAS ASSOCIATED TO TRUNCATED CURRENT LIE ALGEBRAS

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Abstract. Finite W-algebras associated to truncated current Lie algebras are studied in this paper. We show that some properties of finite W-algebras in the semisimple case hold in the truncated current case. In particular, Kostant's theorem and Skryabin equivalence hold in our case. As an application, we give a classification of simple Whittaker modules for truncated current Lie algebras in the $s\ell_2$ case.

1. Introduction

Finite W-algebras appeared firstly in B. Kostant's paper [10], where the author considered finite W-algebras (thought at that time, not this name) associated to principal nilpotent elements of semisimple Lie algebras, and proved that the result algebra is in fact isomorphic to the center of the universal enveloping algebra. Then Kostant's student Lynch generalized the construction to even grading nilpotent elements in his thesis paper [11]. After more than twenty years later, A. Premet's gave the general definition of finite W-algebras associated to an arbitrary nilpotent element in [16], and in the appendix of Premet's paper, S. Skryabin proved an equivalence between a category of W-algebra modules and a subcategory of Lie algebra modules, hence established a close relation between the representation theory of Lie algebras and that of finite W-algebras. More related research on finite W-algebras can be found in [2, 7, 16].

Truncated current Lie algebras are quotients of current algebras, they are also called generalized Takiff algebras or polynomial algebras. They have important applications in physics [1, 3], and are interesting research objects

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in mathematics too [14, 20]. In the definition of finite W-algebras, a non-degenerate invariant bilinear form and a good $\mathbb{Z}$-grading are essential. We show that these two essential ingredients exist on truncated current Lie algebras hence allow us to define finite W-algebras. Moreover, we show that finite W-algebras associated to truncated current Lie algebras can also be realized a quantization of Slodowy slices, and both Kostant's theorem and Skryabin equivalence hold for them.

The organization of the paper is as follows. In Section 2, we show that non-degenerate invariant bilinear forms exist on truncated current Lie algebras. In Section 3, we define finite W-algebras associated to truncated current Lie algebras via Whittaker model and show that they are quantizations of Slodowy slices. Finally, in Section 4, we show that Kostant's theorem and Skryabin equivalence hold in the truncated current case. As an example, we classify irreducible Whittaker modules for truncated current Lie algebra in the $s\ell_2$ case.

All vector spaces and algebras are considered over complex numbers $\mathbb{C}$, except when we mention explicitly.

2. Truncated current Lie algebra and good $\mathbb{Z}$-grading

2.1. *Truncated current Lie algebra.* Given a finite-dimensional Lie algebra $\mathfrak{a}$, the *current algebra* associated to $\mathfrak{a}$ is the Lie algebra $\mathfrak{a} \otimes \mathbb{C}[t]$ with Lie bracket:

$$[a \otimes t^m, b \otimes t^n] := [a, b] \otimes t^{m+n}, \text{ for all } a, b \in \mathfrak{a}, m, n \in \mathbb{Z}_{\geq 0}.$$

The subspace $\mathfrak{a} \otimes t^p\mathbb{C}[t]$ is an ideal of $\mathfrak{a} \otimes \mathbb{C}[t]$ for any nonnegative integer p.

Definition 2.1. *The* level p truncated current Lie algebra *associated to* $\mathfrak{a}$ *is the quotient*

$$\mathfrak{a}_p := \frac{\mathfrak{a} \otimes \mathbb{C}[t]}{\mathfrak{a} \otimes t^{p+1}\mathbb{C}[t]} \cong \mathfrak{a} \otimes \frac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}.$$

The Lie bracket of $\mathfrak{a}_p$ *is*

$$[a \otimes t^i, b \otimes t^j] = [a, b] \otimes t^{i+j}, \text{ where } t^{i+j} \equiv 0 \text{ when } i + j > p.$$

Remark 2.2. Truncated current Lie algebras are also called *generalized Takiff algebras* or *polynomial Lie algebras.*

For convenience, we write xt^i for $x \otimes t^i$. An element of $\mathfrak{a}_p$ can be uniquely expressed as a sum $\sum_{i=0}^{p} x_i t^i$ with $x_i \in \mathfrak{a}$. Let $(\cdot \mid \cdot)$ be a symmetric bilinear form on $\mathfrak{a}$. Let $\bar{c} := (c_0, \cdots, c_p)$ with $c_i \in \mathbb{C}$. Define a symmetric bilinear form on $\mathfrak{a}_p$ by the formula

$$(x \mid y)_p := \sum_{k=0}^{p} c_k \sum_{i+j=k} (x_i \mid y_j), \tag{2.1}$$

where $x = \sum_{i=0}^{p} x_i t^i$ and $y = \sum_{i=0}^{p} y_i t^i$ with $x_i, y_i \in \mathfrak{a}$.

LEMMA 2.3 ([3]). *Assume that* $(\cdot \mid \cdot)$ *is non-degenerate and invariant on* $\mathfrak{a}$*, then bilinear form* $(\cdot \mid \cdot)_p$ *defined by (2.1) is invariant on* $\mathfrak{a}_p$*. It is non-degenerate if and only if* $c_p \neq 0$*.*

PROOF. Let $x = \sum_i x_i t^i, y = \sum_i y_i t^i, z = \sum_i z_i t^i$ with $x_i, y_i, z_i \in \mathfrak{a}$, then

$$\begin{aligned}
([x, y] \mid z)_p &= \sum_{i,j,k} c_k([x_i, y_j] \mid z_{k-i-j}) \\
&= \sum_{i,j,k} c_k(x_i \mid [y_j, z_{k-i-j}]) \\
&= \sum_{i',j,k} c_k(x_{k-j-i'} \mid [y_j, z_{i'}]) \\
&= (x \mid [y, z])_p.
\end{aligned}$$

If $c_p = 0$, then $(\cdot \mid \cdot)_p$ is degenerate as $\mathfrak{a} \otimes t^p$ lies in its kernel. When $c_p \neq 0$, assume that $a = \sum_{i \geq i_0} a_i t^i$, with $a_{i_0} \neq 0$. By the non-degenerancy of $(\cdot \mid \cdot)$, there exists an element $b \in \mathfrak{a}$, such that $(a_{i_0} \mid b) \neq 0$. Then $(a \mid bt^{p-i_0})_p = c_p(a_{i_0} \mid b) \neq 0$, i.e., $(\cdot \mid \cdot)_p$ is non-degenerate. ∎

2.2. *Good* $\mathbb{Z}$*-grading of finite-dimensional Lie algebras.* A $\mathbb{Z}$*-grading* of a Lie algebra $\mathfrak{a}$ is a $\mathbb{Z}$-gradation $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}(i)$ with $[\mathfrak{a}(i), \mathfrak{a}(j)] \subseteq \mathfrak{a}(i + j)$ for all $i, j \in \mathbb{Z}$.

DEFINITION 2.4. *Let* $\Gamma : \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}(i)$ *be a* $\mathbb{Z}$*-grading of a finite-dimensional Lie algebra* $\mathfrak{a}$*. An element* $e \in \mathfrak{a}(2)$ *is called a* good element *with respect to* Γ *if*

$\operatorname{ad} e : \mathfrak{a}(i) \to \mathfrak{a}(i + 2)$ *is injective for* $i \leq -1$ *and surjective for* $i \geq -1$*.*

A $\mathbb{Z}$*-grading of* $\mathfrak{a}$ *is called* good *if it admits a good element.*

Given a good $\mathbb{Z}$-grading Γ and a good element e, the following properties are immediate:

(i) the element e is nilpotent and its centralizer $\mathfrak{a}^e$ lies in $\bigoplus_{i \geq 0} \mathfrak{a}(i)$;
(ii) $\operatorname{ad} e : \mathfrak{a}(-1) \to \mathfrak{a}(1)$ is bijective.

EXAMPLE 2.5. A *standard* $s\ell_2$*-triple* of a Lie algebra $\mathfrak{a}$ is a triple of elements $\{e, f, h\} \subseteq \mathfrak{a}$ with $[e, f] = h, [h, e] = 2e$ and $[h, f] = -2f$. It follows from the representation theory of $s\ell_2$ that the eigenspace decomposition of $\mathfrak{a}$ with respect to $\operatorname{ad} h$ is a good $\mathbb{Z}$-grading with a good element e. Good $\mathbb{Z}$-gradings thus obtained are called *Dynkin* $\mathbb{Z}$*-gradings*.

THEOREM 2.6 (Jacobson-Morozov). *Let* $\mathfrak{g}$ *be a finite-dimensional semisimple Lie algebra and* $e \in \mathfrak{g}$ *a non-zero nilpotent element. Then* e *can be embedded into a standard* $s\ell_2$*-triple* $\{e, f, h\}$ *of* $\mathfrak{g}$*. If* $h' \in [e, \mathfrak{g}]$ *satisfies that*

$[h', e] = 2e$, then $\{e, h'\}$ can be embedded into a standard $s\ell_2$-triple $\{e, h', f'\}$ of $\mathfrak{g}$.

LEMMA 2.7. *Let $\Gamma : \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading of a semisimple Lie algebra $\mathfrak{g}$ and $e \in \mathfrak{g}(2)$. Then there exist $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$, such that $\{e, h, f\}$ form a standard $s\ell_2$-triple.*

PROOF. By Theorem 2.6, we can embed e into an $s\ell_2$-triple $\{e, h, f\}$. Write $h = \sum_i h_i, f = \sum_i f_i$ with $h_i, f_i \in \mathfrak{g}(i)$. Then $[h_i, e] = \delta_{i,0}2e$ and $[e, f_i] = h_{i+2}$. In particular, we have $[e, f_{-2}] = h_0$. Therefore, by Theorem 2.6 again, we can embed $\{e, h_0\}$ into an $s\ell_2$-triple $\{e, h_0, f'\}$. Write $f' = \sum_i f'_i$ with $f'_i \in \mathfrak{g}(i)$, then $\{e, h_0, f'_{-2}\}$ is a standard $s\ell_2$-triple that we are looking for. □

LEMMA 2.8. *Let $\Gamma : \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading of a semisimple Lie algebra $\mathfrak{g}$. Then there exists an element $h_\Gamma \in \mathfrak{g}$, such that $[h_\Gamma, x] = ix, \forall x \in \mathfrak{g}(i)$.*

PROOF. It is clear that the linear operator $\delta : \mathfrak{g} \to \mathfrak{g}$ defined by $\delta(x) = ix$ for $x \in \mathfrak{g}(i)$ is a derivation of $\mathfrak{g}$. Since all derivations of a semisimple Lie algebra are inner, there exists an element $h_\Gamma \in \mathfrak{g}$ such that $[h_\Gamma, x] = \delta(x) = ix$ for $x \in \mathfrak{g}(i)$. □

REMARK 2.9. A complete classification of good $\mathbb{Z}$-gradings of finite-dimensional simple Lie algebras over $\mathbb{C}$ was given in [6].

3. Finite W-algebras associated to truncated current Lie algebras

3.1. *Finite W-algebras via Whittaker model definition.* In the sequel, we assume that $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra.

LEMMA 3.1. *Let $\Gamma : \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$ with good element e, and $h_\Gamma \in \mathfrak{g}$ such that $[h_\Gamma, x] = ix, \forall x \in \mathfrak{g}(i)$. Let $\mathfrak{g}_p(i) := \{x \in \mathfrak{g}_p \mid [h_\Gamma, x] = ix\ \}$. Then $\Gamma_p : \bigoplus_i \mathfrak{g}_p(i)$ is a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ with good element e.*

PROOF. For a subspace $\mathfrak{b}$ of $\mathfrak{g}$, denote by $\mathfrak{b}_p := \mathfrak{b} \otimes \frac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}$, then $\mathfrak{g}_p(i) = \mathfrak{g}(i)_p$. For the map $\operatorname{ad} e : \mathfrak{g}_p(i) \to \mathfrak{g}_p(i+2)$, we have $\ker \operatorname{ad} e = (\mathfrak{g}(i)^e)_p$ and $\operatorname{im} \operatorname{ad} e = ([\mathfrak{g}(i), e])_p$, so it is injective for $i \le -1$ and surjective for $i \ge -1$ as e is good with respect to Γ. □

We call Γ_p in Lemma 3.1 a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from that of $\mathfrak{g}$. Let $(\cdot \mid \cdot)$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. By choosing $\bar{c} = (c_0, \cdots, c_p)$ with $c_i = \delta_{i,p}$ for $0 \le i \le p$ in Lemma 2.3, we fix a non-degenerate invariant bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$.

LEMMA 3.2. *Let Γ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$, and $\Gamma_p : \bigoplus_i \mathfrak{g}_p(i)$ the good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from Γ. Then $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i + j \neq 0$.*

Proof. Let $h_\Gamma \in \mathfrak{g}$ be such that $[h_\Gamma, x] = ix, \forall x \in \mathfrak{g}_p(i)$ as in Lemma 2.8. Let $x \in \mathfrak{g}_p(i), y \in \mathfrak{g}_p(j)$ and $i+j \neq 0$. Then $([h_\Gamma, x] \mid y)_p = -(x \mid [h_\Gamma, y])_p$, i.e., $(i+j)(x \mid y)_p = 0$. Since $i+j \neq 0$, that implies $(x \mid y)_p = 0$. ∎

Let $\chi_p = (e \mid \cdot)_p \in \mathfrak{g}_p^*$. Define a skew-symmetric bilinear form on $\mathfrak{g}_p(-1)$ by

$$\langle \cdot, \cdot \rangle_p : \mathfrak{g}_p(-1) \times \mathfrak{g}_p(-1) \to \mathbb{C}, \qquad (x, y) \mapsto \langle x, y \rangle_p := \chi_p([x, y]). \tag{3.1}$$

Lemma 3.3. *The bilinear form on $\mathfrak{g}_p(-1)$ defined by (3.1) is non-degenerate.*

Proof. It follows from the surjectivity of $\operatorname{ad} e : \mathfrak{g}_p(-1) \to \mathfrak{g}_p(1)$, the invariance of $(\cdot \mid \cdot)_p$ and the pairing property $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i+j \neq 0$. ∎

Let $\mathfrak{l}_p$ be an isotropic subspace of $\mathfrak{g}_p(-1)$ with respect to (3.1), i.e., $\langle \mathfrak{l}_p, \mathfrak{l}_p \rangle_p = 0$, and $\mathfrak{l}_p^\perp := \{x \in \mathfrak{g}_p(-1) \mid (e \mid [x, y])_p = 0, \forall y \in \mathfrak{l}_p\}$ be its orthogonal complement. Set

$$\mathfrak{m}_p := \bigoplus_{i \leq -2} \mathfrak{g}_p(i), \quad \mathfrak{m}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p, \mathfrak{n}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p^\perp, \quad \mathfrak{n}_p := \bigoplus_{i \leq -1} \mathfrak{g}_p(i), \tag{3.2}$$

which are all nilpotent subalgebras of $\mathfrak{g}_p$. As $(e \mid [\mathfrak{m}_{\mathfrak{l},p}, \mathfrak{n}_{\mathfrak{l},p}])_p = 0$, the character χ_p defines a one-dimensional representation of $\mathfrak{m}_{\mathfrak{l},p}$, which we denote by $\mathbb{C}_{\chi_p}$. Let

$$Q_{\chi_p} := U(\mathfrak{g}_p) \otimes_{U(\mathfrak{m}_{\mathfrak{l},p})} \mathbb{C}_{\chi_p} \cong U(\mathfrak{g}_p)/I_{\chi_p},$$

where I_{χ_p} is the left ideal of $U(\mathfrak{g}_p)$ generated by $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$.

Lemma 3.4. *The adjoint action of $\mathfrak{n}_{\mathfrak{l},p}$ on $U(\mathfrak{g}_p)$ preserves the subspace I_{χ_p}.*

Proof. Let $x \in \mathfrak{n}_{\mathfrak{l},p}, y = \sum_i u_i(a_i - \chi_p(a_i)) \in I_{\chi_p}$ with $u_i \in U(\mathfrak{g}_p)$ and $a_i \in \mathfrak{m}_{\mathfrak{l},p}$. Then

$$[x, y] = \sum_i \left([x, u_i](a_i - \chi_p(a_i)) + u_i[x, a_i - \chi_p(a_i)] \right).$$

As $\chi_p([\mathfrak{n}_{\mathfrak{l},p}, \mathfrak{m}_{\mathfrak{l},p}]) = 0$, we have $[x, a_i - \chi_p(a_i)] = [x, a_i] \in I_{\chi_p}$, hence $[x, y] \in I_{\chi_p}$. ∎

Since $\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}$ preserves I_{χ_p}, it induces a well-defined adjoint action on Q_{χ_p}, such that

$$[x, \bar{u}] = \overline{[x, u]} \text{ for } x \in \mathfrak{n}_{\mathfrak{l},p}, u \in U(\mathfrak{g}_p),$$

where we denote by $\bar{u} := u + I_{\chi_p}$ for the image of $u \in U(\mathfrak{g}_p)$ in Q_{χ_p}. Let

$$H_{\chi_p} := Q_{\chi_p}^{\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}} = \{\bar{u} \in Q_{\chi_p} \mid [x, u] \in I_{\chi_p} \text{ for all } x \in \mathfrak{n}_{\mathfrak{l},p}\}.$$

Lemma 3.5. *There is a well-defined multiplication on H_{χ_p} by*

$$\bar{u} \cdot \bar{v} := \overline{uv} \textit{ for } \bar{u}, \bar{v} \in H_{\chi_p}.$$

Proof. First, we show that the multiplication $\bar{u}\cdot\bar{v}$ does not depend on the representatives. It is obvious that it does not depend on the representatives of v. For that of u, we need to show that $yv \in I_{\chi_p}$ for all $y \in I_{\chi_p}, \bar{v} \in H_{\chi_p}$. Assume that $y = \sum_i u_i(a_i - \chi_p(a_i))$, then

$$yv = [y, v] + vy = \sum_i u_i[a_i - \chi_p(a_i), v] + \sum_i [u_i, v](a_i - \chi_p(a_i)) + vy. \tag{3.3}$$

We have $[a_i - \chi_p(a_i), v] = [a_i, v] \in I_{\chi_p}$ by the definition of H_{χ_p} hence $yv \in I_{\chi_p}$. Next we show that H_{χ_p} is closed under the multiplication. Let $\bar{u}_1, \bar{u}_2 \in H_{\chi_p}$, we need to show that $\overline{u_1 u_2} \in H_{\chi_p}$, i.e., $[x, u_1 u_2] \in I_{\chi_p}, \forall x \in \mathfrak{n}_{\mathfrak{l},p}$. By Leibniz's rule, we have

$$[x, u_1 u_2] = [x, u_1]u_2 + u_1[x, u_2].$$

By the definition of H_{χ_p}, $[x, u_1], [x, u_2] \in I_{\chi_p}$ hence $[x, u_1]u_2 \in I_{\chi_p}$ by (3.3). □

The space H_{χ_p} inherits an associative algebra structure from that of $U(\mathfrak{g}_p)$ by Lemma 3.5.

Definition 3.6. *The* finite W-algebra *associated to the pair* $(\mathfrak{g}_p, e)$ *is defined to be* H_{χ_p}.

Remark 3.7. When $p = 0$, we recover the finite W-algebra defined in [16].

When $\mathfrak{l}_p$ is Lagrangian, i.e., $\mathfrak{l}_p = \mathfrak{l}_p^{\perp}$ hence $\mathfrak{m}_{\mathfrak{l},p} = \mathfrak{n}_{\mathfrak{l},p}$, we can realize H_{χ_p} as the opposite endomorphism algebra $(\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op}$ in the following way. As $Q_{\chi_p} \cong U(\mathfrak{g}_p)/I_{\chi_p}$ is a cyclic $\mathfrak{g}_p$-module, its endomorphism φ is determined by $\varphi(\bar{1})$. As $\bar{1}$ is killed by I_{χ_p}, $\varphi(\bar{1}) = \bar{y}$ gives an endormorphism of Q_{χ_p} if and only if $\bar{y}$ is killed by I_{χ_p}, so

$$\begin{aligned} H_{\chi_p} &= \{\bar{y} \in Q_{\chi_p} \mid [a, y] \in I_{\chi_p} \text{ for all } a \in \mathfrak{n}_{\mathfrak{l},p}\} \\ &= \{\bar{y} \in Q_{\chi_p} \mid (a - \chi_p(a))y \in I_{\chi_p} \text{ for all } a \in \mathfrak{m}_{\mathfrak{l},p}\} \\ &= (\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op}. \end{aligned}$$

3.2. *Poisson structure on Slodowy slice.* Let $\Gamma_p : \bigoplus_i \mathfrak{g}_p(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from that of $\mathfrak{g}$, with good element e. Let $f \in \mathfrak{g}_p(-2), h \in \mathfrak{g}_p(0)$ and $\{e, f, h\}$ forms a standard $s\ell_2$-triple of $\mathfrak{g}_p$ ensured by Lemma 2.7. The non-degenerate form $(\cdot \mid \cdot)_p$ defines a bijection $\kappa_p : \mathfrak{g}_p \to \mathfrak{g}_p^*$ through $x \mapsto (x \mid \cdot)_p$. Let $\mathfrak{g}_p^f$ be the centralizer of f in $\mathfrak{g}_p$. Set

$$\mathcal{S}_{e_p} := e + \mathfrak{g}_p^f \quad \text{and} \quad \mathcal{S}_{\chi_p} := \kappa_p(\mathcal{S}_{e_p}) = \chi_p + \ker \mathrm{ad}^* f.$$

Call S_{e_p} and S_{χ_p} the *Slodowy slice* through e in $\mathfrak{g}_p$ and through χ_p in $\mathfrak{g}_p^*$, respectively.

Remark 3.8. When $p = 0$, $\mathcal{S}_e := \mathcal{S}_{e_0}$ is the Slodowy slice through e in $\mathfrak{g}$. In the language of jet schemes [15], $\mathcal{S}_{e_p}$ is the p-th jet scheme of $\mathcal{S}_e$.

By the representation theory of $s\ell_2$, we have $\mathfrak{g}_p = \mathfrak{g}_p^e \oplus [\mathfrak{g}_p, f] = \mathfrak{g}_p^f \oplus [\mathfrak{g}_p, e]$, which implies that $\operatorname{ad} e : [f, \mathfrak{g}_p] \to [e, \mathfrak{g}_p]$ and $\operatorname{ad} f : [e, \mathfrak{g}_p] \to [f, \mathfrak{g}_p]$ are both bijective.

LEMMA 3.9 ([5, 8]). *Let* $r \in \bigoplus_{i \leq 1} \mathfrak{g}_p(i)$.

(a) *Let* $a \in \mathfrak{g}_p$, *then* $[e + r, [f, a]] = 0$ *only if* $[f, a] = 0$.
(b) *We have* $[e + r, [f, \mathfrak{g}_p]] \cap \mathfrak{g}_p^f = 0$ *and* $[e + r, [f, \mathfrak{g}_p]] \oplus \mathfrak{g}_p^f = \mathfrak{g}_p$.
(c) *Let* $a \in \mathfrak{g}_p$. *If* $[e + r, a] \in \mathfrak{g}_p^f$ *and* $(a \mid [e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)_p = 0$, *then* $[e + r, a] = 0$.

PROOF. (a) Let $a = \sum_i a_i$ with $a_i \in \mathfrak{g}_p(i)$ and $[f, a] \neq 0$. Let i_0 be such that $[f, a_{i_0}] \neq 0$ but $[f, a_i] = 0, \forall i > i_0$. Then the i_0-th component (which lies in $\mathfrak{g}_p(i_0)$) of $[e + r, [f, a]]$ is $[e, [f, a_{i_0}]]$ as $r \in \bigoplus_{i \leq 1} \mathfrak{g}_p(i)$ and $e \in \mathfrak{g}_p(2)$. Now the bijectivity of $\operatorname{ad} e : [f, \mathfrak{g}_p] \to [e, \mathfrak{g}_p]$ ensures that $[e, [f, a_{i_0}]] \neq 0$, in particular, $[e + r, [f, a]] \neq 0$.

(b) Assume that $a = \sum_i a_i$ with $a_i \in \mathfrak{g}_p(i)$ satisfies $[e + r, [f, a]] \neq 0$. Then $[f, a] \neq 0$. Let i_0 be as in (a), then the i_0-th component of $[e + r, [f, a]]$ is $[e, [f, a_{i_0}]] \neq 0$, and the $(i_0 - 2)$-th component of $[f, [e + r, [f, a]]]$ is $[f, [e, [f, a_{i_0}]]]$, which is also nonzero by the bijectivity of $\operatorname{ad} f : [e, \mathfrak{g}_p] \to [f, \mathfrak{g}_p]$. For the second part, let us count dimensions. We have $\dim[e + r, [f, \mathfrak{g}_p]] = \dim[f, \mathfrak{g}_p]$ by (a). Note that $\dim[f, \mathfrak{g}_p] = \dim \mathfrak{g}_p - \dim \mathfrak{g}_p^f$, so $\dim \mathfrak{g}_p = \dim \mathfrak{g}_p^f + \dim[e + r, [f, \mathfrak{g}_p]]$, and (b) is proved.

(c) For a subspace V of $\mathfrak{g}_p$, denote by $V^\perp$ its orthogonal complement with respect to $(\cdot \mid \cdot)_p$. Then $([e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)^\perp = [e + r, \mathfrak{g}_p]^\perp + (\mathfrak{g}_p^f)^\perp$. Note that $(\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p]$ and $[e + r, \mathfrak{g}_p]^\perp = \ker \operatorname{ad}(e + r)$ as $(\cdot \mid \cdot)_p$ is non-degenerate and invariant. Therefore, (c) is equivalent to saying that if $a = u + v$ with $u \in (\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p]$, $v \in [e+r, \mathfrak{g}_p]^\perp$ and $[e+r, a] \in \mathfrak{g}_p^f$, then $[e+r, a] = 0$. Since $u \in [f, \mathfrak{g}_p]$ and $v \in \ker \operatorname{ad}(e+r)$, we have $[e+r, a] = [e+r, u] \in \mathfrak{g}_p^f \cap [e+r, [f, \mathfrak{g}_p]]$, which must be zero by (b). ∎

It is well-known that there is a Poisson structure on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, with the coadjoint orbits as the symplectic leaves. Given a non-degenerate invariant bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{g}$, one can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\cdot \mid \cdot)$, hence equip $\mathfrak{g}$ itself with a Poisson structure, and the symplectic foliation of $\mathfrak{g}$ is given by the adjoint orbits. Let $\mathbb{O}$ be an adjoint orbit and $x \in \mathbb{O}$. The tangent space $T_x\mathbb{O}$ can be identified with $[\mathfrak{g}, x]$, and the symplectic form on $T_x\mathbb{O}$ is

$$\omega_x([a, x], [b, x]) = ([a, b] \mid x) \text{ for } a, b \in \mathfrak{g}. \tag{3.4}$$

THEOREM 3.10 ([18]). *Let* M *be a Poisson manifold with the symplectic foliation* $\sqcup_\alpha S_\alpha$. *Let* N *be a submanifold of* M *such that for all* α,

(i) N *is transversal to* S_α, *i.e.*, $T_xN + T_xS_\alpha = T_xM$ *for all* $x \in N \cap S_\alpha$.

(ii) *the subspace* $T_xN \cap T_xS_\alpha$ *is a symplectic subspace of* T_xS_α*, i.e., the symplectic form on* T_xS_α *is non-degenerate when restricted to* $T_xN \cap T_xS_\alpha$ *for all* $x \in N \cap S_\alpha$*.*

Then there is an induced Poisson structure on N*. The symplectic foliation of* N *is given by* $\sqcup_\alpha(N \cap S_\alpha)$ *and the symplectic form on* $T_x(N \cap S_\alpha)$ *for all* $x \in N \cap S_\alpha$ *is the restriction of the symplectic form on* T_xS_α*.*

Proposition 3.11. *The slice* S_{e_p} *has a Poisson structure.*

Proof. We show that the conditions in Theorem 3.10 are satisfied for the submanifold S_{e_p}. Let $x = e + r \in S_{e_p} \cap \mathbb{O}_x$, where $\mathbb{O}_x$ is the adjoint orbit of $\mathfrak{g}_p$ through x. As $r \in \mathfrak{g}_p^f \subseteq \bigoplus_{i \leq 0} \mathfrak{g}_p(i)$, Lemma 3.9 applies. Note that $T_xS_{e_p} = \mathfrak{g}_p^f$ and $T_x\mathbb{O}_x = [\mathfrak{g}_p, x]$. Part (b) of Lemma 3.9 shows that S_{e_p} is transversal to $\mathbb{O}_x$ at x. Next we show that the restriction of the symplectic form ω_x defined by (3.4) on the subspace $T_x\mathbb{O}_x \cap T_xS_{e_p} = [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ is non-degenerate. Assume that there exists an element $[a, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ such that for all $[b, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$, we have

$$\omega_x([a, x], [b, x]) = (x \mid [a, b])_p = (a \mid [b, x])_p = 0.$$

Part (c) of Lemma 3.9 shows that $[a, x] = 0$. Therefore, ω_x is non-degenerate when restricted to $[\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$. So S_{e_p} inherits a Poisson structure from that of $\mathfrak{g}_p$. ∎

Corollary 3.12. *The Slodowy slice* $\mathcal{S}_{\chi_p}$ *has a Poisson structure.*

The Poisson algebra $\mathbb{C}[\mathcal{S}_{\chi_p}]$ is called the *classical finite W-algebra* associated to $(\mathfrak{g}_p, e)$.

3.3. *An isomorphism of affine varieties.* Keep the notation in Section 3.1. Let G_p be the adjoint group of $\mathfrak{g}_p$ and $N_{\mathfrak{l},p}$ the unipotent subgroup of G_p with Lie algebra $\mathfrak{n}_{\mathfrak{l},p}$. Let

$$\mathfrak{m}_{\mathfrak{l},p}^\perp := \{x \in \mathfrak{g}_p \mid (x \mid y)_p = 0, \forall y \in \mathfrak{m}_{\mathfrak{l},p}\} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_p(i)\right) \oplus [\mathfrak{l}_p^\perp, e]. \tag{3.5}$$

As $\mathfrak{n}_{\mathfrak{l},p}$ is nilpotent, elements of $N_{\mathfrak{l},p}$ can be expressed as $\exp(\operatorname{ad} x)$ for $x \in \mathfrak{n}_{\mathfrak{l},p}$. Consider the adjoint action of $N_{\mathfrak{l},p}$ on S_{e_p}. Let $x \in \mathfrak{n}_{\mathfrak{l},p}, y \in \mathfrak{g}_p^f \subseteq \bigoplus_{i \leq 0} \mathfrak{g}_p(i)$. Then

$$\exp(\operatorname{ad} x)(e + y) = (1 + \operatorname{ad} x + \cdots + \frac{\operatorname{ad}^n x}{n!} + \cdots)(e + y) \in e + \mathfrak{m}_{\mathfrak{l},p}^\perp.$$

Therefore, the image of the adjoint action map $N_{\mathfrak{l},p} \times S_{e_p}$ lies in $e + \mathfrak{m}_{\mathfrak{l},p}^\perp$.

Lemma 3.13. *The adjoint action map* $\beta : N_{\mathfrak{l},p} \times S_{e_p} \to e + \mathfrak{m}_{\mathfrak{l},p}^\perp$ *is an isomorphism of affine varieties.*

Proof. The adjoint action map is obviously a morphism of varieties, we only need to show the bijectivity. We show that given $z \in \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, there is a unique $x \in \mathfrak{n}_{\mathfrak{l},p}$ and a unique $y \in \mathfrak{g}_p^f$, such that $\exp(\operatorname{ad} x)(e+y) = e+z$. Recall the expressions of $\mathfrak{m}_{\mathfrak{l},p}^{\perp}$ and $\mathfrak{n}_{\mathfrak{l},p}$ in (3.5) and (3.2), respectively, and note that $\mathfrak{g}_p^f \subseteq \bigoplus_{i\leq 0}\mathfrak{g}_p(i)$. For $x \in \mathfrak{n}_{\mathfrak{l},p}, y \in \mathfrak{g}_p^f, z \in \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, we can assume that $x = \sum_{i\leq -1} x_i, y = \sum_{j\leq 0} y_j$ and $z = \sum_{i\leq 1} z_i$ with $x_i, y_i, z_i \in \mathfrak{g}_p(i), x_{-1} \in \mathfrak{l}_p^{\perp}$ and $z_1 \in [\mathfrak{l}_p^{\perp}, e]$. Note that

$$\exp(\operatorname{ad} x)(e+y) = e + y + [x,e] + [x,y] + \sum_{n\geq 2}\frac{(\operatorname{ad} x)^n}{n!}(e+y).$$

The equation $\exp(\operatorname{ad} x)(e+y) = e+z$ means that

$$\sum_k z_k = \sum_j y_j + \sum_i [x_i, e] + \sum_{i,j}[x_i, y_j] + \sum_{n\geq 2}\frac{(\sum_i \operatorname{ad} x_i)^n}{n!}(e + \sum_j y_j), \tag{3.6}$$

which is equivalent to a series of equations, namely, for $k \leq 1$,

$$\begin{aligned} z_k - y_k - [x_{k-2}, e] = {} & \sum_{i+j=k}\operatorname{ad} x_i(y_j) + \sum_{n\geq 2}\frac{\sum_{i_1+\cdots+i_n=k-2}\operatorname{ad} x_{i_1}\cdots \operatorname{ad} x_{i_n}(e)}{n!} \\ & + \sum_{n\geq 2}\frac{\sum_{i_1+\cdots+i_n+j=k}\operatorname{ad} x_{i_1}\cdots\operatorname{ad} x_{i_n}(y_j)}{n!}. \end{aligned} \tag{3.7}$$

Given k, note that $\operatorname{ad} x_i, y_j$ appear on the right hand side of (3.7) only when $i > k-2$ and $j > k$. So if $\{x_i, y_j\}_{i\geq k_0-2, j\geq k_0}$ satisfy (3.7) for all $k \geq k_0$, and if we only change the values of $\{x_i, y_j\}_{i<k_0-2, j<k_0}$, then (3.7) is still valid for $k \geq k_0$.

Now we use a decreasing induction on k to show that given z, there is a unique solution (x, y) for (3.6). When $k = 1$, (3.7) reads $[x_{-1}, e] = z_1$, there is a unique solution $x_{-1} \in \mathfrak{l}_p^{\perp}$ as $z_1 \in [\mathfrak{l}_p^{\perp}, e]$ and $\operatorname{ad} e : \mathfrak{l}_p^{\perp} \to [\mathfrak{l}_p^{\perp}, e]$ is injective. For $k = k_0 \leq 0$, we assume that we have uniquely determined $\{x_i, y_j\}_{i\geq k_0-1, j\geq k_0+1}$ such that (3.7) is satisfied for $k \geq k_0+1$. We show that we can uniquely determine (x_{k_0-2}, y_{k_0}), such that (3.7) is satisfied for $k \geq k_0$. Set $k = k_0$ in (3.7), since $\{x_i, y_j\}_{i\geq k_0-1, j\geq k_0+1}$ are already determined, the right hand side of (3.7) is determined, which we denote by w_{k_0}, is an element of $\mathfrak{g}_p(k_0)$. Then (3.7) becomes

$$[e, x_{k_0-2}] = w_{k_0} + y_{k_0} - z_{k_0}.$$

This equation has a unique solution for (x_{k_0-2}, y_{k_0}) when z_{k_0} and w_{k_0} are given, as $\mathfrak{g}_p(k_0) = \mathfrak{g}_p^f(k_0) \oplus [\mathfrak{g}_p(k_0-2), e]$ and $\operatorname{ad} e$ is injective on $\mathfrak{g}_p(k_0-2)$. More precisely, write $w_{k_0} - z_{k_0} = a + b$ with $a \in \mathfrak{g}_p^f(k_0)$ and $b \in [\mathfrak{g}_p(k_0-2), e]$, then $y_{k_0} = -a$ and x_{k_0-2} is the unique element satisfying $[e, x_{k_0-2}] = b$.

By induction, we can find a unique solution (x, y) for (3.6) when z is given. ∎

Remark 3.14. Lemma 3.13 was proved in [10] when e is a principal nilpotent element, and then generalized by W. Gan and V. Ginzburg in [7] for Dynkin good $\mathbb{Z}$-gradings. Their proof involves a $\mathbb{C}^*$-action on both varieties and then applies a general theorem in algebraic geometry. Our proof here is purely algebraic and works for all good $\mathbb{Z}$-gradings.

Corollary 3.15. *The coadjoint action map* $\alpha : N_{\mathfrak{l},p} \times \mathcal{S}_{\chi_p} \to \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$ *is an isomorphism of affine varieties, where* $\mathfrak{m}_{\mathfrak{l},p}^{\perp,*} := \kappa_p(\mathfrak{m}_{\mathfrak{l},p}^{\perp})$.

3.4. *Quantization of Slodowy slices.* Keep the notation of Section 3.1. Denote the canonical PBW-filtration on $U(\mathfrak{g}_p)$ by $\{U_n(\mathfrak{g}_p) \mid n \geq 0\}$, and let

$$U_n(\mathfrak{g}_p)(i) := \{x \in U_n(\mathfrak{g}_p) \mid [h_\Gamma, x] = ix\}.$$

The *Kazhdan filtration* on $U(\mathfrak{g}_p)$ is defined by $K_n U(\mathfrak{g}_p) = \sum_{i+2j\leq n} U_j(\mathfrak{g}_p)(i)$ for $n \in \mathbb{Z}$, which is separated and exhaustive, i.e.,

$$\bigcap_{n\in\mathbb{Z}} K_n U(\mathfrak{g}_p) = \{0\} \text{ and } U(\mathfrak{g}_p) = \bigcup_{n\in\mathbb{Z}} K_n U(\mathfrak{g}_p).$$

The Kazhdan filtration on $U(\mathfrak{g}_p)$ induces filtrations on I_{χ_p}, Q_{χ_p} and H_{χ_p}, which we also denote by K_n. Note that $K_n Q_{\chi_p} = 0$ for $n < 0$ as $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$ contains all the negative-degree generators of $U(\mathfrak{g}_p)$ with respect to the Kazhdan filtration. Let gr_K be the associated graded with respect to the Kazhdan filtration, then $\mathrm{gr}_K I_{\chi_p}$ is exactly the ideal of $\mathbb{C}[\mathfrak{g}_p^*]$ defining the affine subvariety $\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, i.e.,

$$\mathrm{gr}_K U(\mathfrak{g}_p)/I_{\chi_p} = \mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}].$$

Since $H_{\chi_p} \subseteq Q_{\chi_p}$, we have a natural inclusion map

$$\nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathrm{gr}_K Q_{\chi_p}.$$

On the other hand, as $\mathcal{S}_{\chi_p} \subseteq \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, we have a restriction map

$$\nu_2 : \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \to \mathbb{C}[\mathcal{S}_{\chi_p}].$$

Composing these two maps, we get a homomorphism,

$$\nu = \nu_2 \circ \nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathbb{C}[\mathcal{S}_{\chi_p}]. \tag{3.8}$$

We are going to prove that ν is an isomorphism.

The module Q_{χ_p} is a filtered $U(\mathfrak{n}_{\mathfrak{l},p})$-module, where the filtration on $U(\mathfrak{n}_{\mathfrak{l},p})$ is the Kazhdan filtration induced from that of $U(\mathfrak{g}_p)$. This filtration induces filtrations on the cohomologies $H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})$, and there are canonical homomorphisms

$$\phi_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}). \tag{3.9}$$

Theorem 3.16. *The homomorphism* ν *defined in (3.8) is an isomorphism.*

Proof. First, we show that $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = \delta_{i,0}\mathbb{C}[\mathcal{S}_{\chi_p}]$. Recall the isomorphism of affine varieties in Lemma 3.13, which is $N_{\mathfrak{l}_p}$-equivariant (the action on $N_{\mathfrak{l},p}\times S_{e_p}$ is left multiplication on the first component, and the action on $e+\mathfrak{m}_{\mathfrak{l},p}^{\perp}$ is the adjoint action.). Thus we have an $\mathfrak{n}_{\mathfrak{l},p}$-module isomorphism $\mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathbb{C}[\mathcal{S}_{\chi_p}]$. Hence

$$H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathbb{C}[\mathcal{S}_{\chi_p}].$$

The cohomology $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}])$ is equal to the algebraic de Rham cohomology of $N_{\mathfrak{l}_p}$, which is $\mathbb{C}$ for $i = 0$ and trivial for $i > 0$ as $N_{\mathfrak{l}_p}$ is isomorphic to an affine space [4].

Next we show that the homomorphisms ϕ_i in (3.9) are all isomorphisms. The standard cochain complex for computing the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in Q_{χ_p} is

$$0 \to Q_{\chi_p} \to \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots \to \Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots . \tag{3.10}$$

The good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}_p^*$, and the subspace $\mathfrak{n}_{\mathfrak{l},p}^*$ is positively graded as $\mathfrak{n}_{\mathfrak{l},p}$ is negatively graded in $\mathfrak{g}_p$. We write the gradation as $\mathfrak{n}_{\mathfrak{l},p}^* = \bigoplus_{i\geq 1} \mathfrak{n}_{\mathfrak{l},p}^*(i)$. Define a filtration of $\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p}$ by setting $F_s(\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})$ to be the subspace spanned by $(x_1 \wedge \cdots \wedge x_n) \otimes v$ for all $x_i \in \mathfrak{n}_{\mathfrak{l},p}^*(n_i), v \in K_j Q_{\chi_p}$ such that $j + \sum n_i \leq s$, where K_j is the Kazhdan filtration on Q_{χ_p}. This defines a filtered complex on (3.10) whose associated graded complex gives us the standard cochain complex for the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in $\mathrm{gr}_K Q_{\chi_p}$. Consider the spectral sequence with

$$E_0^{s,t} = \frac{F_s(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}{F_{s-1}(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}.$$

Then $E_1^{s,t} = H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, \dfrac{K_s Q_{\chi_p}}{K_{s-1}Q_{\chi_p}})$ and the spectral sequence converges to

$$E_\infty^{s,t} = \frac{F_s H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})}{F_{s-1} H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})},$$

i.e., the maps $\phi_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p})$ are isomorphisms hence

$$\mathrm{gr}_K H_{\chi_p} = \mathrm{gr}_K H^0(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \cong H^0(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) \cong \mathbb{C}[\mathcal{S}_{\chi_p}].$$

□

Remark 3.17. For $p = 0$, the isomorphism in Theorem 3.16 was proved by A. Premet [16] when $\mathfrak{l}$ is a Lagrangian subspace and then generalized by W. Gan and V. Ginzburg [7] for general isotropic subspaces $\mathfrak{l}$. Our method here follows [19].

Corollary 3.18. *The algebra H_{χ_p} does not depend on the choice of $\mathfrak{l}_p$.*

Proof. It is enough to prove that if $\mathfrak{l}_p \subseteq \mathfrak{l}'_p$ are two isotropic subspaces of $\mathfrak{g}_p(-1)$, then the corresponding finite W-algebras H_{χ_p} and H'_{χ_p} are isomorphic. We have an inclusion $\pi : H_{\chi_p} \hookrightarrow H'_{\chi_p}$ hence a map $\mathrm{gr}\,\pi : \mathrm{gr}_K H_{\chi_p} \hookrightarrow \mathrm{gr}_K H'_{\chi_p}$. By Theorem 3.16, the map $\mathrm{gr}\,\pi$ is an isomorphism as they are both isomorphic to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, so π is itself an isomorphism. □

4. Kostant's theorem and Skryabin equivalence

4.1. *Kostant's theorem.* A nilpotent element $x \in \mathfrak{g}$ is called *regular* (or *principal nilpotent*) if its centralizer $\mathfrak{g}^x$ has minimal dimension, i.e., $\dim \mathfrak{g}^x \leq \dim \mathfrak{g}^{x'}$ for all $x' \in \mathfrak{g}$. We show in this section that the finite W-algebra H_{χ_p} associated to $(\mathfrak{g}_p, e)$, when e is regular, is isomorphic to $Z(\mathfrak{g}_p)$, the center of the universal enveloping algebra $U(\mathfrak{g}_p)$.

Let $S(\mathfrak{g}_p)$ be the symmetric algebra of $\mathfrak{g}_p$. It is well known that there is an isomorphism of $\mathfrak{g}_p$-modules $\varphi : S(\mathfrak{g}_p) \to \mathrm{gr}U(\mathfrak{g}_p)$, where gr is the associated graded of the PBW filtration of $U(\mathfrak{g}_p)$. Let $I(\mathfrak{g}_p) := \{g \in S(\mathfrak{g}_p) \mid [x, g] = 0, \forall x \in \mathfrak{g}_p\}$ be the $\mathfrak{g}_p$-invariants in $S(\mathfrak{g}_p)$ and $Z(\mathfrak{g}_p)$ be the center of $U(\mathfrak{g}_p)$. Then the restriction of φ to $I(\mathfrak{g}_p)$ yields an isomorphism of vector spaces

$$\varphi : I(\mathfrak{g}_p) \to \mathrm{gr}Z(\mathfrak{g}_p).$$

Recall that $S_{e_p} = e + \mathfrak{g}_p^f$ and $\mathcal{S}_{\chi_p} = \kappa_p(S_{e_p})$. Since $\mathcal{S}_{\chi_p} \subseteq \mathfrak{g}_p^*$, we have a canonical restriction $\iota_p : \mathbb{C}[\mathfrak{g}_p^*] \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Identifying $\mathbb{C}[\mathfrak{g}_p^*]$ with $S(\mathfrak{g}_p)$ and restricting ι_p to $I(\mathfrak{g}_p)$, we get a natural map from $I(\mathfrak{g}_p)$ to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, which we still denote by ι_p.

Lemma 4.1 ([12, 17]). *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and e be a regular nilpotent element of $\mathfrak{g}$. Then the following statements hold.*

(1) *Every element of S_{e_p} is regular. Moreover, the adjoint orbit of every regular element intersects S_{e_p} in a unique point.*
(2) *The map $\iota_p : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$ is an isomorphism of vector spaces.*

Theorem 4.2. *Let e be a regular nilpotent element of $\mathfrak{g}$. Then the finite W-algebra H_{χ_p} associated to the pair $(\mathfrak{g}_p, e)$ is isomorphic to the center of $U(\mathfrak{g}_p)$.*

Proof. Since $Z(\mathfrak{g}_p) \subseteq U(\mathfrak{g}_p)$ is obviously invariant under the adjoint action of $\mathfrak{n}_{\mathfrak{l},p}$, we have a natural map $j_p : Z(\mathfrak{g}_p) \to H_{\chi_p}$, which preserves the Kazhdan filtrations on $Z(\mathfrak{g}_p)$ and H_{χ_p}. Passing to their associated graded, we have $\mathrm{gr}\, j_p : \mathrm{gr}Z(\mathfrak{g}_p) \to \mathrm{gr}H_{\chi_p}$, which is the isomorphism $\iota : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Since the associated graded of j_p is an isomorphism, j_p itself is an isomorphism of algebras.

$$\begin{array}{ccc} Z(\mathfrak{g}_p) & \xrightarrow{j_p} & H_{\chi_p} \\ \big\downarrow{\scriptstyle \mathrm{gr}} & & \big\downarrow{\scriptstyle \mathrm{gr}} \\ I(\mathfrak{g}_p) & \xrightarrow[\cong]{\mathrm{gr}\, j_p} & \mathbb{C}[\mathcal{S}_{\chi_p}] \end{array}$$

□

Remark 4.3. When $p = 0$, Lemma 4.1 and Theorem 4.2 were proved by B. Kostant [10]. Moreover, T. Macedo and A. Savage [12] generalized Lemma 4.1 to truncated multicurrent Lie algebras. In fact, finite W-algebras associated to truncated multicurrent Lie algebras can be defined and Kostant's theorem holds there.

4.2. *Skryabin equivalence.* Keep the notation of Section 3.1.

Definition 4.4. *A $\mathfrak{g}_p$-module M is called a* Whittaker module *if $a-\chi_p(a)$ acts locally nilpotently on M for all $a \in \mathfrak{m}_{\mathfrak{l},p}$, an element $m \in M$ is called a* Whittaker vector *if $(a - \chi_p(a)) \cdot m = 0$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. For a Whittaker module M, denote by* $\mathrm{Wh}(M)$ *the collection of the Whittaker vectors of M.*

The $\mathfrak{g}_p$-module Q_{χ_p} is a Whittaker module and $\mathrm{Wh}(Q_{\chi_p}) = H_{\chi_p}$.

Denote by $\mathfrak{g}_p$-Wmod^{χ_p} the category of finitely generated Whittaker $\mathfrak{g}_p$-modules, and H_{χ_p}-Mod the category of finitely generated left H_{χ_p}-modules. Note that Q_{χ_p} admits a right H_{χ_p}-module structure as we have $H_{\chi_p} \cong (\mathrm{End}_{\mathfrak{g}_p} Q_{\chi_p})^{op}$.

Lemma 4.5. *Let $M \in \mathfrak{g}_p$-$Wmod^{\chi_p}$ and $N \in H_{\chi_p}$-Mod, then*

(1) $\mathrm{Wh}(M) = 0$ *implies that $M = 0$.*

(2) $\mathrm{Wh}(M)$ *admits an H_{χ_p}-module structure, with*

$$(y + I_{\chi_p}) \cdot v = y \cdot v \ \text{ for } \ y + I_{\chi_p} \in H_{\chi_p}, v \in \mathrm{Wh}(M).$$

(3) $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p$-$Wmod^{\chi_p}$.

Proof. By definition, a Whittaker $\mathfrak{g}_p$-module M is locally $U(\mathfrak{m}_{\mathfrak{l},p})$-finite as $U(\mathfrak{m}_{\mathfrak{l},p})$ is generated by 1 and $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. Given a nonzero vector $v \in M$, we have $\dim U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v < \infty$. Since $a - \chi_p(a)$ are nilpotent operators on $U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v$, by Engel's theorem, we can find a nonzero common eigenvector for them, which is a Whittaker vector, so $\mathrm{Wh}(M) \neq 0$ if $M \neq 0$. This proves part (1).

For (2), we need to show $y \cdot v \in \mathrm{Wh}(M), \forall y + I_{\chi_p} \in H_{\chi_p}, v \in \mathrm{Wh}(M)$. We have

$$(a - \chi_p(a))y \cdot v = [a - \chi_p(a), y] \cdot v + y(a - \chi_p(a)) \cdot v = [a - \chi_p(a), y] \cdot v.$$

By Lemma 3.4, we have $[a, y] \in I_{\chi_p}$, so $(a-\chi_p(a))y \cdot v = 0$, i.e., $y \cdot v \in \mathrm{Wh}(M)$.

For (3), as Q_{χ_p} is a Whittaker $\mathfrak{g}_p$-module, $a-\chi_p(a)$ acts locally nilpotently on it. But the $U(\mathfrak{g}_p)$-action on the tensor product is from the left side, so $a-\chi_p(a)$ acts automatically locally nilpotently on the tensor product $Q_{\chi_p}\otimes_{H_{\chi_p}} N, \forall a\in\mathfrak{m}_{\mathfrak{l},p}$. ∎

By Lemma 4.5, we have two functors,

$$\begin{aligned} \mathrm{Wh}: \quad & \mathfrak{g}_p\text{-Wmod}^{\chi_p} \longrightarrow H_{\chi_p}\text{-Mod}, \qquad M \longmapsto \mathrm{Wh}(M),\\ Q_{\chi_p}\otimes_{H_{\chi_p}} - : \quad & H_{\chi_p}\text{-Mod} \longrightarrow \mathfrak{g}_p\text{-Wmod}^{\chi_p}, \qquad N \longmapsto Q_{\chi_p}\otimes_{H_{\chi_p}} N. \end{aligned}$$

The functor $\mathrm{Wh}(-)$ is left exact and the functor $Q_{\chi_p}\otimes_{H_{\chi_p}} -$ is right exact.

THEOREM 4.6. *The two functors* $\mathrm{Wh}(-)$ *and* $Q_{\chi_p}\otimes_{H_{\chi_p}} -$ *give an equivalence of categories between* $\mathfrak{g}_p$*-Wmod*$^{\chi_p}$ *and* H_{χ_p}*-Mod.*

PROOF. Let $\mathfrak{l}_p$ be a Lagrangian subspace of $\mathfrak{g}_p(-1)$, so we have $\mathfrak{m}_{\mathfrak{l},p}=\mathfrak{n}_{\mathfrak{l},p}$. First of all, we show $\mathrm{Wh}(Q_{\chi_p}\otimes_{H_{\chi_p}} N)\cong N$ for all $N\in H_{\chi_p}$-Mod. Assume that N is generated by a finite-dimensional subspace N_0. Setting $K_nN:=(K_nH_{\chi_p})N_0$ gives a filtration on N and it becomes a filtered H_{χ_p}-module. Twist the $\mathfrak{m}_{\mathfrak{l},p}$-action on $Q_{\chi_p}\otimes_{H_{\chi_p}} N$ by $-\chi_p$, i.e., for $a\in\mathfrak{m}_{\mathfrak{l},p}, u\in Q_{\chi_p}, v\in N$, define a new action

$$a\cdot(u\otimes v)=(a-\chi_p(a))u\otimes v=\mathrm{ad}(a-\chi_p(a))(u)\otimes v.$$

Then $\mathrm{Wh}(Q_{\chi_p}\otimes_{H_{\chi_p}} N)=H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p}\otimes_{H_{\chi_p}} N)$ with respect to this new action. The Kazhdan filtrations on Q_{χ_p} and N induce a Kazhdan filtration on $Q_{\chi_p}\otimes_{H_{\chi_p}} N$, with

$$K_n(Q_{\chi_p}\otimes_{H_{\chi_p}} N)=\sum_{i+j=n} K_iQ_{\chi_p}\otimes_{H_{\chi_p}} K_jN.$$

Since both $K_nQ_{\chi_p}=0$ and $K_nN=0$ for $n<0$ as we noted in Section 3.4, the filtration gives us homomorphisms for $i\geq 0$,

$$\phi_i:\mathrm{gr}_K H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p}\otimes_{H_{\chi_p}} N)\to H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p}\otimes_{H_{\chi_p}} N)). \tag{4.1}$$

Remember that $\mathrm{gr}_K Q_{\chi_p}\cong\mathbb{C}[\chi_p+\mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$ and $\mathrm{gr}_K H_{\chi_p}\cong\mathbb{C}[\chi_p+\ker\mathrm{ad}^* f]$. Since $\chi_p+\ker\mathrm{ad}^* f$ is an affine subspace of $\chi_p+\mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, $\mathrm{gr}_K Q_{\chi_p}$ is free over $\mathrm{gr}_K H_{\chi_p}$, and we have an isomorphism

$$\mathrm{gr}_K(Q_{\chi_p}\otimes_{H_{\chi_p}} N)\cong\mathrm{gr}_K Q_{\chi_p}\otimes_{\mathrm{gr}_K H_{\chi_p}}\mathrm{gr}_K N.$$

By Corollary 3.15, we have $\mathfrak{m}_{\mathfrak{l},p}$-module (precisely, $\mathfrak{n}_{\mathfrak{l},p}$-module) isomorphisms

$$\mathrm{gr}_K Q_{\chi_p}\cong\mathbb{C}[N_{\mathfrak{l}_p}]\otimes\mathbb{C}[S_{\chi_p}]\cong\mathbb{C}[N_{\mathfrak{l}_p}]\otimes\mathrm{gr}_K H_{\chi_p}.$$

Therefore,

$$\begin{aligned} H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N)) &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p} \otimes_{\mathrm{gr}_K H_{\chi_p}} \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathrm{gr}_K N \\ &= \delta_{i,0} \mathrm{gr}_K N. \end{aligned}$$

There is a spectral sequence as that in the proof of Theorem 3.16, which asserts that those ϕ_i in (4.1) are all isomorphisms. Therefore, we have (note that $\mathrm{gr}_K N = N$)

$$H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong \begin{cases} N & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases} \tag{4.2}$$

In particular, we have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong N$. Next we show that $Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \cong M$ for all $M \in \mathfrak{g}_p\text{-Wmod}^{\chi_p}$. Define a map

$$\varphi : Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M, \qquad (y + I_{\chi_p}) \otimes v \mapsto y \cdot v,$$

which is a $\mathfrak{g}_p$-module homomorphism. Then we have the following exact sequence,

$$0 \to \ker \varphi \to Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M \to \operatorname{coker} \varphi \to 0. \tag{4.3}$$

Applying $\mathrm{Wh}(-)$ to (4.3), the identity $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$ and the left exactness of $\mathrm{Wh}(-)$ imply that $\mathrm{Wh}(\ker \varphi) = 0$, hence $\ker \varphi = 0$ by Lemma 4.5. The long exact sequence of the cohomology of $\mathfrak{m}_{\mathfrak{l},p}$ associated to (4.3) gives

$$\begin{aligned} 0 &\to H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) \\ &\to H^0(\mathfrak{m}_{\mathfrak{l},p}, M) \to H^0(\mathfrak{m}_{\mathfrak{l},p}, \operatorname{coker} \varphi) \to 0. \end{aligned} \tag{4.4}$$

We stop at $H^0(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{coker}\varphi)$ because $H^1(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = 0$ by (4.2). Note that $H^0(\mathfrak{m}_{\mathfrak{l},p}, -) = \mathrm{Wh}(-)$ and we have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$, so (4.4) implies that $\mathrm{Wh}(\operatorname{coker} \varphi) = 0$ hence $\operatorname{coker} \varphi = 0$, i.e., the map φ is an isomorphism. ∎

Corollary 4.7. *There is a one-to-one correspondence between simple Whittaker modules of $\mathfrak{g}_p$ and simple modules of the finite W-algebra.*

4.3. *Whittaker modules for truncated current Lie algebras in the $s\ell_2$ case.* Consider the case when $\mathfrak{g} = s\ell_2$. Let $\{e, f, h\}$ be the canonical basis of $s\ell_2$ such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Denote by $x_i := x \otimes t^i$ for $x \in \mathfrak{g}$ (the notation is different from what we used before). We allow to use e_i, h_i, f_i for all non-negative integers i in $\mathfrak{g}_p$, but will consider them as zero when $i > p$. It is obvious that $\{e_i, f_i, h_i \mid 0 \leq i \leq p\}$ form a basis of $\mathfrak{g}_p$. Fix a non-degenerate invariant bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$.

There is a natural good $\mathbb{Z}$-grading for the nilpotent element f_0 on $\mathfrak{g}_p$ given by $\mathrm{ad}(-h_0)$. Explicitly, the $\mathbb{Z}$-grading is

$$\mathfrak{g}_p(-2) = \mathrm{span}\{e_i\}_{0\leq i\leq p}, \mathfrak{g}_p(0) = \mathrm{span}\{h_i\}_{0\leq i\leq p}, \mathfrak{g}_p(2) = \mathrm{span}\{f_i\}_{0\leq i\leq p}.$$

Keep the notation of Section 3.1. We have $\mathfrak{m}_p = \mathfrak{g}_p(-2)$. Let $\chi_p := (f_0 \mid \cdot)_p$ and I_{χ_p} be the left ideal of $U(\mathfrak{g}_p)$ generated by $\{e_i - \chi_p(e_i)\}_{0\leq i\leq p}$. A Whittaker module for $\mathfrak{g}_p$ is a module on which $e_i - \chi_p(e_i)$ acts locally nilpotently for all i. Consider the $\mathfrak{g}_p$-module $Q_{\chi_p} = U(\mathfrak{g}_p)/I_{\chi_p}$. As f_0 is regular nilpotent, by Kostant's Theorem 4.2, the finite W-algebra is isomorphic to the center of $U(\mathfrak{g}_p)$, i.e.,

$$H_{\chi_p} \cong Z(\mathfrak{g}_p).$$

Following the idea of A. Molev [13, 14], for $0 \leq k \leq p$, define

$$C_k = \sum_{0\leq j\leq k} \big(\frac{h_{p-k+j}}{2} (\frac{h_{p-j}}{2} + (p+1)\delta_{p,j}) + f_{p-k+j} e_{p-j} \big) \in U(\mathfrak{g}_p).$$

Proposition 4.8 ([9, 14]). *The center of* $U(\mathfrak{g}_p)$ *is* $Z(\mathfrak{g}_p) = \mathbb{C}[C_0, \cdots, C_p]$.

It is well known that simple modules of $\mathbb{C}[C_0, \cdots, C_p]$ are all one-dimensional and one-to-one correspond to its maximal ideals. Let $\mathbb{C}_\varepsilon$ be such a module, while C_k acts on it as ε_k for $0 \leq k \leq p$, and $\varepsilon_1, \cdots, \varepsilon_p \in \mathbb{C}$ are arbitrary constants. By Skryabin equivalence (Theorem 4.6), we have the following result.

Theorem 4.9. *Simple Whittaker modules of* $\mathfrak{g}_p$ *are of the form* $Q_{\chi_p} \otimes_{Z(\mathfrak{g}_p)} \mathbb{C}_\varepsilon$.

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