GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 35 – 47

HÖLDER CONTINUITY FOR THE SOLUTIONS OF THE $p(x)$-LAPLACE EQUATION WITH GENERAL RIGHT-HAND SIDE

Abdeslem Lyaghfouri
United Arab Emirates University, UAE

Abstract. We show that bounded solutions of the quasilinear elliptic equation $\Delta_{p(x)}u = g + div(\mathbf{F})$ are locally Hölder continuous provided that the functions g and $\mathbf{F}$ are in suitable Lebesgue spaces.

1. Introduction

We consider the following equation

$$\Delta_{p(x)}u = g + div(\mathbf{F}) \quad \text{in } W^{-1,q(x)}(\Omega), \tag{1.1}$$

where $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian, Ω is an open bounded domain of $\mathbb{R}^n$, $n \geq 2$, $x = (x_1, \dots, x_n)$, $q(x) = \frac{p(x)}{p(x)-1}$, and $p : \Omega \to \mathbb{R}$ is a measurable function which satisfies for some positive constants $p_+ > p_- > 1$ and $s > n$

$$\begin{aligned} &p_- \leq p(x) \leq p_+ \quad \text{a.e. } x \in \Omega, \\ &\nabla p \in \left(L^s(\Omega)\right)^n. \end{aligned} \tag{1.2}$$

As a consequence of (1.2), we have $p \in W^{1,s}(\Omega)$. Moreover, due to Sobolev embedding $W^{1,s}(\Omega) \subset C^{0,\beta}(\Omega)$, $\left(\beta = 1 - \dfrac{n}{s}\right)$, p is Hölder continuous in Ω.

We call a solution of equation (1.1) any function $u \in W^{1,p(x)}(\Omega)$ that fulfills

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\zeta dx = -\int_\Omega g(x)\zeta dx + \int_\Omega \mathbf{F}(x) \cdot \nabla\zeta dx \quad \forall \zeta \in W_0^{1,p(x)}(\Omega).$$

2020 *Mathematics Subject Classification.* 35B65, 35J92.
Key words and phrases. $p(x)$-Laplacian, Hölder continuity.

The Lebesgue and Sobolev spaces with variable exponents are defined (see for example [2, 15] and [18]) by:

$$L^{p(x)}(\Omega)=\left\{u:\Omega\to\mathbb{R}\ \text{ measurable }:\int_\Omega |u(x)|^{p(x)}dx<\infty\right\},$$
$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}\Omega):\nabla u\in\left(L^{p(x)}(\Omega)\right)^n\right\},$$
$$W_0^{1,p(x)}(\Omega)=\overline{C_0^\infty(\Omega)}_{W^{1,p(x)}(\Omega)}.$$

These spaces are separable, complete and reflexive, when equipped with the following norms

$$\|u\|_{L^{p(x)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u}{\lambda}\right|^{p(x)}dx\le 1\right\},$$
$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)},\quad \|\nabla u\|_{p(x)}=\sum_{i=1}^n\left\|\frac{\partial u}{\partial x_i}\right\|_{p(x)}.$$

Our aim is to establish Hölder continuity for bounded solutions of (1.1). We observe that if $p(x)>n$ in an open set $U\subset\subset\Omega$, then by Sobolev embedding $W^{1,p(x)}(U)\subset W^{1,p_m(U)}(U)\subset C^{0,\alpha}(U)$, where $p_m(U)=\min_{x\in\overline{U}}p(x)$ and $\alpha=1-\dfrac{n}{p_m(U)}$. Therefore any solution of (1.1) is Hölder continuous in U. In this paper, we assume that

$$p(x)\le n\quad\forall x\in\Omega. \tag{1.3}$$

g is a real valued function that satisfies for a positive number t_1

$$\begin{aligned}&t_1>\frac{n}{p(x)}\quad\forall x\in\Omega,\\&g\in L^{t_1}(\Omega).\end{aligned}\tag{1.4}$$

$\mathbf{F}=(F_1,\ldots,F_n)$ is a vector function that satisfies for a positive number t_2

$$\begin{aligned}&t_2>\frac{n}{p(x)-1}\quad\forall x\in\Omega,\\&\mathbf{F}\in L^{t_2}(\Omega).\end{aligned}\tag{1.5}$$

Among problems that fit in the equation (1.1) setting, is the dam problem ($g=0$, $\mathbf{F}=\chi\mathbf{e}$, with $\mathbf{e}=(0,\ldots,0,1)$, $\chi\in L^\infty(\Omega)$), and $p(x)$ a constant, [4, 5, 9]). It is know that the solution in this case is $C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha\in(0,1)$ (see [7]). In fact, due to the particularity of the problem (i.e., because $u\ge 0$ and $\chi=1$ a.e. in $\{u>0\}$), we actually have $u\in C^{0,1}_{\text{loc}}(\Omega)$ (see [12]). Another problem is the obstacle problem, [10, 11, 13, 14, 23]). Indeed, because the solution of the obstacle problem satisfies the Levy-Stampacchia inequality, i.e., $f\chi([u>0])\le\Delta_{p(x)}u\le f$ a.e. in Ω (see for example [24]), where

$f \in L^{q(x)}(\Omega)$, one can write $\Delta_{p(x)} u = g$ a.e. in Ω, where $g = \theta f$ and $0 \leq \theta \leq 1$ a.e. in Ω. For a more general framework, we refer to [8, 12, 19].

Equations involving variable exponents are called equations with nonstandard growth. For an overview of such equations, we refer to [16], where the authors establish global boundedness and Hölder continuity for a class of elliptic problems.

The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that (1.2)-(1.5) hold. Then any bounded solution of (1.1) is locally Hölder continuous in* Ω.

It is expected that solutions of (1.1) are locally and even globally bounded if a Dirichlet or Neumann boundary condition is added. This can be established for example by adapting the so-called De Giorgi-Nash-Moser theory. However, it is not the purpose of this paper to discuss this question. The interested reader is refereed for example to the papers [16] and [25], where global boundedness for a class of elliptic problems are established for both homogeneous Dirichlet and nonhomogeneous Neumann boundary conditions.

In the context of constant exponent, the regularity result in Theorem 1.1 was established in [22] for a more general quasi-linear elliptic operator that includes the $p-$Laplacian under the assumptions: $g \in L^{\frac{n}{p+\epsilon}}(\Omega)$ and $\mathbf{F} \in L^t(\Omega)$ with $\epsilon > 0$ and $t > \dfrac{n}{p-1}$, which coincide with assumptions (1.5). Moreover, when the right-hand side of (1.1) is a nonnegative Radon measure with a suitable growth condition, local Hölder continuity was established in [17] for the $p-$Laplacian and in a more general framework in [22]. This result was later extended in [7] and [20] respectively for the $A-$Laplacian and $p(x)-$Laplacian.

Throughout this paper, we will denote by $B_r(x)$ (resp. $\overline{B}_r(x)$) the open (resp. closed) ball of center x and radius r in $\mathbb{R}^n$, with $\omega_n = |B_1|$ standing for the measure of the unit open ball B_1.

2. PROOF OF THEOREM 1.1

In this section, we denote by u a bounded solution of (1.1) with $M = \|u\|_\infty$, and will show that it is locally Hölder continuous in Ω. The proof is based on Lemma 2.1.

LEMMA 2.1. *Let* $\delta = \frac{s-n}{2sn}$, $q = \frac{n+s}{2}$, $m = \frac{sq(1+\delta)}{s-q}$, *and*

$$R_1 = \min\left(\mathrm{diam}(\Omega)/16, c(n,s,p_-,p_+)\left(\int_\Omega |\nabla u|^{p(x)}dx + 1\right)^{\frac{-m}{\beta}}\right).$$

Assume that (1.2)-(1.5) hold. Then there exists a positive constant

$$C = C(n, s, p_-, p_+, M, \mathrm{diam}(\Omega), \|\nabla u\|_{L^{p(x)}(\Omega)}, \|\nabla p\|_{L^s(\Omega)}, \|g\|_{L^{t_1}(\Omega)}, \|F\|_{L^{t_2}(\Omega)}),$$

such that we have for $R \in (0, R_1)$ *with* $B_{2R}(x_0) \subset \Omega$

$$\int_{B_r(x_0)} |\nabla u|^{p_m} dx \leq C r^{n-p_m+\alpha p_m} \qquad \forall r \in (0, R), \tag{2.1}$$

where $p_m = \min_{x \in \overline{B}_R(x_0)} p(x)$ *and* $\alpha = 1 - \max\left(\frac{n}{t_1 p_m}, \frac{n}{t_2(p_m - 1)}\right)$.

The proof of Lemma 2.1 is based on Lemma 2.2.

LEMMA 2.2. *Assume that (1.2)-(1.5) hold and let* R_1 *be the positive number in Lemma 2.1. Then there exists a positive constant*

$$C_1 = C_1(n, s, p_-, p_+, \mathrm{diam}(\Omega), \|\nabla u\|_{L^{p(x)}(\Omega)}, \|\nabla p\|_{L^s(\Omega)}),$$

such that we have for $R \in (0, R_1)$, $B_{2R}(x_0) \subset \Omega$, $v \in W^{1,p(x)}(B_R(x_0))$ *with* $\Delta_{p(x)} v = 0$ *in* $B_R(x_0)$ *and* $v = u$ *on* $\partial B_R(x_0)$

$$\int_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq C_1 \left(\left(\left(\frac{r}{R}\right)^n + R^{\frac{\beta}{2}} \right) \int_{B_R(x_0)} |\nabla u|^{p(x)} dx + R^{n+\beta} \right) \tag{2.2}$$

for all $r \in (0, R)$.

PROOF. First, observe that we have $v \in C^{1,\sigma}_{loc}(B_R(x_0))$ (see for example [3]). Next, since $\int_{B_R(x_0)} \frac{|\nabla v|^{p(x)}}{p(x)} dx$ minimizes the energy $\int_{B_R(x_0)} \frac{|\nabla \xi|^{p(x)}}{p(x)} dx$ over all functions $\xi \in W^{1,p(x)}(B_R(x_0))$ such that $\xi = u$ on $\partial B_R(x_0)$, we get

$$\begin{aligned} \int_{B_R(x_0)} |\nabla v|^{p(x)} dx &= \int_{B_R(x_0)} p(x) \frac{|\nabla v|^{p(x)}}{p(x)} dx \\ &\leq p_+ \int_{B_R(x_0)} \frac{|\nabla v|^{p(x)}}{p(x)} dx \leq p_+ \int_{B_R(x_0)} \frac{|\nabla u|^{p(x)}}{p(x)} dx \\ &\leq \frac{p_+}{p_-} \int_{B_R(x_0)} |\nabla u|^{p(x)} dx. \end{aligned} \tag{2.3}$$

Using (2.3), we get for $r \in \left[\frac{R}{8}, R\right)$

$$\begin{aligned} \int_{B_r(x_0)} |\nabla v|^{p(x)} dx &= \left(\frac{R}{r}\right)^n \left(\frac{r}{R}\right)^n \int_{B_r(x_0)} |\nabla v|^{p(x)} dx \\ &\leq 8^n \left(\frac{r}{R}\right)^n \int_{B_r(x_0)} |\nabla v|^{p(x)} dx \\ &\leq 8^n \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla v|^{p(x)} dx \\ &\leq 8^n \frac{p_+}{p_-} \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla u|^{p(x)} dx. \end{aligned}$$

Therefore it is enough to prove (2.2) for $r \in \left(0, \frac{R}{8}\right)$. Since $\Delta_{p(x)} v = 0$ in $B_R(x_0)$, $v = u$ on $\partial B_R(x_0)$, we obtain by the maximum principle $\|v\|_{L^\infty(B_R(x_0))} \leq M$. Moreover, we know (see [6], Corollary 2.1) for $\gamma = \beta/2$ and $\epsilon_0 = \frac{1}{2}\min(1, m-1)$, that there exist two positive constants $c_1 = c_1(n, s, p_-, p_+)$ and

$$R_0 = R_0 \left(n, s, p_-, p_+, \mathrm{diam}(\Omega), \int_{B_R(x_0)} |\nabla v|^{p(x)} dx \right)$$

such that we have for each $R \in (0, \min(R_0, \mathrm{diam}(\Omega)/16))$ and $r \in (0, R/8)$,

$$\begin{aligned} \fint_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq c_1 \Bigg(& \fint_{B_R(x_0)} |\nabla v|^{p(x)} dx \\ & + KR^\beta \left(1 + \int_{B_R(x_0)} |\nabla v|^{p(x)} dx \right)^{\epsilon_0(1+\delta)} \left(1 + \fint_{B_R(x_0)} |\nabla v|^{p(x)} dx \right)^{1+\delta} \Bigg), \end{aligned}$$

where $K = \|\nabla p\|_{L^s(\Omega)} \left(1 + (\mathrm{diam}(\Omega)/2)^\beta \|\nabla p\|_{L^s(\Omega)}\right)$ and $\fint_E f dx = \frac{1}{|E|} \int_E f dx$ denotes the mean value of the function f on the measurable set E.

Using (2.3), we obtain for

$$c_2 = c_1 \left(\frac{p_+}{p_-} \right)^{(1+\epsilon_0)(1+\delta)} \max \left(1, 2^\delta K \left(1 + \frac{p_+}{p_-} \int_\Omega |\nabla u|^{p(x)} dx \right)^{\epsilon_0(1+\delta)} \right)$$

that

$$\begin{aligned} \fint_{B_r(x_0)} |\nabla v|^{p(x)} dx & \leq c_1 \Bigg(\frac{p_+}{p_-} \fint_{B_R(x_0)} |\nabla u|^{p(x)} dx \\ & \quad + KR^\beta \left(1 + \frac{p_+}{p_-} \int_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{\epsilon_0(1+\delta)} \left(1 + \frac{p_+}{p_-} \fint_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta} \Bigg) \\ & \leq c_1 \Bigg(\frac{p_+}{p_-} \fint_{B_R(x_0)} |\nabla u|^{p(x)} dx \\ & \quad + KR^\beta \left(1 + \frac{p_+}{p_-} \int_\Omega |\nabla u|^{p(x)} dx \right)^{\epsilon_0(1+\delta)} \left(1 + \frac{p_+}{p_-} \fint_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta} \Bigg) \\ & \leq c_2 \left(\fint_{B_R(x_0)} |\nabla u|^{p(x)} dx + R^\beta + R^\beta \left(\fint_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta} \right), \end{aligned}$$

which leads to

$$\int_{B_r(x_0)} |\nabla v|^{p(x)}dx \le c_2\Bigg(\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |\nabla u|^{p(x)}dx + \omega_n r^n R^\beta + \omega_n^{-\delta} r^n R^{\beta-n(1+\delta)} \left(\int_{B_R(x_0)} |\nabla u|^{p(x)}dx\right)^{1+\delta}\Bigg)$$

or

(2.4)

$$\begin{aligned}\int_{B_r(x_0)} |\nabla v|^{p(x)}dx &\le c_2\Bigg(\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |\nabla u|^{p(x)}dx + \omega_n R^{n+\beta} \\ &\qquad + \omega_n^{-\delta} R^{\beta-n\delta} \left(\int_{B_R(x_0)} |\nabla u|^{p(x)}dx\right)^{1+\delta}\Bigg) \\ &\le c_2\Bigg(\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |\nabla u|^{p(x)}dx + \omega_n R^{n+\beta} \\ &\qquad + \omega_n^{-\delta} R^{\frac{\beta}{2}} \left(\int_\Omega |\nabla u|^{p(x)}dx\right)^{\delta} \left(\int_{B_R(x_0)} |\nabla u|^{p(x)}dx\right)\Bigg).\end{aligned}$$

We point out (see [1, 6]) that

$$R_0 \approx \left(\frac{\epsilon_0}{c(n,p_-,p_+)^m c(\beta)}\right)^{2/\beta} \left(\int_{B_R(x_0)} |\nabla v|^{p(x)}dx + 1\right)^{\frac{-2m\epsilon_0}{\beta}}.$$

Using (2.3) again, we get for some positive constant $c_0 = c_0(n,s,p_-,p_+)$ independent of R

$$\begin{aligned} R_0 &\ge c_0 \left(\int_{B_R(x_0)} |\nabla v|^{p(x)}dx + 1\right)^{\frac{-2m\epsilon_0}{\beta}} \\ &\ge c_0 \left(\frac{p_+}{p_-}\int_{B_R(x_0)} |\nabla u|^{p(x)}dx + 1\right)^{\frac{-2m\epsilon_0}{\beta}} \\ &\ge c_0 \left(\frac{p_+}{p_-}\right)^{\frac{-2m\epsilon_0}{\beta}} \left(\int_\Omega |\nabla u|^{p(x)}dx + 1\right)^{\frac{-m}{\beta}} \\ &= c(n,s,p_-,p_+) \left(\int_\Omega |\nabla u|^{p(x)}dx + 1\right)^{\frac{-m}{\beta}}. \end{aligned}$$

Thus, we can take

$$R_1 = \min\Bigg(\operatorname{diam}(\Omega)/16, c(n,s,p_-,p_+)\left(\int_\Omega |\nabla u|^{p(x)}dx + 1\right)^{\frac{-m}{\beta}}\Bigg),$$

and (2.2) follows from (2.4) if we choose

$$C_1 = c_2 \max\left(1, \omega_n, \omega_n^{-\delta}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^\delta\right)$$
$$= C_1(n, s, p_-, p_+, \operatorname{diam}(\Omega), \|\nabla p\|_{L^s(\Omega)}, \|\nabla u\|_{L^{p(x)}(\Omega)}).$$

□

Proof of Lemma 2.1. Let R_1 be as in Lemma 2.1, v as in Lemma 2.2, and $R \in (0, R_1)$. First, we have for $r \in (0, R)$

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u|^{p(x)} &= \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u-v)dx \\ &\quad + \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx \\ &= \int_{B_r(x_0)}\Big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\Big)\cdot\nabla(u-v)dx \\ &\quad + \int_{B_r(x_0)} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla(u-v)dx \\ &\quad + \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v dx = I_1 + I_2 + I_3.\end{aligned} \tag{2.5}$$

Next, using the monotonicity of $\xi \to |\xi|^{p(x)-2}\xi$, the fact that $\Delta_{p(x)}v = 0$ in $B_R(x_0)$ and $u = v$ on $\partial B_R(x_0)$, we obtain

$$\begin{aligned} I_1 &\le \int_{B_R(x_0)} \Big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\Big)\cdot\nabla(u-v)dx \\ &= \int_{B_R(x_0)} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u-v)dx \\ &\quad - \int_{B_R(x_0)} |\nabla v|^{p(x)-2}\nabla v\cdot\nabla(u-v)dx \\ &= -\int_{B_R(x_0)} g(u-v)dx + \int_{B_R(x_0)} F\cdot\nabla(u-v)dx.\end{aligned} \tag{2.6}$$

Applying Young's inequality, we derive for some positive constant c depending only on p_- and p_+

$$\begin{aligned} I_2 &= \int_{B_r(x_0)} |\nabla v|^{p(x)-2}\nabla v\cdot\nabla u dx - \int_{B_r(x_0)} |\nabla v|^{p(x)} \\ &\le \int_{B_r(x_0)} |\nabla v|^{p(x)-1}\cdot|\nabla u|dx \\ &\le \frac{1}{4}\int_{B_r(x_0)} |\nabla u|^{p(x)}dx + c\int_{B_r(x_0)} |\nabla v|^{p(x)}dx.\end{aligned} \tag{2.7}$$

Similarly, we obtain

$$\begin{aligned} I_3 &\le \int_{B_r(x_0)} |\nabla u|^{p(x)-1} \cdot |\nabla v| dx \\ &\le \frac{1}{4}\int_{B_r(x_0)} |\nabla u|^{p(x)} dx + c\int_{B_r(x_0)} |\nabla v|^{p(x)} dx. \end{aligned} \tag{2.8}$$

Using (2.5)-(2.8), we get

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^{p(x)} dx \le 4c\int_{B_r(x_0)} |\nabla v|^{p(x)} dx - 2\int_{B_R(x_0)} g(u-v)dx \\ + 2\int_{B_R(x_0)} F\cdot\nabla(u-v)dx. \end{aligned} \tag{2.9}$$

Combining (2.2), and (2.9), we get for a positive constant

$$C_1 = C_1(n, s, p_-, p_+, \mathrm{diam}(\Omega), \|\nabla p\|_{L^s(\Omega)}, \|\nabla u\|_{L^{p(x)}(\Omega)})$$

and

$$\begin{aligned} &\int_{B_r(x_0)} |\nabla u|^{p(x)} dx \\ &\le C_1\left(\left(\left(\frac{r}{R}\right)^n + R^{\frac{\beta}{2}}\right)\int_{B_R(x_0)} |\nabla u|^{p(x)} dx + R^{n+\beta}\right) \\ &\quad - 2\int_{B_R(x_0)} g(u-v)dx + 2\int_{B_R(x_0)} F\cdot\nabla(u-v)dx. \end{aligned} \tag{2.10}$$

Applying Hölder's inequality and using (1.4) and the fact that $|u|, |v| \le M$ in $B_R(x_0)$, we obtain for

$$\alpha_1 = 1 - \frac{n}{t_1 p_m} \text{ and } C_2 = 4M\omega_n^{1-\frac{1}{t_1}} \cdot \|g\|_{L^{t_1}(\Omega)}$$

that

$$\begin{aligned} \left|-2\int_{B_R(x_0)} g(u-v)dx\right| &\le 4M\int_{B_R(x_0)} |g|dx \\ &\le 4M\|g\|_{L^{t_1}(B_R(x_0))} \cdot |B_R(x_0)|^{1-\frac{1}{t_1}} \\ &= 4M\|g\|_{L^{t_1}(B_R(x_0))} \cdot \omega_n^{1-\frac{1}{t_1}} \cdot R^{n-\frac{n}{t_1}} \\ &= C_2 R^{n-p_m+\alpha_1 p_m}. \end{aligned} \tag{2.11}$$

Due to (1.4) and the continuity of $p(x)$, we have $t_1 > \dfrac{n}{p_m}$. Hence $\alpha_1 \in (0,1)$.

We observe from (1.3) and (1.5) that we have

$$t_2 \ge \frac{t_2 p(x)}{n} > \frac{p(x)}{p(x)-1} \ge \frac{p_m}{p_m - 1}.$$

Applying Young's and Hölder's inequalities and taking into account (1.5) and using the convexity of $|\xi|^{p_m}$, we obtain for $\epsilon \in (0,1)$ and a constant C_3 depending only on p_- and p_+

$$
\begin{aligned}
&\left| 2\int_{B_R(x_0)} F\cdot\nabla(u-v)dx \right| \\
&\quad \le C_3 \int_{B_R(x_0)} |F|^{\frac{p_m}{p_m-1}}dx + \epsilon\int_{B_R(x_0)} |\nabla(u-v)|^{p_m}dx \\
&\quad \le C_3 \|F\|^{\frac{p_m}{p_m-1}}_{L^{t_2}(\Omega)} \cdot |B_R(x_0)|^{\frac{t_2(p_m-1)-p_m}{t_2(p_m-1)}} \\
&\qquad + \epsilon \int_{B_R(x_0)} |\nabla(u-v)|^{p_m}dx \\
&\quad = C_3\omega_n^{\frac{t_2(p_m-1)-p_m}{t_2(p_m-1)}} \|F\|^{\frac{p_m}{p_m-1}}_{L^{t_2}(\Omega)} R^{n-\frac{np_m}{t_2(p_m-1)}} \\
&\qquad + \epsilon \int_{B_R(x_0)} |\nabla(u-v)|^{p_m}dx \\
&\quad \le C_3' R^{n-p_m+\alpha_2 p_m} + \epsilon 2^{p_m-1}\int_{B_R(x_0)} |\nabla u|^{p_m}dx \\
&\qquad + \epsilon 2^{p_m-1}\int_{B_R(x_0)} |\nabla v|^{p_m}dx,
\end{aligned} \tag{2.12}
$$

where

$$
\alpha_2 = 1 - \frac{n}{t_2(p_m-1)} \qquad \text{and} \qquad C_3' = C_3.\omega_n^{\frac{t_2(p_m-1)-p_m}{t_2(p_m-1)}} \cdot \|F\|^{\frac{p_m}{p_m-1}}_{L^{t_2}(\Omega)}.
$$

Due to (1.5), and the continuity of $p(x)$, we have $t_2 > \dfrac{n}{p_m-1}$, and therefore $\alpha_2 \in (0,1)$.

Observe that we have for $w \in W^{1,p(x)}(B_R(x_0))$

$$
\begin{aligned}
\int_{B_R(x_0)} |\nabla w|^{p_m}dx &= \int_{B_R(x_0)\cap[|\nabla w|\le 1]} |\nabla w|^{p_m}dx \\
&\quad + \int_{B_R(x_0)\cap[|\nabla w|>1]} |\nabla w|^{p_m}dx \\
&\le \int_{B_R(x_0)} |\nabla w|^{p(x)}dx + \omega_n R^n.
\end{aligned} \tag{2.13}
$$

Using (2.3) and (2.13), we get from (2.12)

$$\begin{aligned}
\left|2\int_{B_R(x_0)} F\cdot\nabla(u-v)dx\right| &\le C_3' R^{n-p_m+\alpha_2 p_m}\\
&+ \epsilon 2^{p_m-1}(1+\frac{p_+}{p_-})\int_{B_R(x_0)}|\nabla u|^{p(x)}dx + \epsilon 2^{p_m}\omega_n R^n\\
&= (C_3' + \epsilon 2^{p_m}\omega_n R^{(1-\alpha_2)p_m})R^{n-p_m+\alpha_2 p_m}\\
&+ \epsilon 2^{p_m-1}(1+\frac{p_+}{p_-})\int_{B_R(x_0)}|\nabla u|^{p(x)}dx\\
&\le C_4\left(R^{n-p_m+\alpha_2 p_m} + \epsilon\int_{B_R(x_0)}|\nabla u|^{p(x)}dx\right),
\end{aligned} \tag{2.14}$$

where $C_4 = \max\left(C_3' + 2^{p_m}\omega_n(\mathrm{diam}(\Omega)/2)^{(1-\alpha_2)p_m}, 2^{p_m-1}(1+\frac{p_+}{p_-})\right)$.

Combing (2.10), (2.11) and (2.14), we get

$$\begin{aligned}
\int_{B_r(x_0)}|\nabla u|^{p(x)}dx &\le C_1\left(\left(\left(\frac{r}{R}\right)^n + R^{\frac{\beta}{2}}\right)\int_{B_R(x_0)}|\nabla u|^{p(x)}dx\right)\\
&+ C_1 R^{n+\beta} + C_2 R^{n-p_m+\alpha_1 p_m} + C_4 R^{n-p_m+\alpha_2 p_m}\\
&+ C_4\epsilon\int_{B_R(x_0)}|\nabla u|^{p(x)}dx.
\end{aligned} \tag{2.15}$$

Setting $\phi(r) = \int_{B_r(x_0)}|\nabla u|^{p(x)}dx$ and $C_4' = \max(C_1, C_2, C_4)$, we obtain from (2.15) that

$$\phi(r) \le C_4'\left(\left(\left(\frac{r}{R}\right)^n + R^{\frac{\beta}{2}} + \epsilon\right)\phi(R) + R^{n+\beta} + R^{n-p_m+\alpha_1 p_m} + R^{n-p_m+\alpha_2 p_m}\right). \tag{2.16}$$

Now, observe that we have for $\alpha = \min(\alpha_1, \alpha_2)$ and $a = \mathrm{diam}(\Omega)/2$

$$\begin{aligned}
&R^{n+\beta} + R^{n-p_m+\alpha_1 p_m} + R^{n-p_m+\alpha_2 p_m}\\
&= (R^{\beta+(1-\alpha)p_m} + R^{(\alpha_1-\alpha)p_m} + R^{(\alpha_2-\alpha)p_m})R^{n-p_m+\alpha p_m}\\
&\le \left(a^{\beta+(1-\alpha)p_m} + a^{(\alpha_1-\alpha)p_m} + a^{(\alpha_2-\alpha)p_m}\right)R^{n-p_m+\alpha p_m}\\
&= C_4'' R^{n-p_m+\alpha p_m}.
\end{aligned} \tag{2.17}$$

Using (2.16) and (2.17), we get for $C_5 = C_4'\max(1, C_4'')$

$$\phi(r) \le C_5\left(\left(\left(\frac{r}{R}\right)^n + R^{\frac{\beta}{2}} + \epsilon\right)\phi(R) + R^{n-p_m+\alpha p_m}\right) \quad \forall r\in(0,R). \tag{2.18}$$

If we assume that $R^{\frac{\beta}{2}} < \epsilon$, then given that ϕ is a nonnegative and nondecreasing function on $(0, R)$, we infer from (2.18) and Lemma 5.12

p. 248 of [21] applied with $\delta = 2\epsilon$, that we have for two positive constants $C_6 = C_6(C_5, n, \alpha, p_m)$ and $\delta_0 = \delta_0(C_5, n, \alpha, p_m)$

$$\phi(r) \le C_6 \left(\frac{r}{R}\right)^{n-p_m+\alpha p_m} \left(\phi(R) + R^{n-p_m+\alpha p_m}\right) \qquad \forall r \in (0, R) \tag{2.19}$$

provided that $\delta < \delta_0$ and $R < (\delta/2)^{\frac{2}{\beta}} = R_2$.

Using (2.19) and (2.13), for $R = r$ and $w = u$, we get for

$$C_7 = \frac{C_6(\phi(R) + R^{n-p_m+\alpha p_m})}{R^{n-p_m+\alpha p_m}}$$

that

$$\begin{aligned}\int_{B_r(x_0)} |\nabla u|^{p_m} dx &\le \int_{B_r(x_0)} |\nabla u|^{p(x)} dx + \omega_n r^n \le C_7 r^{n-p_m+\alpha p_m} + \omega_n r^n \\ &= (C_7 + \omega_n r^{(1-\alpha)p_m}) r^{n-p_m+\alpha p_m} \le C r^{n-p_m+\alpha p_m},\end{aligned}$$

where $C = C_7 + \omega_n(\mathrm{diam}(\Omega)/2)^{(1-\alpha)p_m}$. This completes the proof of the lemma. □

PROOF OF THEOREM 1.1. Obviously we can choose δ small enough so that $R_2 < R_1$. Then, by using (2.1) and Hölder's inequality, we obtain for all $R \in (0, R_2)$ and all $r \in (0, R)$

$$\begin{aligned}\int_{B_r} |\nabla u| dx &\le |B_r|^{1-\frac{1}{p_m}} \left(\int_{B_r} |\nabla u|^{p_m} dx\right)^{\frac{1}{p_m}} \\ &\le \omega_n^{1-\frac{1}{p_m}} \cdot r^{n-\frac{n}{p_m}} \cdot (C r^{n-p_m+\alpha p_m})^{\frac{1}{p_m}} \\ &= C^{\frac{1}{p_m}} \omega_n^{1-\frac{1}{p_m}} \cdot r^{n-\frac{n}{p_m}} \cdot r^{\frac{n-p_m+\alpha p_m}{p_m}} \\ &\le C' r^{n-1+\alpha},\end{aligned}$$

where $C' = C^{\frac{1}{p_m}} \omega_n^{1-\frac{1}{p_m}}$. We conclude ([21, Theorem 1.53 (Morrey) p. 30]) that $u \in C^{0,\alpha}_{loc}(\Omega)$, which completes the proof of the theorem. □

REMARK 2.3. If $g, F \in L^\infty(\Omega)$, then (1.4)-(1.5) are satisfied for any $t_1 > \frac{n}{p(x)}$ and $t_2 > \frac{n}{p(x)-1}$. Therefore, we obtain $u \in C^{0,\alpha}_{loc}(B_R(x_0))$ for any $B_R(x_0) \subset\subset \Omega$ with $\alpha = 1 - \max\left(\frac{n}{t_1 p_m}, \frac{n}{t_2(p_m - 1)}\right)$, $p_m = \min_{x \in \overline{B}_R(x_0)} p(x)$ and R small enough. Given that t_1 and t_2 can be chosen arbitrarily large, we obtain $u \in C^{0,\alpha}_{loc}(\Omega)$ for any $0 < \alpha < 1$.

References

[1] E. Acerbi and G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117–148.

[2] S. Antontsev and S. Shmarev, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, in: Handbook of differential equations: Stationary partial differential equations, Vol 3, Elsevier-North Holland, 2006, 1–100.

[3] A. Coscia and G. Mingione, *Hölder continuity of the gradient of $p(x)$-harmonic mappings*, C. R. Acad. Sci. Paris Sér. I Math. **328**, (1999), 363–368.

[4] J. Carrillo and A. Lyaghfouri, *The dam problem for nonlinear Darcy's laws and Dirichlet boundary conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 453–505.

[5] S. Challal and A. Lyaghfouri, *A filtration problem through a heterogeneous porous medium*, Interfaces Free Bound. **6** (2004), 55–79.

[6] S. Challal and A. Lyaghfouri, *Gradient estimates for $p(x)$-harmonic functions*, Manuscripta Math. **131** (2010), 403–414.

[7] S. Challal and A. Lyaghfouri, *Hölder continuity of solutions to the A-Laplace equation involving measures*, Commun. Pure Appl. Anal. **8**, (2009), 1577–1583.

[8] S. Challal and A. Lyaghfouri, *Lipschitz continuity of solutions of a free boundary problem involving the p-Laplacian*, J. Math. Anal. Appl. **355** (2009), 700–707.

[9] S. Challal and A. Lyaghfouri, *On the dam problem with two fluids governed by a nonlinear Darcy's law*, Adv. Differential Equations **11** (2006), 841–892.

[10] S. Challal and A. Lyaghfouri, *Porosity of free boundaries in A-obstacle problems*, Nonlinear Anal. **70** (2009), 2772–2778.

[11] S. Challal and A. Lyaghfouri, *On the porosity of the free boundary in the $p(x)$-obstacle problem*, Port. Math. **68** (2011), 109–123.

[12] S. Challal and A. Lyaghfouri, *Regularity results for a quasilinear free boundary problem*, Math. Model. Anal. **25**, (2020), 338–350.

[13] S. Challal, A. Lyaghfouri and J. F. Rodrigues, *On the A-obstacle problem and the Hausdorff measure of its free boundary*, Ann. Mat. Pura Appl. (4) **191** (2012), 113–165.

[14] S. Challal, A. Lyaghfouri, J. F. Rodrigues and R. Teymurazyan, *On the regularity of the free boundary for a class of quasilinear obstacle problems*, Interfaces Free Bound. **16** (2014), 359–394.

[15] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424–446.

[16] K. Ho, Y.-H. Kim, P. Winkert and C. Zhang, *The boundedness and Hölder continuity of solutions to elliptic equations involving variable exponents and critical growth*, J. Differential Equations **313** (2022), 503–532.

[17] T. Kilpeläinen, *Hölder continuity of solutions to quasilinear elliptic equations involving measures*, Potential Anal. **3** (1994), 265–272.

[18] O. Kováčik and J. Rákosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41(116)** (1991), 592–618.

[19] A. Lyaghfouri, *A continuity result for a quasilinear elliptic free boundary problem*, Appl. Math. **65** (2020), 67–87.

[20] A. Lyaghfouri, *Hölder continuity of $p(x)$-superharmonic functions*, Nonlinear Anal. **73** (2010), 2433–2444.

[21] J. Malý and W. P. Ziemer, Fine regularity of solutions of elliptic partial differential equations, American Mathematical Society, Providence, 1997.

[22] J. M. Rakotoson and W. P. Ziemer, *Local behavior of solutions of quasilinear elliptic equations with general structure*, Trans. Amer. Math. Soc. **319** (1990), 747–764.

[23] J. F. Rodrigues and R. Teymurazyan, *On the two obstacles problem in Orlicz-Sobolev spaces and applications*, Complex Var. Elliptic Equ. **56** (2011), 769–787.

[24] J. F. Rodrigues, M. Sanchon and J. M. Urbano, *The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data*, Monatsh. Math. **154**, (2008), 303–322.

[25] P. Winkert and R. Zacher, *A priori bounds for weak solutions to elliptic equations with nonstandard growth*, Discrete Contin. Dyn. Syst. Ser. S **5**, (2012), 865–878.

A. Lyaghfouri
Department of Mathematical Sciences
United Arab Emirates University
Al Ain, Abu Dhabi
UAE
E-mail: a.lyaghfouri@uaeu.ac.ae

Received: 15.3.2021.
Revised: 28.3.2022.