HÖLDER CONTINUITY FOR THE SOLUTIONS OF THE
\(p(x)\)-LAPLACE EQUATION WITH GENERAL RIGHT-HAND SIDE

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Abstract. We show that bounded solutions of the quasilinear elliptic equation
\(\Delta_{p(x)} u = g + \text{div}(F)\) are locally Hölder continuous provided that
the functions \(g\) and \(F\) are in suitable Lebesgue spaces.

1. Introduction

We consider the following equation
\[(1.1) \quad \Delta_{p(x)} u = g + \text{div}(F) \quad \text{in} \quad W^{-1,q(x)}(\Omega),\]
where \(\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2}\nabla u)\) is the \(p(x)\)-Laplacian, \(\Omega\) is an open bounded domain of \(\mathbb{R}^n\), \(n \geq 2\), \(x = (x_1, \ldots, x_n)\), \(q(x) = \frac{p(x)}{p(x)+1}\), and \(p : \Omega \to \mathbb{R}\) is a measurable function which satisfies for some positive constants \(p_+ > p_- > 1\) and \(s > n\)
\[(1.2) \quad p_- \leq p(x) \leq p_+ \quad \text{a.e.} \quad x \in \Omega, \quad \nabla p \in (L^s(\Omega))^n.\]
As a consequence of (1.2), we have \(p \in W^{1,s}(\Omega)\). Moreover, due to Sobolev embedding \(W^{1,s}(\Omega) \subset C^{0,\beta}(\Omega), \quad (\beta = 1 - \frac{n}{s})\), \(p\) is Hölder continuous in \(\Omega\).

We call a solution of equation (1.1) any function \(u \in W^{1,p(x)}(\Omega)\) that fulfills
\[
\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \zeta dx = - \int_\Omega g(x) \zeta dx + \int_\Omega F(x) \cdot \nabla \zeta dx \quad \forall \zeta \in W^{1,p(x)}_0(\Omega).
\]

2020 Mathematics Subject Classification. 35B65, 35J92.
Key words and phrases. \(p(x)\)-Laplacian, Hölder continuity.
The Lebesgue and Sobolev spaces with variable exponents are defined (see for example [2, 15] and [18]) by:

\[ L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}, \]

\[ W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \nabla u \in \left( L^{p(x)}(\Omega) \right)^n \right\}, \]

\[ W^{1,p(x)}_0(\Omega) = C_0^\infty(\Omega). \]

These spaces are separable, complete and reflexive, when equipped with the following norms

\[ \| u \|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u|^{p(x)}}{\lambda} \, dx \leq 1 \right\}, \]

\[ \| u \|_{1,p(x)} = \| u \|_{p(x)} + \| \nabla u \|_{p(x)}, \quad \| \nabla u \|_{p(x)} = \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{p(x)}. \]

Our aim is to establish Hölder continuity for bounded solutions of (1.1). We observe that if \( p(x) > n \) in an open set \( U \subset \subset \Omega \), then by Sobolev embedding \( W^{1,p(x)}(U) \subset W^{1,p_m(U)}(U) \subset C^{0,\alpha}(U) \), where \( p_m(U) = \min_{x \in U} p(x) \) and \( \alpha = 1 - \frac{n}{p_m(U)} \). Therefore any solution of (1.1) is Hölder continuous in \( U \).

In this paper, we assume that

\[ p(x) \leq n \quad \forall x \in \Omega. \]

\[ g \text{ is a real valued function that satisfies for a positive number } t_1 \]

\[ t_1 > \frac{n}{p(x)} \quad \forall x \in \Omega, \]

\[ g \in L^{t_1}(\Omega). \]

\[ F = (F_1, \ldots, F_n) \text{ is a vector function that satisfies for a positive number } t_2 \]

\[ t_2 > \frac{n}{p(x) - 1} \quad \forall x \in \Omega, \]

\[ F \in L^{t_2}(\Omega). \]

Among problems that fit in the equation (1.1) setting, is the dam problem \( (g = 0, F = \chi e, \text{ with } e = (0, \ldots, 0, 1), \chi \in L^\infty(\Omega), \text{ and } p(x) \text{ a constant, } [4, 5, 9]) \). It is know that the solution in this case is \( C^{0,\alpha}_{\text{loc}}(\Omega) \) for any \( \alpha \in (0,1) \) (see [7]). In fact, due to the particularity of the problem (i.e., because \( u \geq 0 \) and \( \chi = 1 \text{ a.e. in } \{u > 0\} \)), we actually have \( u \in C^{0,1}_{\text{loc}}(\Omega) \) (see [12]). Another problem is the obstacle problem, [10, 11, 13, 14, 23]). Indeed, because the solution of the obstacle problem satisfies the Levy-Stampacchia inequality, i.e., \( f\chi(\{u > 0\}) \leq \Delta_{p(x)}u \leq f \text{ a.e. in } \Omega \) (see for example [24]), where
f ∈ L^{q(x)}(Ω), one can write ∆_{p(x)} u = g a.e. in Ω, where g = θf and 0 ≤ θ ≤ 1 a.e. in Ω. For a more general framework, we refer to [8, 12, 19].

Equations involving variable exponents are called equations with non-standard growth. For an overview of such equations, we refer to [16], where the authors establish global boundedness and Hölder continuity for a class of elliptic problems.

The main result of this paper is the following theorem.

**Theorem 1.1.** Assume that (1.2)-(1.5) hold. Then any bounded solution of (1.1) is locally Hölder continuous in Ω.

It is expected that solutions of (1.1) are locally and even globally bounded if a Dirichlet or Neumann boundary condition is added. This can be established for example by adapting the so-called De Giorgi-Nash-Moser theory. However, it is not the purpose of this paper to discuss this question. The interested reader is referred for example to the papers [16] and [25], where global boundedness for a class of elliptic problems are established for both homogeneous Dirichlet and nonhomogeneous Neumann boundary conditions.

In the context of constant exponent, the regularity result in Theorem 1.1 was established in [22] for a more general quasi-linear elliptic operator that includes the p−Laplacian under the assumptions: g ∈ L^{q(1)}(Ω) and F ∈ L^t(Ω) with ε > 0 and t > n/p, which coincide with assumptions (1.5). Moreover, when the right-hand side of (1.1) is a nonnegative Radon measure with a suitable growth condition, local Hölder continuity was established in [17] for the p−Laplacian and in a more general framework in [22]. This result was later extended in [7] and [20] respectively for the A−Laplacian and p(x)−Laplacian.

Throughout this paper, we will denote by B_r(x) (resp. B_r(x)) the open (resp. closed) ball of center x and radius r in R^n, with ω_n = |B_1| standing for the measure of the unit open ball B_1.

2. Proof of Theorem 1.1

In this section, we denote by u a bounded solution of (1.1) with M = ∥u∥_∞, and will show that it is locally Hölder continuous in Ω. The proof is based on Lemma 2.1.

**Lemma 2.1.** Let δ = \frac{2-n}{2m}, q = \frac{n+δ}{2}, m = \frac{sq(1+δ)}{s-q}, and

R_1 = \min \left( \frac{\text{diam}(Ω)}{16}, c(n, s, p_-, p_+) \left( \int_Ω |∇u|^{p(x)} dx + 1 \right)^{-\frac{m}{s}} \right).

Assume that (1.2)-(1.5) hold. Then there exists a positive constant

C = C(n, s, p_-, p_+, M, \text{diam}(Ω), ∥∇u∥_{L^{p(x)}(Ω)}, ∥∇p∥_{L^t(Ω)}, ∥g∥_{L^t(Ω)}, ∥F∥_{L^t(Ω)}),
such that we have for $R \in (0, R_1)$ with $B_{2R}(x_0) \subset \Omega$

\begin{equation}
\int_{B_r(x_0)} |\nabla u|^{p_m} \, dx \leq C r^{n - p_m + \alpha p_m} \quad \forall r \in (0, R),
\end{equation}

where $p_m = \min_{x \in B_R(x_0)} p(x)$ and $\alpha = 1 - \max \left( \frac{n}{t_1 p_m}, \frac{n}{t_2(p_m - 1)} \right)$.

The proof of Lemma 2.1 is based on Lemma 2.2.

**Lemma 2.2.** Assume that (1.2)-(1.5) hold and let $R_1$ be the positive number in Lemma 2.1. Then there exists a positive constant

\[ C_1 = C_1(n, s, p_-, p_+, \text{diam}(\Omega), \| \nabla u \|_{L^p(\Omega)}, \| \nabla p \|_{L^p(\Omega)}), \]

such that we have for $R \in (0, R_1)$, $B_{2R}(x_0) \subset \Omega$, $v \in W^{1,p(x)}(B_R(x_0))$ with $\Delta p(x) v = 0$ in $B_R(x_0)$ and $v = u$ on $\partial B_R(x_0)$

\begin{equation}
\int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx \leq C_1 \left( \left( \frac{r}{R} \right)^n + R^{\frac{n}{2}} \right) \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + R^{n + \beta}
\end{equation}

for all $r \in (0, R)$.

**Proof.** First, observe that we have $v \in C^{1,\sigma}_{loc}(B_R(x_0))$ (see for example [3]). Next, since $\int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx$ minimizes the energy $\int_{B_r(x_0)} |\nabla \xi|^{p(x)} \, dx$ over all functions $\xi \in W^{1,p(x)}(B_R(x_0))$ such that $\xi = u$ on $\partial B_R(x_0)$, we get

\begin{equation}
\int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx = \int_{B_r(x_0)} p(x) \frac{|\nabla v|^{p(x)}}{p(x)} \, dx
\end{equation}

\begin{align*}
&\leq \frac{p_+}{p_-} \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx \\
&\leq \frac{p_+}{p_-} \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx.
\end{align*}

Using (2.3), we get for $r \in \left[ \frac{R}{8}, R \right]$

\begin{align*}
\int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx &= \left( \frac{R}{r} \right)^n \left( \frac{r}{R} \right)^n \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx \\
&\leq 8^n \left( \frac{r}{R} \right)^n \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx \\
&\leq 8^n \left( \frac{r}{R} \right)^n \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx \\
&\leq \frac{p_+}{p_-} 8^n \left( \frac{r}{R} \right)^n \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx.
\end{align*}
Therefore it is enough to prove (2.2) for $r \in \left(0, \frac{R}{8}\right)$. Since $\Delta_{p(x)}v = 0$ in $B_R(x_0)$, $v = u$ on $\partial B_R(x_0)$, we obtain by the maximum principle $\|v\|_{L^\infty(B_R(x_0))} \leq M$. Moreover, we know (see [6], Corollary 2.1) for $\gamma = \frac{\beta}{2}$ and $\epsilon_0 = \frac{1}{2} \min(1, m - 1)$, that there exist two positive constants $c_1 = c_1(n, s, p_-, p_+)$ and

$$R_0 = R_0\left(n, s, p_-, p_+, \text{diam}(\Omega), \int_{B_R(x_0)} |\nabla v|^{p(x)} dx\right)$$

such that we have for each $R \in \left(0, \min(R_0, \text{diam}(\Omega)/16\right))$ and $r \in (0, R/8)$,

$$\int_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq c_1 \left(\int_{B_R(x_0)} |\nabla v|^{p(x)} dx\right)
$$

$$+ KR^\beta \left(1 + \int_{B_R(x_0)} |\nabla v|^{p(x)} dx\right)^{\epsilon_0(1+\delta)} \left(1 + \int_{B_R(x_0)} |\nabla v|^{p(x)} dx\right)^{1+\delta},$$

where $K = \|\nabla p\|_{L^\infty(\Omega)} \left(1 + (\text{diam}(\Omega)/2)^\beta \|\nabla p\|_{L^\infty(\Omega)}\right)$ and $\int_E f dx = \frac{1}{|E|} \int_E f dx$ denotes the mean value of the function $f$ on the measurable set $E$.

Using (2.3), we obtain for

$$c_2 = c_1 \left(\frac{p_+}{p_-}\right)^{(1+\epsilon_0)(1+\delta)} \max \left(1, 2^\delta K \left(1 + \frac{p_+}{p_-} \int \Omega |\nabla u|^{p(x)} dx\right)^{\epsilon_0(1+\delta)}\right)$$

that

$$\int_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq c_1 \left(\frac{p_+}{p_-} \int_{B_R(x_0)} |\nabla u|^{p(x)} dx\right)
$$

$$+ KR^\beta \left(1 + \frac{p_+}{p_-} \int_{B_R(x_0)} |\nabla u|^{p(x)} dx\right)^{\epsilon_0(1+\delta)} \left(1 + \frac{p_+}{p_-} \int_{J_B(x_0)} |\nabla u|^{p(x)} dx\right)^{1+\delta)}$$

$$\leq c_1 \left(\frac{p_+}{p_-} \int_{J_{B_R}(x_0)} |\nabla u|^{p(x)} dx\right)
$$

$$+ KR^\beta \left(1 + \frac{p_+}{p_-} \int_{\Omega} |\nabla u|^{p(x)} dx\right)^{\epsilon_0(1+\delta)} \left(1 + \frac{p_+}{p_-} \int_{J_B(x_0)} |\nabla u|^{p(x)} dx\right)^{1+\delta)}$$

$$\leq c_2 \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} dx + R^\beta + R^\beta \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} dx\right)^{1+\delta}\right).$$
which leads to
\[
\int_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq c_2 \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |\nabla u|^{p(x)} dx + \omega_n r^\beta
\]
\[
+ \omega_n^{-\delta} R^{\beta-n(1+\delta)} \left( \int_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta}
\]
or
(2.4)
\[
\int_{B_r(x_0)} |\nabla v|^{p(x)} dx \leq c_2 \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |\nabla u|^{p(x)} dx + \omega_n R^{n+\beta}
\]
\[
+ \omega_n^{-\delta} R^{\beta-n\delta} \left( \int_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta}
\]
\[
\leq c_2 \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |\nabla u|^{p(x)} dx + \omega_n R^{n+\beta}
\]
\[
+ \omega_n^{-\delta} R^{\beta} \left( \int_{B_R(x_0)} |\nabla u|^{p(x)} dx \right)^{1+\delta} \left( \int_{B_R(x_0)} |\nabla u|^{p(x)} dx \right).
\]
We point out (see [1, 6]) that
\[
R_0 \approx \left( \frac{\epsilon_0}{c(n,p_-, p_+)^m c(\beta)} \right)^{2/\beta} \left( \int_{B_R(x_0)} |\nabla v|^{p(x)} dx + 1 \right)^{-2m\epsilon_0}. \tag{2.3}
\]
Using (2.3) again, we get for some positive constant \( c_0 = c_0(n, s, p_-, p_+) \) independent of \( R \)
\[
R_0 \geq c_0 \left( \int_{B_R(x_0)} |\nabla v|^{p(x)} dx + 1 \right)^{-2m\epsilon_0}
\]
\[
\geq c_0 \left( \frac{p_+}{p_-} \int_{B_R(x_0)} |\nabla u|^{p(x)} dx + 1 \right)^{-2m\epsilon_0}
\]
\[
\geq c_0 \left( \frac{p_+}{p_-} \right)^{-2m\epsilon_0} \left( \int_{\Omega} |\nabla u|^{p(x)} dx + 1 \right)^{-m}
\]
\[
= c(n, s, p_-, p_+) \left( \int_{\Omega} |\nabla u|^{p(x)} dx + 1 \right)^{-\frac{m}{p_-}}
\]
Thus, we can take
\[
R_1 = \min \left( \frac{\text{diam}(\Omega)}{16}, c(n, s, p_-, p_+) \left( \int_{\Omega} |\nabla u|^{p(x)} dx + 1 \right)^{-\frac{m}{p_-}} \right).
\]
and (2.2) follows from (2.4) if we choose

\[ C_1 = c_2 \max \left( 1, \omega_n, \omega_n^{-\delta} \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\delta} \right) \]

\[ = C_1(n, s, p_-, p_+, \text{diam}(\Omega), \|\nabla p\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^{p(x)}(\Omega)}). \]

\[ \square \]

**Proof of Lemma 2.1.** Let \( R_1 \) be as in Lemma 2.1, \( v \) as in Lemma 2.2, and \( R \in (0, R_1) \). First, we have for \( r \in (0, R) \)

\[
\int_{B_r(x_0)} |\nabla u|^{p(x)} = \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla (u - v) \, dx \\
\quad + \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot v \, dx \\
\quad = \int_{B_r(x_0)} \left( |\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v \right) \cdot \nabla (u - v) \, dx \\
\quad + \int_{B_r(x_0)} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla (u - v) \, dx \\
\quad + \int_{B_r(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot v \, dx = I_1 + I_2 + I_3. \]

Next, using the monotonicity of \( \xi \to |\xi|^{p(x)-2}\xi \), the fact that \( \Delta_{p(x)} v = 0 \) in \( B_R(x_0) \) and \( u = v \) on \( \partial B_R(x_0) \), we obtain

\[
I_1 \leq \int_{B_R(x_0)} \left( |\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v \right) \cdot \nabla (u - v) \, dx \\
\quad = \int_{B_R(x_0)} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla (u - v) \, dx \\
\quad - \int_{B_R(x_0)} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla (u - v) \, dx \\
\quad = - \int_{B_R(x_0)} g(u - v) \, dx + \int_{B_R(x_0)} F \cdot \nabla (u - v) \, dx. \]

Applying Young’s inequality, we derive for some positive constant \( c \) depending only on \( p_- \) and \( p_+ \)

\[
I_2 = \int_{B_r(x_0)} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla u \, dx \quad - \quad \int_{B_r(x_0)} |\nabla v|^{p(x)} \\
\quad \leq \int_{B_r(x_0)} |\nabla v|^{p(x)-1} \cdot |\nabla u| \, dx \\
\quad \leq \frac{1}{4} \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + c \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx. \]
Similarly, we obtain
\[ I_3 \leq \int_{B_r(x_0)} |\nabla u|^{p(x) - 1} \cdot |\nabla v| \, dx \]
(2.8)
\[ \leq \frac{1}{4} \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + c \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx. \]

Using (2.5)-(2.8), we get
\[ \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx \leq \frac{1}{4} \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + c \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx. \]
(2.9)
\[ \leq 1 \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx + c \int_{B_r(x_0)} |\nabla v|^{p(x)} \, dx. \]

Combining (2.2), and (2.9), we get for a positive constant
\[ C_1 = C_1(n, s, p_-, p_+, \text{diam}(\Omega), \|\nabla p\|_{L^s(\Omega)}, \|\nabla u\|_{L^{p_-(\Omega)}}) \]
and
\[ \int_{B_r(x_0)} |\nabla u|^{p(x)} \, dx \]
(2.10)
\[ \leq C_1 \left( \left( \frac{r}{R} \right)^n + R^{\frac{\beta}{s}} \right) \int_{B_R(x_0)} |\nabla u|^{p(x)} \, dx + R^{n+\beta} \right) \]
\[ - 2 \int_{B_R(x_0)} g(u - v) \, dx + 2 \int_{B_R(x_0)} F \cdot \nabla (u - v) \, dx. \]

Applying Hölder’s inequality and using (1.4) and the fact that $|u|, |v| \leq M$ in $B_R(x_0)$, we obtain for
\[ \alpha_1 = 1 - \frac{n}{t_1 p_m} \quad \text{and} \quad C_2 = 4M \omega_n^{1-\frac{1}{t_1}} \cdot \|g\|_{L^{t_1}(\Omega)} \]
that
\[ \left| -2 \int_{B_R(x_0)} g(u - v) \, dx \right| \leq 4M \int_{B_R(x_0)} |g| \, dx \]
(2.11)
\[ \leq 4M \|g\|_{L^{t_1}(B_R(x_0))} \cdot |B_R(x_0)|^{1-\frac{1}{t_1}} \]
\[ = 4M \|g\|_{L^{t_1}(B_R(x_0))} \cdot \omega_n^{1-\frac{1}{t_1}} \cdot R^{n-\frac{t_1}{n}} \]
\[ = C_2 R^{n-p_m+\alpha_1 p_m}. \]

Due to (1.4) and the continuity of $p(x)$, we have $t_1 > \frac{n}{p_m}$. Hence $\alpha_1 \in (0, 1)$.

We observe from (1.3) and (1.5) that we have
\[ t_2 \geq \frac{t_2 p(x)}{n} > \frac{p(x)}{p(x) - 1} \geq \frac{p_m}{p_m - 1}. \]
Applying Young’s and Hölder’s inequalities and taking into account (1.5) and using the convexity of $|\xi|^{p_m}$, we obtain for $\varepsilon \in (0,1)$ and a constant $C_3$ depending only on $p_-$ and $p_+$

\[
2 \int_{B_R(x_0)} F \cdot \nabla (u - v) \, dx \\
\leq C_3 \int_{B_R(x_0)} |F|^{\frac{p_m}{p_m - \varepsilon}} \, dx + \varepsilon \int_{B_R(x_0)} |\nabla (u - v)|^{p_m} \, dx \\
\leq C_3 \|F\|_{L^2(\Omega)}^{\frac{p_m}{p_m - 1}} \cdot |B_R(x_0)|^{\frac{p_m}{2(p_m - 1)}} + \varepsilon \int_{B_R(x_0)} |\nabla (u - v)|^{p_m} \, dx \\
= C_3 \omega_n \frac{t_2(p_m - 1) - p_m}{t_2(p_m - 1)} \|F\|_{L^2(\Omega)}^{\frac{p_m}{p_m - 1}} \cdot \frac{R^{n - \frac{n p_m}{2(p_m - 1)}}}{R^{n - \frac{p_m}{2(p_m - 1)}}} \\
+ \varepsilon \int_{B_R(x_0)} |\nabla (u - v)|^{p_m} \, dx \\
\leq C'_3 R^{n - p_m + \alpha_2 p_m} + \varepsilon 2^{p_m - 1} \int_{B_R(x_0)} |\nabla u|^{p_m} \, dx \\
+ \varepsilon 2^{p_m - 1} \int_{B_R(x_0)} |\nabla v|^{p_m} \, dx,
\]

where

\[
\alpha_2 = 1 - \frac{n}{t_2(p_m - 1)} \quad \text{and} \quad C'_3 = C_3 \omega_n \frac{t_2(p_m - 1) - p_m}{t_2(p_m - 1)} \cdot \|F\|_{L^2(\Omega)}^{\frac{p_m}{p_m - 1}}.
\]

Due to (1.5), and the continuity of $p(x)$, we have $t_2 > \frac{n}{p_m - 1}$, and therefore $\alpha_2 \in (0,1)$.

Observe that we have for $w \in W^{1,p(x)}(B_R(x_0))$

\[
\int_{B_R(x_0)} |\nabla w|^{p_m} \, dx = \int_{B_R(x_0) \cap \{|\nabla w| \leq 1\}} |\nabla w|^{p_m} \, dx \\
+ \int_{B_R(x_0) \cap \{|\nabla w| > 1\}} |\nabla w|^{p_m} \, dx \\
\leq \int_{B_R(x_0)} |\nabla w|^{p(x)} \, dx + \omega_n R^{n}.
\]
Using (2.3) and (2.13), we get from (2.12)
\[
\left| 2 \int_{B_R(x_0)} F \cdot \nabla (u - v) \, dx \right| \leq C_3' R^{n-p_m+\alpha_2p_m} \\
+ \epsilon 2^{p_m-1} \left( 1 + \frac{p_+}{p_-} \right) \int_{B_R(x_0)} |\nabla u|^p(x) \, dx + \epsilon 2^{p_m} \omega_n R^n \\
= (C_3' + \epsilon 2^{p_m} \omega_n R^{(1-\alpha_2)p_m}) R^{n-p_m+\alpha_2p_m} \\
+ \epsilon 2^{p_m-1} \left( 1 + \frac{p_+}{p_-} \right) \int_{B_R(x_0)} |\nabla u|^p(x) \, dx \\
\leq C_4 \left( R^{n-p_m+\alpha_2p_m} + \epsilon \int_{B_R(x_0)} |\nabla u|^p(x) \, dx \right),
\]
where \( C_4 = \max \left( C_3' + 2^{p_m} \omega_n (\text{diam}(\Omega)/2)^{(1-\alpha_2)p_m}, \epsilon 2^{p_m-1} \left( 1 + \frac{p_+}{p_-} \right) \right) \).

Combining (2.10), (2.11) and (2.14), we get
\[
\int_{B_r(x_0)} |\nabla u|^p(x) \, dx \leq C_1 \left( \left( \frac{r}{R} \right)^n + R^+ \right) \int_{B_R(x_0)} |\nabla u|^p(x) \, dx \\
+ C_1 R^{n+p+\beta} + C_2 R^{n-p_m+\alpha_1 p_m} + C_4 R^{n+p_m+\alpha_2 p_m} \\
+ C_4 \epsilon \int_{B_R(x_0)} |\nabla u|^p(x) \, dx.
\]
Setting \( \phi(r) = \int_{B_r(x_0)} |\nabla u|^p(x) \, dx \) and \( C_4' = \max(C_1, C_2, C_4) \), we obtain from (2.15) that
\[
\phi(r) \leq C_4' \left( \left( \frac{r}{R} \right)^n + R^+ + \epsilon \right) \phi(R) + R^{n+p+\beta} + R^{n-p_m+\alpha_1 p_m} + R^{n+p_m+\alpha_2 p_m}.
\]
Now, observe that we have for \( \alpha = \min(\alpha_1, \alpha_2) \) and \( a = \text{diam}(\Omega)/2 \)
\[
R^{n+p+\beta} + R^{n-p_m+\alpha_1 p_m} + R^{n-p_m+\alpha_2 p_m} \\
= R^{\beta+1-\alpha} p_m + R^{(\alpha_1-\alpha) p_m} + R^{(\alpha_2-\alpha) p_m} R^{n-p_m+\alpha_p m} \\
\leq \left( a^{\beta+1-\alpha} p_m + a^{(\alpha_1-\alpha) p_m} + a^{(\alpha_2-\alpha) p_m} \right) R^{n-p_m+\alpha_p m} \\
= C_4'' R^{n-p_m+\alpha_p m}.
\]
Using (2.16) and (2.17), we get for \( C_5 = C_4' \max(1, C_4'') \)
\[
(2.18) \quad \phi(r) \leq C_5 \left( \left( \frac{r}{R} \right)^n + R^+ + \epsilon \right) \phi(R) + R^{n-p_m+\alpha_p m} \quad \forall r \in (0, R).
\]
If we assume that \( R^+ < \epsilon \), then given that \( \phi \) is a nonnegative and nondecreasing function on \( (0, R) \), we infer from (2.18) and Lemma 5.12.
HOLDER CONTINUITY FOR THE $p(x)$-LAPLACE EQUATION

p. 248 of [21] applied with $\delta = 2\epsilon$, that we have for two positive constants $C_6 = C_6(C_5, n, \alpha, p_m)$ and $\delta_0 = \delta_0(C_5, n, \alpha, p_m)$

(2.19) $\phi(r) \leq C_6 \left( \frac{r}{R} \right)^{n-p_m+\alpha p_m} \left( \phi(R) + R^{n-p_m+\alpha p_m} \right) \quad \forall r \in (0, R)$

provided that $\delta < \delta_0$ and $R < (\delta/2)^{\frac{1}{\alpha}} = R_2$.

Using (2.19) and (2.13), for $R = r$ and $w = u$, we get for $C_7 = C_6(\phi(R) + R^{n-p_m+\alpha p_m})$

\[
\int_{B_r(x_0)} |\nabla u|^p dx \leq \left( \int_{B_r(x_0)} |\nabla u|^p(x) dx \right)^{\frac{1}{p_m}} \leq \omega_n r^n \cdot \left( C_7 + \omega_n \right)^{\frac{1}{p_m}} \cdot \left( C_7 + \omega_n \right)^{\frac{1}{p_m}} = C' \cdot r^{n-1+\alpha},
\]

where $C' = C_7 \cdot \omega_n^{1-\frac{1}{p_m}}$. We conclude ([21, Theorem 1.53 (Morrey) p. 30]) that $u \in C^{0,\alpha}_{loc}(\Omega)$, which completes the proof of the theorem.

PROOF OF THEOREM 1.1. Obviously we can choose $\delta$ small enough so that $R_2 < R_1$. Then, by using (2.1) and Hölder’s inequality, we obtain for all $R \in (0, R_2)$ and all $r \in (0, R)$

\[
\int_{B_r} |\nabla u|^p dx \leq |B_r|^{\frac{1}{p_m}} \left( \int_{B_r} |\nabla u|^p dx \right)^{\frac{1}{p_m}} \leq \omega_n r^n \cdot \left( C_7 + \omega_n \right)^{\frac{1}{p_m}} \cdot \left( C_7 + \omega_n \right)^{\frac{1}{p_m}} = C' \cdot r^{n-1+\alpha},
\]

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Received: 15.3.2021.
Revised: 28.3.2022.