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THREE KINDS OF NUMERICAL INDICES OF l_p-SPACES

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Abstract. In this paper, we investigate the polynomial numerical index $n^{(k)}(l_p)$, the symmetric multilinear numerical index $n_s^{(k)}(l_p)$, and the multilinear numerical index $n_m^{(k)}(l_p)$ of l_p spaces, for $1 \le p \le \infty$. First we prove that $n_s^{(k)}(l_1) = n_m^{(k)}(l_1) = 1$, for every $k \ge 2$. We show that for $1 < p < \infty$, $n_I^{(k)}(l_p^{j+1}) \le n_I^{(k)}(l_p^j)$, for every $j \in \mathbb{N}$ and $n_I^{(k)}(l_p) = \lim_{j\to\infty} n_I^{(k)}(l_p^j)$, for every $I = s, m$, where $l_p^j = (\mathbb{C}^j, \|\cdot\|_p)$ or $(\mathbb{R}^j, \|\cdot\|_p)$. We also show the following inequality between $n_s^{(k)}(l_p^j)$ and $n^{(k)}(l_p^j)$: let $1 < p < \infty$ and $k \in \mathbb{N}$ be fixed. Then

$$c(k : l_p^j)^{-1}\ n^{(k)}(l_p^j) \le n_s^{(k)}(l_p^j) \le n^{(k)}(l_p^j),$$

for every $j \in \mathbb{N} \cup \{\infty\}$, where $l_p^\infty := l_p$,

$$c(k : l_p) = \inf\Big\{M > 0 : \|\check{Q}\| \le M\|Q\|,\ \text{for every } Q \in \mathcal{P}({}^k l_p)\Big\}$$

and $\check{Q}$ denotes the symmetric k-linear form associated with Q. From this inequality, we deduce that if l_p is a complex space, then $\lim_{j\to\infty} n_s^{(j)}(l_p) = \lim_{j\to\infty} n_m^{(j)}(l_p) = 0$, for every $1 < p < \infty$.

1. Introduction

Throughout this paper $\mathbb{K}$ denotes either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$. If the field is not specified the results are valid in both cases. Let E and F be Banach spaces over the field $\mathbb{K}$. We write B_E and S_E for the closed unit ball and the unit sphere of E, respectively. The dual space of E is denoted by E^*. We write E^k for the product $E \times \cdots \times E$ with k factors, for some natural number k. We denote by $\mathcal{L}({}^k E : F)$ the Banach space of

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continuous k-linear mappings of E^k into F endowed with the norm

$$\|A\| = \sup \{\|A(x_1, \dots, x_k)\| \ : x_j \in B_E, j = 1, \dots, k\}.$$

$A \in \mathcal{L}(^kE : F)$ is said to be symmetric if

$$A(x_1, \dots, x_k) = A(x_{\sigma(1)}, \dots, x_{\sigma(k)}),$$

for any $x_1, \dots, x_k$ in E and any permutation σ of the first k natural numbers. We denote by $\mathcal{L}_s(^kE : F)$ the closed subspace of all symmetric k-linear maps in $\mathcal{L}(^kE : F)$. Given $A \in \mathcal{L}(^kE : F)$, we define the symmetric k-linear mapping $A_s : E^k \to F$ (which we call the *symmetrization* of A) by

$$A_s(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma} A(x_{\sigma(1)}, \dots, x_{\sigma(k)}),$$

for any $x_1, \dots, x_k$ in E, where the summation is over the $k!$ permutations σ of the first k natural numbers. We denote $\mathcal{L}(^kE : \mathbb{K})$ and $\mathcal{L}_s(^kE : \mathbb{K})$ by $\mathcal{L}(^kE)$ and $\mathcal{L}_s(^kE)$ respectively. A mapping $P : E \to F$ is said to be a continuous k-homogeneous polynomial if there exists an $A \in \mathcal{L}(^kE : F)$ such that $P(x) = A(x, \dots, x)$, for all $x \in E$. For $A \in \mathcal{L}(^kE : F)$, we define the associated polynomial $\widehat{A} : E \to F$ by $\widehat{A}(x) = A(x, \dots, x)$ for $x \in E$. It is obvious that $\widehat{A} = \widehat{A_s}$. We denote by $\mathcal{P}(^kE : F)$ the Banach space of continuous k-homogeneous polynomials of E into F endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We denote $\mathcal{P}(^kE : \mathbb{K})$ by $\mathcal{P}(^kE)$. We also note that $\|\hat{A}\| \leq \|A_s\| \leq \|A\|$ for any A in $\mathcal{L}(^kE : F)$. We refer to [7] for a general background on the theory of polynomials on an infinite dimensional Banach space.

In this paper we consider the spaces $\mathcal{L}(^kE : E), \mathcal{L}_s(^kE : E)$ and $\mathcal{P}(^kE : E)$. Let

$$\Pi(E^k) = \Big\{ [x^*, x_1, \dots, x_k] : x^* \in E^*, x_j \in E, \\ 1 = x^*(x_j) = \|x_j\| = \|x^*\|, 1 \leq j \leq k \Big\}.$$

The numerical range of $A \in \mathcal{L}(^kE : E)$ is defined by

$$W(A) := \ \{ \ x^*(A(x_1, \dots, x_k)) \ : \ (x_1, \dots, x_k, x^*) \in \Pi(E^k)\}$$

and the numerical radius of $A \in \mathcal{L}(^kE : E)$ is defined by

$$v(A) := \sup \ \{ \ |x^*(A(x_1, \dots, x_k))| \ : \ (x_1, \dots, x_k, x^*) \in \Pi(E^k)\}.$$

Similarly, for each $P \in \mathcal{P}(^kE : E)$, the numerical range of P is defined by

$$W(P) := \{x^*(Px) : (x, x^*) \in \Pi(E^1)\}$$

and the numerical radius of P is defined by

$$v(P) := \sup \ \{|\lambda| : \lambda \in W(P)\}.$$

Clearly we have $v(A) \leq \|A\|, v(A_s) \leq \|A_s\|$ and $v(\widehat{A}) \leq \|\widehat{A}\|$, for any A in $\mathcal{L}(^k E : E)$. It is obvious that

$$(*) \qquad v(\widehat{A}) \leq v(A_s) \leq v(A) \quad (A \in \mathcal{L}(^k E : E)),$$

as in the case of norms of them. The following example shows that the inequalities in $(*)$ can be strict. In fact, we define a continuous 2-linear map $A \in \mathcal{L}(^2 l_1 : l_1)$ by

$$A(x, y) = (\frac{1}{2}x_1 y_1 + 2x_1 y_2)e_1 + (-\frac{1}{2}x_2 y_2 - x_1 y_2)e_2,$$

for any $x = (x_i), y = (y_i) \in l_1$, where $e_1 = (1, 0, 0, \ldots)$ and $e_2 = (0, 1, 0, 0, \ldots)$. Then we have

$$A_s(x, y) = (\frac{1}{2}x_1 y_1 + x_1 y_2 + x_2 y_1)e_1 + (-\frac{1}{2}x_2 y_2 - \frac{1}{2}x_1 y_2 - \frac{1}{2}x_2 y_1)e_2$$

and

$$\widehat{A}(x) = (\frac{1}{2}x_1^2 + 2x_1 x_2)e_1 + (-\frac{1}{2}x_2^2 - x_1 x_2)e_2.$$

It is not difficult to show that $v(\widehat{A}) = \frac{1}{2}, \|\widehat{A}\| = 1, v(A_s) = \frac{3}{2} = \|A_s\|$ and $v(A) = 3 = \|A\|$. Thus

$$v(\frac{\widehat{A}}{\|\widehat{A}\|}) < v(\frac{A_s}{\|A_s\|}) = v(\frac{A}{\|A\|}).$$

Note that $\|\widehat{A}\| < \|A_s\| < \|A\|$ and $v(\widehat{A}) < v(A_s) < v(A)$.

In [4] the k-th polynomial numerical index of E, the constant $n_p^{(k)}(E)$ is defined by

$$n_p^{(k)}(E) := \inf \{v(P) : P \in S_{\mathcal{P}(^k E : E)}\}.$$

Clearly $0 \leq n_p^{(k)}(E) \leq 1$ (see [1, 2, 3, 4, 5] and [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] for general information and background on the theory of numerical index of Banach spaces).

In connection to $n_p^{(k)}(E)$, Kim ([11, 13]) introduced the new concepts of the k-th multilinear numerical index and k-th symmetric multilinear numerical index of E, generalizing to k-linear and symmetric k-linear maps, respectively the classical numerical index defined by G. Lumer ([16]) in the sixties for linear operators. The k-th *multilinear numerical index* of E was defined ([11, 13]) by

$$n_m^{(k)}(E) := \inf \{v(A) : A \in S_{\mathcal{L}(^k E : E)}\}.$$

We define the k-th *symmetric multilinear numerical index* of E by

$$n_s^{(k)}(E) := \inf \{v(A) : A \in S_{\mathcal{L}_s(^k E : E)}\}.$$

Clearly $0 \leq n_m^{(k)}(E) \leq 1, 0 \leq n_s^{(k)}(E) \leq 1$. Since $\mathcal{L}_s(^k E : E)$ is a closed subspace of $\mathcal{L}(^k E : E)$, we have $n_m^{(k)}(E) \leq n_s^{(k)}(E)$. Clearly $n_m^{(k)}(E)$ ($n_s^{(k)}(E)$ resp.) is the greatest constant $c \geq 0$ such that $c\|A\| \leq v(A)$ for every $A \in$

$\mathcal{L}({}^kE:E)$ ($A\in\mathcal{L}_s({}^kE:E)$ resp.). Note that $n_m^{(k)}(E)>0$ ($n_s^{(k)}(E)>0$ resp) if and only if v and $\|\cdot\|$ are equivalent norms on $\mathcal{L}({}^kE:E)$ ($\mathcal{L}_s({}^kE:E)$ resp). It is easy to verify that if E_1,E_2 are isometrically isomorphic Banach spaces, then $n_m^{(k)}(E_1)=n_m^{(k)}(E_2)$ and $n_s^{(k)}(E_1)=n_s^{(k)}(E_2)$. Kim ([9, 10]) investigated properties and the inequalities between $n_m^{(k)}(E), n_s^{(k)}(E)$ and $n_p^{(k)}(E)$. In this paper, we first prove that $n_s^{(k)}(l_1)=n_m^{(k)}(l_1)=1$, for every $k\ge 2$. We show that for $1<p<\infty$,

$$n_I^{(k)}(l_p^{j+1})\le n_I^{(k)}(l_p^j),$$

for every $j\in\mathbb{N}$, and

$$n_I^{(k)}(l_p)=\lim_{j\to\infty} n_I^{(k)}(l_p^j),$$

for every $I=s,m$, where $l_p^j=(\mathbb{C}^j,\|\cdot\|_p)$ or $(\mathbb{R}^j,\|\cdot\|_p)$. We also show the following inequality between $n_s^{(k)}(l_p^j)$ and $n^{(k)}(l_p^j)$: let $1<p<\infty$ and $k\in\mathbb{N}$ be fixed. Then

$$c(k:l_p^j)^{-1}\ n^{(k)}(l_p^j)\le n_s^{(k)}(l_p^j)\le n^{(k)}(l_p^j) \text{ for every } j\in\mathbb{N}\cup\{\infty\},$$

where $l_p^\infty:=l_p$,

$$c(k:l_p)=\inf\Big\{M>0:\|\check{Q}\|\le M\|Q\| \text{ for every } Q\in\mathcal{P}({}^kl_p)\Big\}$$

and $\check{Q}$ denotes the symmetric k-linear form associated with Q. From this inequality, we deduce that if l_p is a complex space, then $\lim_{j\to\infty} n_s^{(j)}(l_p)=\lim_{j\to\infty} n_m^{(j)}(l_p)=0$, for every $1<p<\infty$.

2. The multilinear numerical index of l_1 is one

For $1<p<\infty$ and $j\in\mathbb{N}$, l_p^j denotes $\mathbb{K}^j$ endowed with the usual p-norm, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We may consider l_p^j as a subspace of l_p. Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of l_p and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$. The following theorem presents explicit formulas for the numerical radius and the norm of T, for every $T\in\mathcal{L}({}^kl_1:l_1)$ and every $k\ge 2$.

Theorem 2.1. *Let $k\ge 2$. Let $T=\sum_{j\in\mathbb{N}}T_je_j\in\mathcal{L}({}^kl_1:l_1)$ be such that*

$$T_j\Big(\sum_{i\in\mathbb{N}}x_i^{(1)}e_i,\cdots,\sum_{i\in\mathbb{N}}x_i^{(k)}e_i\Big)=\sum_{i_1,\dots,i_k\in\mathbb{N}}a_{i_1\cdots i_k}^{(j)}\ x_{i_1}^{(1)}\cdots x_{i_k}^{(k)}\in\mathcal{L}({}^kl_1),$$

for some $a_{i_1\cdots i_k}^{(j)}\in\mathbb{R}$. Then

$$\sup\Big\{\sum_{j\in\mathbb{N}}\Big|a_{i_1\cdots i_k}^{(j)}\Big|:i_1,\dots,i_k\in\mathbb{N}\Big\}=v(T)=\|T\|.$$

Consequently, $n_s^{(k)}(l_1)=n_m^{(k)}(l_1)=1$, for every $k\ge 2$.

PROOF. CLAIM. The following inequality holds:

$$\sup\Big\{\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big| : i_1,\ldots,i_k\in\mathbb{N}\Big\}\le v(T).$$

Let $i_1,\ldots,i_k\in\mathbb{N}$ be fixed. Let $A=\{i_1,\ldots,i_k\}$. Notice that

$$\Big[\Big(\sum_{l\in A}\lambda_l e_l^* + \sum_{j\in\mathbb{N}\setminus A}\operatorname{sign}(a^{(j)}_{i_1\cdots i_k})e_j^*\Big), \overline{\lambda_{i_1}}e_{i_1},\ldots,\overline{\lambda_{i_k}}e_{i_k}\Big)\Big]\in\Pi((l_1)^k),$$

for every $\lambda_l\in\mathbb{C}$ and $l\in A$, where $\overline{\lambda_l}$ is the conjugate complex number of λ_l. It follows that

$$\begin{aligned}
v(T) &\ge \sup\Big\{\Big|\Big(\sum_{l\in A}\lambda_l e_l^* + \sum_{j\in\mathbb{N}\setminus A}\operatorname{sign}(a^{(j)}_{i_1\cdots i_k})e_j^*\Big)\Big(T\Big(\overline{\lambda_{i_1}}e_{i_1},\ldots,\overline{\lambda_{i_k}}e_{i_k}\Big)\Big)\Big| :\\
&\qquad |\lambda_l|=1, \lambda_l\in\mathbb{C} \text{ for } l\in A\Big\}\\
&= \sup\Big\{\Big|\Big(\sum_{l\in A}\lambda_l e_l^* + \sum_{j\in\mathbb{N}\setminus A}\operatorname{sign}(a^{(j)}_{i_1\cdots i_k})e_j^*\Big)\Big(T\Big(e_{i_1},\ldots,e_{i_k}\Big)\Big)\Big| :\\
&\qquad |\lambda_l|=1, \lambda_l\in\mathbb{C} \text{ for } l\in A\Big\}\\
&= \sup\Big\{\Big|\sum_{l\in A}\lambda_l a^{(l)}_{i_1\cdots i_k} + \sum_{j\in\mathbb{N}\setminus A}\Big|a^{(j)}_{i_1\cdots i_k}\Big|\Big| : |\lambda_l|=1, \lambda_l\in\mathbb{C} \text{ for } l\in A\Big\}\\
&= \Big|\sum_{l\in A}\operatorname{sign}(a^{(l)}_{i_1\cdots i_k})a^{(l)}_{i_1\cdots i_k} + \sum_{j\in\mathbb{N}\setminus A}\Big|a^{(j)}_{i_1\cdots i_k}\Big|\Big|\\
&= \Big|\sum_{l\in A}\Big|a^{(l)}_{i_1\cdots i_k}\Big| + \sum_{j\in\mathbb{N}\setminus A}\Big|a^{(j)}_{i_1\cdots i_k}\Big|\Big|\\
&= \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big|.
\end{aligned}$$

Hence, $\sup\big\{\sum_{j\in\mathbb{N}}\big|a^{(j)}_{i_1\cdots i_k}\big| : i_1,\ldots,i_k\in\mathbb{N}\big\}\le v(T)$, which concludes the claim.

Let $\epsilon>0$. Choose $i_1',\ldots,i_k'\in\mathbb{N}$ be such that

$$\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1'\cdots i_k'}\Big| > \sup\Big\{\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big| : i_1,\ldots,i_k\in\mathbb{N}\Big\}-\epsilon.$$

Let $\sum_{i\in\mathbb{N}}x_i^{(1)}e_i,\cdots,\sum_{i\in\mathbb{N}}x_i^{(k)}e_i\in S_{l_1}$.

It follows that

$$\begin{aligned}
&\Big\|T\Big(\sum_{i\in\mathbb{N}} x_i^{(1)}e_i,\cdots,\sum_{i\in\mathbb{N}} x_i^{(k)}e_i\Big)\Big\|_1\\
&\quad=\sum_{j\in\mathbb{N}}\Big|T_j\Big(\sum_{i\in\mathbb{N}} x_i^{(1)}e_i,\cdots,\sum_{i\in\mathbb{N}} x_i^{(k)}e_i\Big)\Big|\\
&\quad\le\sum_{j\in\mathbb{N}}\Big(\sum_{i_1,\dots,i_k\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big|\,\Big|x^{(1)}_{i_1}\Big|\cdots\Big|x^{(k)}_{i_k}\Big|\Big)\\
&\quad=\sum_{i_1,\dots,i_k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big|\Big)\,\Big|x^{(1)}_{i_1}\Big|\cdots\Big|x^{(k)}_{i_k}\Big|\\
&\quad<\sum_{i_1,\dots,i_k\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i'_1\cdots i'_k}\Big|+\epsilon\Big)\,\Big|x^{(1)}_{i_1}\Big|\cdots\Big|x^{(k)}_{i_k}\Big|\\
&\quad=\Big(\sum_{j\in\mathbb{N}}|a^{(j)}_{i'_1\cdots i'_k}|+\epsilon\Big)\sum_{i_1,\dots,i_k\in\mathbb{N}}\Big|x^{(1)}_{i_1}|\cdots|x^{(k)}_{i_k}\Big|\\
&\quad=\Big(\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i'_1\cdots i'_k}\Big|+\epsilon\Big)\Big(\sum_{i_1\in\mathbb{N}}\Big|x^{(1)}_{i_1}\Big|\Big)\cdots\Big(\sum_{i_k\in\mathbb{N}}\Big|x^{(k)}_{i_k}\Big|\Big)\\
&\quad=\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i'_1\cdots i'_k}\Big|+\epsilon\le v(T)+\epsilon\text{ (by Claim)},
\end{aligned}$$

which shows that

$$\sup\Big\{\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_k}\Big| : i_1,\dots,i_k\in\mathbb{N}\Big\}=v(T)=\|T\|.$$

Therefore, we complete the proof. □

Notice that $n_p^{(2)}(l_1)\le\frac{1}{2}$ from the example in the introduction. Kim ([11]) showed that $n_s^{(k)}(l_\infty)=n_m^{(k)}(l_\infty)=1$, for every $k\ge 2$.

3. Three kinds of numerical indices of l_p-spaces for $1<p<\infty$

Theorem 3.1. *Let $1<p<\infty$ and $k\in\mathbb{N}$ be fixed. Then*

$$n_I^{(k)}(l_p^{j+1})\le n_I^{(k)}(l_p^j),\quad\textit{for every } j\in\mathbb{N},\textit{ and } n_I^{(k)}(l_p)=\lim_{j\to\infty}n_I^{(k)}(l_p^j),$$

for every $I=s,m$.

Proof. Let $I=s$. Let $j\in\mathbb{N}$ be fixed. We define $P_{\{1,\dots,j\}}: l_p\to l_p^j$ by

$$P_{\{1,\dots,j\}}\Big(\sum_{l=1}^{\infty}\lambda_l e_l\Big)=\sum_{l=1}^{j}\lambda_l e_l.$$

Obviously, $P_{\{1,\dots,j\}}$ is linear. Let $T \in \mathcal{L}_s({}^k l_p^j : l_p^j)$ with $\|T\| = 1$. We define $T_1 \in \mathcal{L}_s({}^k l_p^{j+1} : l_p^{j+1})$ by

$$T_1(x_1, \dots, x_k) = T\Big(P_{\{1,\dots,j\}}(x_1), \dots, P_{\{1,\dots,j\}}(x_k)\Big)$$

for $x_1, \dots, x_k \in l_p^{j+1}$. It is obvious that $T_1 \in \mathcal{L}_s({}^k l_p^{j+1} : l_p^{j+1})$ with $\|T_1\| = 1$.

Claim 1. $v(T) = v(T_1)$

Let $[x^*, y_1, \dots, y_k] \in \Pi((l_p^j)^k)$. Then $[x^*, y_1, \dots, y_k] \in \Pi((l_p^{j+1})^k)$ and

$$(*) \qquad \begin{aligned} \Big|x^*(T(y_1, \dots, y_k))\Big| &= \Big|x^*\Big(T\Big(P_{\{1,\dots,k\}}(y_1), \dots, P_{\{1,\dots,k\}}(y_k)\Big)\Big)\Big| \\ &= \Big|x^*(T_1(y_1, \dots, y_k))\Big| \le v(T_1). \end{aligned}$$

By taking the infimum in the left side of $(*)$ over $[x^*, y_1, \dots, y_k] \in \Pi((l_p^j)^k)$, we have $v(T) \le v(T_1)$. For the reverse inequality, let $\epsilon > 0$. By the Hölder inequality, there exist $z_0 := \sum_{l=1}^{j+1} a_l e_l \in S_{l_p^{j+1}}$ such that

$$\Big[\sum_{l=1}^{j+1} \mathrm{sign}(a_l)|a_l|^{p-1} e_l^*,\ z_0, \dots, z_0\Big] \in \Pi((l_p^{j+1})^k)$$

and

$$v(T_1) - \epsilon < \Big|\Big(\sum_{l=1}^{j+1} \mathrm{sign}(a_l)|a_l|^{p-1} e_l^*\Big)\Big(T_1(z_0, \dots, z_0)\Big)\Big|.$$

Let $c := (\sum_{l=1}^{j} |a_l|^p)^{\frac{1}{p}} \le 1$. It follows that

$$\begin{aligned} v(T_1) - \epsilon &< \Big|\Big(\sum_{l=1}^{j+1} \mathrm{sign}(a_l)|a_l|^{p-1} e_l^*\Big)\Big(T_1(z_0, \dots, z_0)\Big)\Big| \\ &= \Big|\Big(\sum_{l=1}^{j+1} \mathrm{sign}(a_l)|a_l|^{p-1} e_l^*\Big)\Big(T\Big(\sum_{l=1}^{j} a_l e_l, \dots, \sum_{l=1}^{j} a_l e_l\Big)\Big)\Big| \\ &= c^{k+p-1}\Big|\Big(\sum_{l=1}^{j} \mathrm{sign}(a_l)\frac{|a_l|}{c}^{p-1} e_l^*\Big)\Big(T\Big(\frac{1}{c}\sum_{l=1}^{j} a_l e_l, \dots, \frac{1}{c}\sum_{l=1}^{j} a_l e_l\Big)\Big)\Big| \\ &\le \Big|\Big(\sum_{l=1}^{j} \mathrm{sign}(a_l)\frac{|a_l|}{c}^{p-1} e_l^*\Big)\Big(T\Big(\frac{1}{c}\sum_{l=1}^{j} a_l e_l, \dots, \frac{1}{c}\sum_{l=1}^{j} a_l e_l\Big)\Big)\Big| \\ &\quad \text{(since } c^{k+p-1} \le 1) \\ &\le v(T) \\ &\quad \Big(\text{since } \Big[\Big(\sum_{l=1}^{j} \mathrm{sign}(a_l)\frac{|a_l|}{c}^{p-1} e_l^*\Big),\ \sum_{l=1}^{j} \frac{a_l}{c} e_l, \dots, \sum_{l=1}^{m} \frac{a_l}{c} e_l\Big] \\ &\quad \in \Pi((l_p^j)^k)\Big), \end{aligned}$$

which shows that $v(T_1) \le v(T)$. Thus, Claim 1 holds.

CLAIM 2. $n_s^{(k)}(l_p^{j+1}) \le n_s^{(k)}(l_p^j)$ for every $j \in \mathbb{N}$.

It follows that

$$\begin{aligned} n_s^{(k)}(l_p^j) &= \inf_{T \in S_{\mathcal{L}_s({}^k l_p^j : l_p^j)}} v(T) = \inf_{T \in S_{\mathcal{L}_s({}^k l_p^j : l_p^j)}} v(T_1) \\ &\ge \inf_{R \in S_{\mathcal{L}_s({}^k l_p^{j+1} : l_p^{j+1})}} v(R) = n_s^{(k)}(l_p^{j+1}). \end{aligned}$$

Thus, Claim 2 holds.

We define $T_2 \in \mathcal{L}_s({}^k l_p : l_p)$ by

$$T_2(z_1, \ldots, z_k) = T\Big(P_{\{1,\ldots,j\}}(z_1), \ldots, P_{\{1,\ldots,j\}}(z_k)\Big)$$

for $z_1, \ldots, z_k \in l_p$. It is obvious that $T_2 \in S_{\mathcal{L}_s({}^k l_p : l_p)}$. By analogous argument as in Claim 1, we have $v(T) = v(T_2)$. It follows that

$$\begin{aligned} n_s^{(k)}(l_p^j) &= \inf_{T \in S_{\mathcal{L}_s({}^k l_p^j : l_p^j)}} v(T) = \inf_{T \in S_{\mathcal{L}_s({}^k l_p^j : l_p^j)}} v(T_2) \\ &\ge \inf_{R \in S_{\mathcal{L}_s({}^k l_p : l_p)}} v(R) = n_s^{(k)}(l_p). \end{aligned}$$

Hence, $n_s^{(k)}(l_p) \le n_s^{(k)}(l_p^j)$, for every $j \in \mathbb{N}$.

CLAIM 3. $n_s^{(k)}(l_p) = \lim_{j\to\infty} n_s^{(k)}(l_p^j)$.

Let $R \in \mathcal{L}_s({}^k l_p : l_p)$ with $\|R\| = 1$. For each $j \in \mathbb{N}$, we define $R_j \in \mathcal{L}_s({}^k l_p^j : l_p^j)$ by

$$R_j(x_1, \ldots, x_k) = P_{\{1,\ldots,j\}}\Big(R(x_1, \ldots, x_k)\Big),$$

for $x_1, \ldots, x_k \in l_p^j$. It is obvious that $\|R_j\| \le 1, \|R_j\| \le \|R_{j+1}\|$ and $v(R_j) \le v(R)$. For each $j \in \mathbb{N}$, we define $R_j' \in \mathcal{L}_s({}^k l_p : l_p)$ by

$$R_j'(z_1, \ldots, z_k) = R_j\Big(P_{\{1,\ldots,j\}}(z_1), \ldots, P_{\{1,\ldots,j\}}(z_k)\Big),$$

for $z_1, \ldots, z_k \in l_p$. By analogous arguments as in Claim 1, $v(R_j') = v(R_j)$.

We claim that $\lim_{j\to\infty} \|R_j\| = 1$.

Indeed, let $\epsilon > 0$. Choose $z_1, \dots, z_k \in S_{l_p}$ such that $\|R(z_1, \dots, z_k)\| > 1 - \epsilon$. By continuity of R at $z_1, \dots, z_k$, it follows that

$$\begin{aligned} &\Big\|R_j\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big) - R(z_1, \dots, z_k)\Big\| \\ &\quad\le \Big\|P_{\{1,\dots,j\}}\Big(R\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big)\Big) \\ &\qquad - R\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big)\Big\| \\ &\qquad + \Big\|R\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big) - R(z_1, \dots, z_k)\Big\| \\ &\quad\le \Big\|\Big(I - P_{\{1,\dots,j\}}\Big)\Big(R\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big)\Big)\Big\| \\ &\qquad + \sum_{1\le l\le k} \Big\|P_{\{1,\dots,j\}}(z_l) - z_l\Big\| \to 0 \text{ as } j \to \infty. \end{aligned}$$

Choose $N_0 \in \mathbb{N}$ such that

$$\Big\|R_j\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big) - R(z_1, \dots, z_k)\Big\| < \epsilon$$

for all $j \ge N_0$. Then for all $j \ge N_0$,

$$1 \ge \|R_j\| \ge \Big\|R_j\Big(P_{\{1,\dots,j\}}(z_1), \dots, P_{\{1,\dots,j\}}(z_k)\Big) - R(z_1, \dots, z_k)\Big\| > 1 - 2\epsilon,$$

which shows that $\lim_{j\to\infty} \|R_j\| = 1$.

We claim that $\lim_{j\to\infty} v(R_j) = v(R)$.

Indeed, let $\epsilon > 0$. Choose $[y^*, y_0, \dots, y_0] \in \Pi((l_p)^k)$ such that

$$\Big|y^*(R(y_0, \dots, y_0))\Big| > v(R) - \epsilon.$$

Let $y_0 := \sum_{l=1}^{\infty} b_l e_l$. By the Hölder inequality, $y^* = \sum_{l=1}^{\infty} \text{sign}(b_l)\ |b_l|^{p-1} e_l^*$. For $j \in \mathbb{N}$, we define

$$y_0^{(j)} := \sum_{l=1}^{j-1} b_l e_l + (\sum_{l=j}^{\infty} |b_l|^p)^{\frac{1}{p}} e_l$$

and

$$y_j^* := \sum_{l=1}^{j-1} sign(b_l)\ |b_l|^{p-1} e_l^* + (\sum_{l=j}^{\infty} |b_l|^p)^{\frac{p-1}{p}} e_l^*.$$

Let $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$. It is obvious that $\Big[y_j^*, y_0^{(j)}, \dots, y_0^{(j)}\Big] \in \Pi((l_p)^k)$ and

$$\lim_{j\to\infty} \|y_0 - y_0^{(j)}\|_p = 0 = \lim_{j\to\infty} \|y^* - y_j^*\|_q.$$

Notice that

$$\lim_{j\to\infty} y_j^*(R(y_0^{(j)}, \dots, y_0^{(j)})) = y^*(R(y_0, \dots, y_0)).$$

Indeed,

$$\begin{aligned}
&\left|y_j^*(R(y_0^{(j)},\dots,y_0^{(j)}))-y^*(R(y_0,\dots,y_0))\right| \\
&\qquad\le \left|y_j^*(R(y_0^{(j)},\dots,y_0^{(j)}))-y^*(R(y_0^{(j)},\dots,y_0^{(j)}))\right| \\
&\qquad\quad+\left|y^*(R(y_0^{(j)},\dots,y_0^{(j)}))-y^*(R(y_0,\dots,y_0))\right| \\
&\qquad\le \|y_j^*-y^*\|_q \left\|R(y_0^{(j)},\dots,y_0^{(j)})\right\|_p \\
&\qquad\quad+\left\|R(y_0^{(j)},\dots,y_0^{(j)})-R(y_0,\dots,y_0)\right\|_p \to 0 \text{ as } j\to\infty.
\end{aligned}$$

Choose $N_1 \in \mathbb{N}$ such that

$$\left|y_j^*(R(y_0^{(j)},\dots,y_0^{(j)}))\right| > v(R)-\epsilon,$$

for all $j \ge N_1$. It is easy to show that for all $j \ge N_1$,

$$y_{N_1}^*(R_j'(y_0^{(N_1)},\dots,y_0^{(N_1)})) = y_{N_1}^*(R(y_0^{(N_1)},\dots,y_0^{(N_1)})).$$

It follows that for all $j \ge N_1$,

$$\begin{aligned}
v(R)-\epsilon &< \left|y_{N_1}^*(R(y_0^{(N_1)},\dots,y_0^{(N_1)}))\right| \\
&= \left|y_{N_1}^*(R_j'(y_0^{(N_1)},\dots,y_0^{(N_1)}))\right| \\
&\le v(R_j') = v(R_j) \le v(R),
\end{aligned}$$

which shows that $\lim_{j\to\infty} v(R_j) = v(R)$. It follows that

$$(**) \qquad v(R) = \lim_{j\to\infty} v\Big(\frac{R_j}{\|R_j\|}\Big) \ge \lim_{j\to\infty} n_s^{(k)}(l_p^j) \ge n_s^{(k)}(l_p).$$

Taking the infimum in the left side of $(**)$ over $R \in \mathcal{L}_s({}^k l_p : l_p)$ with $\|R\| = 1$, we have

$$n_s^{(k)}(l_p) = \lim_{j\to\infty} n_s^{(k)}(l_p^j).$$

If $I = m$, analogous arguments give the proof. We complete the proof. ∎

For a Banach space X, the k-th polarization constant of X is defined by

$$c(k:X) := \inf\Big\{M>0 : \|\check{Q}\| \le M\|Q\| \text{ for every } Q \in \mathcal{P}({}^k X)\Big\},$$

where $\check{Q}$ denotes the symmetric k-linear form associated with Q. The polarization constant of X is defined by $c(X) := \liminf_{k\to\infty} c(k:X)^{\frac{1}{k}}$. Recently, Dimant et al. ([6]) proved that $c(X) = 1$ if X is a finite dimensional complex space and $c(X) \le 2$ if X is a finite dimensional real space.

The following theorem shows some relation between $n_s^{(k)}(l_p^j)$ and $n^{(k)}(l_p^j)$.

THEOREM 3.2. *Let $1 < p < \infty$ and $k \in \mathbb{N}$ be fixed. Then*

$$c(k : l_p^j)^{-1}\ n^{(k)}(l_p^j) \le n_s^{(k)}(l_p^j) \le n^{(k)}(l_p^j) \text{ for every } j \in \mathbb{N} \cup \{\infty\},$$

where $l_p^\infty := l_p$.

PROOF. Let $P \in \mathcal{P}({}^k l_p : l_p)$ with $\|P\| = 1$. Let $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$. It is enough to show the theorem for $j = \infty$. It follows that

$$\begin{aligned}
v(P) &= \sup\Big\{|y^*(P(x))| : [y^*, x] \in \Pi(l_p)\Big\} \\
&= \sup\Big\{\Big|y^*(\check{P}(x_1, \ldots, x_k))\Big| : [y^*, x_1, \ldots, x_k] \in \Pi((l_p)^k)\Big\} \\
&= \sup\Big\{\Big|y^*(\check{P}(x, \ldots, x))\Big| : [y^*, x] \in \Pi(l_p)\Big\} \\
&\qquad \text{(by the Hölder inequality)} \\
&= v(\check{P}) = \|\check{P}\|\ v\Big(\frac{\check{P}}{\|\check{P}\|}\Big) \\
&\ge \|P\|\ n_s^{(k)}(l_p) = n_s^{(k)}(l_p).
\end{aligned} \tag{†}$$

Taking the infimum in the left side of (†) over $P \in \mathcal{P}({}^k l_p : l_p)$ with $\|P\| = 1$, we obtain $n_s^{(k)}(l_p) \le n^{(k)}(l_p)$. Let $T \in \mathcal{L}_s({}^k l_p : l_p)$ with $\|T\| = 1$. It follows that

$$\begin{aligned}
v(T) &= \sup\Big\{\Big|y^*(T(x_1, \ldots, x_k))\Big| : [y^*, x_1, \ldots, x_k] \in \Pi((l_p)^k)\Big\} \\
&= \sup\Big\{\Big|y^*(T(x, \ldots, x))\Big| : [y^*, x] \in \Pi(l_p)\Big\} \\
&\qquad \text{(by the Hölder inequality)} \\
&= \sup\Big\{|y^*(\hat{T}(x))| : [y^*, x] \in \Pi(l_p)\Big\}, \\
&= \|\hat{T}\|\ \sup\Big\{\Big|y^*\Big(\frac{\hat{T}}{\|\hat{T}\|}(x)\Big)\Big| : [y^*, x] \in \Pi(l_p)\Big\} \\
&\ge \frac{1}{\inf\Big\{M > 0 : \|\check{P}\| \le M\|P\| \text{ for every } P \in \mathcal{P}({}^k l_p : l_p)\Big\}} v\Big(\frac{\hat{T}}{\|\hat{T}\|}\Big) \\
&\ge c(k : l_p)^{-1}\ n^{(k)}(l_p),
\end{aligned} \tag{††}$$

where $\hat{T}$ denotes the k-homogeneous polynomial associated with T. Taking the infimum in the left side of (††) over $T \in \mathcal{L}_s({}^k l_p : l_p)$ with $\|T\| = 1$, we obtain

$$c(k : l_p)^{-1} n^{(k)}(l_p) \le n_s^{(k)}(l_p).$$

Therefore, we complete the proof. ∎

THEOREM 3.3. *Let $1 < p < \infty$. The following assertions hold:*

(a) *if l_p is a complex space, then given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$(1+\epsilon)^{-k}\ n^{(k)}(l_p) \leq n_s^{(k)}(l_p) \leq n^{(k)}(l_p) \text{ for every } k \geq N;$$

(b) *if l_p is a real space, then given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$(2+\epsilon)^{-k}\ n^{(k)}(l_p) \leq n_s^{(k)}(l_p) \leq n^{(k)}(l_p) \text{ for every } k \geq N.$$

PROOF. (a) Let $j \in \mathbb{N}$ be fixed. Since l_p^j is a finite dimensional complex space, by [6, Theorem 2.1],

$$\limsup_{k\to\infty}\ c(k : l_p^j)^{\frac{1}{k}} = 1.$$

Let $\epsilon > 0$. There is $N \in \mathbb{N}$ such that

$$\sup\{c(k : l_p^j)^{\frac{1}{k}} : k \geq N\} < 1 + \epsilon.$$

Hence,

$$c(k : l_p^j)^{-1} > (1+\epsilon)^{-k} \text{ for every } k \geq N.$$

By Theorems A and B, it follows that

$$\begin{aligned} n_s^{(k)}(l_p) &= \inf\{n_s^{(k)}(l_p^i) : i \in \mathbb{N}\} \geq n_s^{(k)}(l_p^j) \geq c(k : l_p^j)^{-1}\ n^{(k)}(l_p^j) \\ &> (1+\epsilon)^{-k} n^{(k)}(l_p^j) \geq (1+\epsilon)^{-k} n^{(k)}(l_p). \end{aligned}$$

(b) Let $j \in \mathbb{N}$ be fixed. Since l_p^j is a finite dimensional real space, by [6, Proposition 2.7],

$$\limsup_{k\to\infty}\ c(k : l_p^j)^{\frac{1}{k}} \leq 2.$$

The proof follows by analogous arguments to the ones given in the proof of (a). ☐

COROLLARY 3.4. *Let $k \in \mathbb{N}$. If l_{2k} is a real space, then $n_s^{(2k+1)}(l_{2k}) = 0$. Hence, $\lim_{j\to\infty} n_s^{(j)}(l_{2k}) = 0$, for every $j \in \mathbb{N}$.*

PROOF. It follows by [12, Theorem 3.6] and Theorem 3.2. ☐

COROLLARY 3.5. *Let $1 < p < \infty$ and $k \in \mathbb{N}$ be fixed. If l_p is a complex space, then $n_s^{(k)}(l_p) \leq 2^{\frac{1-k}{p}}$. Hence, $\lim_{j\to\infty} n_s^{(j)}(l_p) = \lim_{j\to\infty} n_m^{(j)}(l_p) = 0$, for every $1 < p < \infty$.*

PROOF. It follows by [12, Theorem 3.8] and Theorem 3.2. ☐

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