GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 63 – 71

APPROXIMATELY ORTHOGONALITY PRESERVING MAPPINGS ON HILBERT $C_0(Z)$-MODULES

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Abstract. In this paper, we will use the categorical approach to Hilbert C^*-modules over a commutative C^*-algebra to investigate the approximately orthogonality preserving mappings on Hilbert C^*-modules over a commutative C^*-algebra.

Indeed, we show that if $\Psi : \Gamma \to \Gamma'$ is a nonzero $C_0(Z)$-linear (δ, ε)-orthogonality preserving mapping between the continuous fields of Hilbert spaces on a locally compact Hausdorff space Z, then Ψ is injective, continuous and also for every $x, y \in \Gamma$ and $z \in Z$,

$$|\langle \Psi(x), \Psi(y)\rangle(z) - \varphi^2(z)\langle x, y\rangle(z)| \leq \frac{4(\varepsilon - \delta)}{(1-\delta)(1+\varepsilon)}\|\Psi(x)\|\|\Psi(y)\|,$$

where $\varphi(z) = \sup\{\|\Psi(u)(z)\| : u \text{ is a unit vector in } \Gamma\}$.

1. Introduction

Recently, some authors studied orthogonality and approximately orthogonality preserving mappings in the framework of Hilbert C^*-modules [1, 4, 6, 7, 8, 10]. We recall that in a Hilbert C^*-module $(E, \langle\cdot,\cdot\rangle)$, elements x, y are said to be orthogonal, denoted by $x \perp y$, if $\langle x, y\rangle = 0$, and also for a given $\varepsilon \in [0, 1)$, they are called ε-orthogonal, denoted by $x \perp^{\varepsilon} y$, if $\|\langle x, y\rangle\| \leq \varepsilon\|x\|\|y\|$.

Let $\delta, \varepsilon \in [0, 1)$. A mapping $\Phi : E \to F$ between Hilbert C^*-modules is called

2020 *Mathematics Subject Classification.* 46L08, 47B49.

Key words and phrases. Approximately orthogonality preserving, Hilbert C^*-module, Continuous field of Hilbert spaces.

- *orthogonality preserving* if for every $x, y \in E$,
$$x \perp y \quad \Rightarrow \quad \Phi(x) \perp \Phi(y);$$
- *approximately orthogonality preserving* or (δ, ε)-*orthogonality preserving* if for every $x, y \in E$,
$$x \perp^{\delta} y \quad \Rightarrow \quad \Phi(x) \perp^{\varepsilon} \Phi(y);$$
- ε-*orthogonality preserving* if for every $x, y \in E$,
$$x \perp y \quad \Rightarrow \quad \Phi(x) \perp^{\varepsilon} \Phi(y).$$

It is very easy to see that a linear orthogonality preserving mapping between Hilbert spaces must be a scalar multiple of an isometry. Similarly, Leung et al. [7, 8], showed that the module structure and the orthogonality structure of a Hilbert C^*-module determine its inner product structure. In fact, they proved that if $\Phi : E \longrightarrow F$ is a A-linear orthogonality preserving mapping between Hilbert C^*-modules E and F over a C^*-algebra A, then there exists a (unique) positive central element u in the multiplier algebra $M(J_E)$, such that
$$\langle \Phi(x), \Phi(y) \rangle = u \langle x, y \rangle \qquad (x, y \in E),$$
where J_E is the closed two-sided ideal of A generated by all the A-valued inner products of elements in E.

Approximately orthogonality preserving mappings between Hilbert spaces have been studied in [2, 3, 12, 13]. Some authors studied approximately orthogonality preserving mappings on Hilbert C^*-modules over standard C^*-algebras. In fact, ε-orthogonality preserving and (δ, ε)-orthogonality preserving has been explored for A-linear maps on Hilbert C^*-modules over a standard C^*-algebra A, by Ilišević, Turnšek [6], and Moslehian and Zamani [10], respectively.

In this paper, we investigate approximately orthogonality preserving property for mappings on Hilbert C^*-modules over commutative C^*-algebras. In fact, we will use the categorical approach which says that the category of (left) Hilbert C^*-modules over a commutative C^*-algebra $A = C_0(Z)$ is equivalent to the category of continuous fields of Hilbert spaces over the locally compact Hausdorff space Z, see [5, 11].

The following theorem is the main result of this paper and it will be proved by some lemmas in the next sections. We recall that for a given linear operator T on a Hilbert space H the *minimum modulus* $[T]$ of T is defined by
$$[T] = inf\,\{\|Tx\| : \|x\| = 1\} = \sup\{m \geq 0 : m\|x\| \leq \|Tx\|\}.$$

Main Theorem. *Let $\delta, \varepsilon \in [0, 1)$ and $\Psi : \Gamma \to \Gamma'$ be a nonzero $C_0(Z)$-linear (δ, ε)-orthogonality preserving mapping between the continuous fields of Hilbert spaces over a locally compact Hausdorff space Z. Then*

(1) Ψ is injective and continuous,

(2) maps $\varphi(z) = \|\Psi_z\|$ *and* $\phi(z) = [\Psi_z]$ *are bounded on* Z*, where the linear map* Ψ_z *is given by* $\Psi_z(x(z)) = \Psi(x)(z)$*.*

Moreover, for every map $\gamma : Z \to [0, \infty)$ *satisfying* $\phi \leq \gamma \leq \varphi$ *on* Z *and every* $x, y \in \Gamma$*,* $z \in Z$ *we have*

(3) $\frac{1}{\theta}\gamma(z)\|x(z)\| \leq \|\Psi(x)(z)\| \leq \theta\gamma(z)\|x(z)\|,$

(4) $\frac{1}{\theta^2}\gamma^2(z)\langle x, x\rangle(z) \leq \langle \Psi(x), \Psi(x)\rangle(z) \leq \theta^2\gamma^2(z)\langle x, x\rangle(z),$

(5) $$|\langle \Psi(x), \Psi(y)\rangle(z) - \gamma^2(z)\langle x, y\rangle(z)| \leq 4\left(1 - \frac{1}{\theta^2}\right)\min\{\gamma^2(z)\|x\|\|y\|, \|\Psi(x)\|\|\Psi(y)\|\},$$

where $\theta = \theta(\delta, \varepsilon) = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} + 2\varepsilon\sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}}$.

2. Preliminaries

Definition 2.1. *Let* Z *be a locally compact Hausdorff space. Consider* $((H_z)_{z\in Z}, \Gamma)$*, where* $(H_z)_{z\in Z}$ *is a family of Hilbert spaces and* Γ *is a subset of* $\prod_{z\in Z} H_z$*. Also, we set*

$$C_0 - \prod_{z\in Z} H_z = \left\{x \in \prod_{z\in Z} H_z : \ [z \mapsto \|x(z)\|] \in C_0(Z)\right\}.$$

The pair $((H_z)_{z\in Z}, \Gamma)$ *satisfying the following properties is said to be a continuous field of Hilbert spaces.*

1) Γ *is a linear subspace of* $C_0 - \prod_{z\in Z} H_z$*.*
2) *The set* $\{x(z) : x \in \Gamma\}$ *equals to* H_z*, for every* $z \in Z$*.*
3) *If* $x \in C_0 - \prod_{z\in Z} H_z$ *and for every* $z \in Z$ *and every* $\varepsilon > 0$ *there is a* $x' \in \Gamma$ *such that* $\|x(s) - x'(s)\| < \varepsilon$ *in some neighbourhood of* z*, then* $x \in \Gamma$*.*

If there is no confusion, we denote a continuous field of Hilbert spaces $((H_z)_{z\in Z}, \Gamma)$ by Γ.

If Γ is a continuous field of Hilbert spaces, then the function $z \mapsto \langle x(z), y(z)\rangle$ is an element of $C_0(Z)$, for every $x, y \in \Gamma$. In fact, Γ is a (left) Hilbert $C_0(Z)$-module equipped with the following pointwise multiplication and $C_0(Z)$-valued inner product

$$(f \cdot x)(z) = f(z)x(z) \qquad \& \qquad \langle x, y\rangle(z) = \langle x(z), y(z)\rangle,$$

for all $f \in C_0(Z)$, $x, y \in \Gamma$ and $z \in Z$. Moreover, corresponding to every Hilbert $C_0(Z)$-module E, there is a unique continuous field of Hilbert spaces isomorphic to E. The following lemma determines the structure of $C_0(Z)$-linear mappings between continuous fields of Hilbert spaces.

LEMMA 2.2. *Let $\Psi : (\{H_z\}_{z\in Z}, \Gamma) \to (\{K_z\}_{z\in Z}, \Gamma')$ be a nonzero $C_0(Z)$-linear mapping. For every $z \in Z$, the map $\Psi_z : H_z \longrightarrow K_z$ defined by $\Psi_z(x(z)) = (\Psi(x))(z)$ is well-defined and linear. Moreover the $C_0(Z)$-linear map Ψ is bounded if and only if for every $z \in Z$, Ψ_z is linear and bounded and $\sup_{z\in Z} \|\Psi_z\| < \infty$. Indeed, $\|\Psi\| = \sup_{z\in Z} \|\Psi_z\|$.*

PROOF. By [9, Proposition 1.3.10], for every $x \in \Gamma$ and $\alpha \in (0, \frac{1}{2})$ there is $y \in \Gamma$ such that $x = \langle x, x\rangle^\alpha y$ and so $\Psi(x) = \langle x, x\rangle^\alpha \Psi(y)$. Hence, obviously $x(z) = 0$ implies that $\Psi(x)(z) = 0$. The rest of the proof is straightforward. □

The following fact about elements of a continuous field of Hilbert spaces can be concluded by the locally compact version of Tietze extension theorem.

LEMMA 2.3. *Let $z_0 \in Z$ and $y \in \Gamma$. If $y(z_0) \neq 0$, then there is $g \in C_0(Z)$ such that $\|gy\| = \|g(z_0)y(z_0)\| = 1$. Consequently, if $H_{z_0} \neq \{0\}$, then for any unit vector $h \in H_{z_0}$, there is $x \in \Gamma$ such that $\|x\| = 1$ and $x(z_0) = h$.*

PROOF. Let $\lambda = \|y(z_0)\|$. Since $f(z) = \|y(z)\|$ is a member of $C_0(Z)$, then the set $K = \{z \in Z : f(z) \geq \lambda\}$ is compact. Let $g_0(k) = \frac{1}{f(k)}$, for every $k \in K$. Clearly, $g_0 \in C(K)$ and $0 \leq g_0(k) \leq \frac{1}{\lambda}$, for all $k \in K$. By the locally compact version of Tietze extension theorem, there is a $g \in C_0(Z)$ extending g_0 and $0 \leq g(z) \leq \frac{1}{\lambda}$, for all $z \in Z$. Hence, we have

$$\sup_{z\in Z} g(z)f(z) = \sup_{z\in K} g(z)f(z) = 1 = g(z_0)f(z_0).$$

For second part, we note that for any unit vector $h \in H_{z_0}$ there is a $y \in \Gamma$ such that $y(z_0) = h$. Then, by the previous step, there is $g \in C_0(Z)$ such that $\|gy\| = \|g(z_0)y(z_0)\| = 1$ and $0 \leq g(z) \leq 1$, for all $z \in Z$. Let $x = gy$. Then we have $\|x\| = 1$ and $x(z_0) = h$, since $g(z_0) = 1$. □

3. ε-ORTHOGONALITY PRESERVING MAPPINGS

In this section, we prove the main theorem in the case $\delta = 0$. That is, throughout this section, we suppose that $\Psi : \Gamma \to \Gamma'$ is a nonzero $C_0(Z)$-linear ε-orthogonality preserving mapping between the continuous fields of Hilbert spaces over a locally compact Hausdorff space Z.

The first step to prove the main theorem is to observe that for every $z \in Z$, $\Psi_z : H_z \to K_z$ is ε-orthogonality preserving. Hence, some results that hold in the setting of Hilbert spaces and Hilbert C^*-modules over standard C^*-algebras can be generalized to Hilbert $C_0(Z)$-modules.

LEMMA 3.1. *For every $z \in Z$, the linear map $\Psi_z : H_z \longrightarrow K_z$ is ε-orthogonality preserving and so continuous.*

PROOF. Suppose that $z_0 \in Z$, $x, y \in \Gamma$ and also $x(z_0) \perp y(z_0)$. We show that $\Psi_{z_0}(x(z_0)) \perp^\varepsilon \Psi_{z_0}(y(z_0))$. If $x(z_0) = 0$ (or $y(z_0) = 0$), then

$\Psi(z)(z_0) = 0$ (or $\Psi(y)(z_0) = 0$) and so $\Psi_{z_0}(x(z_0)) \perp^{\varepsilon} \Psi_{z_0}(y(z_0))$. Otherwise, let $f = \langle y, x\rangle \in C_0(Z)$ and define $u = \langle y, x\rangle x \in \Gamma$. Obviously, $u - \langle x, x\rangle y \perp x$. Since Ψ is ε-orthogonality preserving, then for every $z \in Z$, we have

$$\begin{aligned} &|\langle \Psi(u)(z) - \|x(z)\|^2 \Psi(y)(z), \Psi(x)(z)\rangle| \\ &\quad \leq \varepsilon \|\Psi(u)(z) - \|x(z)\|^2 \Psi(y)(z)\| \|\Psi(x)(z)\|. \end{aligned}$$

Now, by definition of u, we have

$$\begin{aligned} &|\|\Psi(x)(z)\|^2 f(z) - \|x(z)\|^2 \langle \Psi(y)(z), \Psi(x)(z)\rangle| \\ &\quad \leq \varepsilon \|f(z)\Psi(x)(z) - \|x(z)\|^2 \Psi(y)(z)\| \|\Psi(x)(z)\|. \end{aligned}$$

Since, $f(z_0) = 0$ and also $x(z_0) \neq 0$, then

$$|\langle \Psi(y)(z_0), \Psi(x)(z_0)\rangle| \leq \varepsilon \|\Psi(y)(z_0)\| \|\Psi(x)(z_0)\|.$$

That is, $\Psi_{z_0}(x(z_0)) \perp^{\varepsilon} \Psi_{z_0}(y(z_0))$. ∎

The following corollary, which is essentially the main theorem with $\delta = 0$, follows from the preceding lemma and [10, Theorem 3.6].

COROLLARY 3.2. *For every $x \in \Gamma$ and $z \in Z$, we have*

$$\frac{1}{\theta}\varphi(z)\|x(z)\| \leq \|\Psi(x)(z)\| \leq \theta\phi(z)\|x(z)\|,$$

where $\varphi(z) = \|\Psi_z\|$, $\phi(z) = [\Psi_z]$ and $\theta = \sqrt{\frac{1+\varepsilon}{1-\varepsilon} + 2\varepsilon\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}}$.

Consequently, Ψ is injective and also for every map $\gamma : Z \to [0, \infty)$ satisfying $\phi \leq \gamma \leq \varphi$ on Z and every $x, y \in \Gamma$, $z \in Z$ we have

1. $\frac{1}{\theta}\gamma(z)\|x(z)\| \leq \|\Psi(x)(z)\| \leq \theta\gamma(z)\|x(z)\|$,
2. $\frac{1}{\theta^2}\gamma^2(z)\langle x, x\rangle(z) \leq \langle \Psi(x), \Psi(x)\rangle(z) \leq \theta^2\gamma^2(z)\langle x, x\rangle(z)$,
3. $|\langle \Psi(x), \Psi(y)\rangle(z) - \gamma^2(z)\langle x, y\rangle(z)|$
$$\leq 4\left(1 - \frac{1}{\theta^2}\right)\min\{\gamma^2(z)\|x\|\|y\|, \|\Psi(x)\|\|\Psi(y)\|\}.$$

Also, some other inequalities can be obtained from the main results of [6, 10] and [12].

COROLLARY 3.3. *For every $x \in \Gamma$ and every $z \in Z$,*

1. $\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\|\Psi_z\|\|x(z)\| \leq \|\Psi_z(x(z))\| \leq \|\Psi_z\|\|x(z)\|$,
2. $|\langle \Psi(x), \Psi(y)\rangle(z) - \varphi^2(z)\langle x, y\rangle(z)| \leq \frac{4\varepsilon}{1+\varepsilon}\|\Psi(x)\|\|\Psi(y)\|$.

LEMMA 3.4. *Ψ is continuous. Consequently, the maps φ and ϕ, defined in the previous lemma, are bounded.*

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of Γ converging to zero and $\{\Psi(x_n)\}_{n=1}^{\infty}$ converges to $u \in \Gamma'$. Let $z \in Z$. We show that $u(z) = 0$. Without loss of generality, we can assume that $x_n(z) \neq 0$, for all $n \in \mathbb{N}$. For every $y \in \Gamma$, $\langle y, y\rangle(z)x_n(z) - \langle x_n, y\rangle(z)y(z) \perp y(z)$. Also, by Lemma 3.1, the map $\Psi_z : H_z \to K_z$ is ε-orthogonality preserving. Consequently,

$$\begin{aligned}&\|\langle y, y\rangle(z)\langle \Psi(x_n), \Psi(y)\rangle(z) - \langle x_n, y\rangle(z)\langle \Psi(y), \Psi(y)\rangle(z)\| \\ &\quad \leq \varepsilon \|\langle y, y\rangle(z)\Psi(x_n) - \langle x_n, y\rangle(z)\Psi(y)(z)\|\|\Psi(y)(z)\|.\end{aligned}$$

Since, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \Psi(x_n) = u$, we have

$$\|\langle u, \Psi(y)\rangle(z)\| \leq \varepsilon\|u(z)\|\|\Psi(y)(z)\|.$$

The preceding inequality holds for every $y \in \Gamma$. Hence, for every $n \in \mathbb{N}$ the following holds:

$$\|\langle u, \Psi(x_n)\rangle(z)\| \leq \varepsilon\|u(z)\|\|\Psi(x_n)(z)\|.$$

Now, since $\{\Psi(x_n)\}_{n=1}^{\infty}$ converges to u, we have

$$\|\langle u(z), u(z)\rangle\| \leq \varepsilon\|u(z)\|\|u(z)\|.$$

This implies $u(z) = 0$, because $\varepsilon < 1$. Finally, φ is bounded, since

$$\sup_{z\in Z} \varphi(z) = \sup_{z\in Z} \|\Psi_z\| = \|\Psi\| < \infty.$$

Also, the map ϕ $(\leq \varphi)$ is bounded. ∎

The proofs of Lemmas 3.1 and 3.4, provide a more direct proof for the main result of [7].

Corollary 3.5. [7, Corollary 3.7] *Suppose that E and F are two Hilbert $C_0(Z)$-modules and $\Psi : E \longrightarrow F$ is an orthogonality preserving $C_0(Z)$-module map. Then Ψ is bounded and there exists a bounded nonnegative function φ on Z that is continuous on $Z_E = \{z \in Z : \langle x, x\rangle(z) \neq 0$ for some $x \in E\}$ and satisfies*

$$\langle \Psi(x), \Psi(y)\rangle = \varphi \cdot \langle x, y\rangle,$$

for all $x, y \in E$.

Remark 3.6. In [10], the authors show that if a nonzero module map T is a ε-orthogonality preserving mapping between Hilbert C^*-modules over a standard C^*-algebra, then $[T] > 0$. However, this is in general not true when we deal with commutative C^*-algebras. For instance, let $Z = \mathbb{N}$ and $\Gamma = \Gamma' = C_0(Z)$. Then $\Psi : \Gamma \longrightarrow \Gamma'$ defined by $\Psi((h_n)_{n\in\mathbb{N}}) = (\frac{1}{n}h_n)_{n\in\mathbb{N}}$ is a nonzero $C_0(Z)$-linear orthogonality preserving mapping, but $[\Psi] = 0$.

4. Approximately orthogonality preserving mappings

In this section, we prove the main theorem in the case $\delta \neq 0$. That is, throughout this section, we assume that the nonzero $C_0(Z)$-linear map $\Psi : \Gamma \to \Gamma'$ is a (δ, ε)-orthogonality preserving mapping, for some $\delta \in (0, 1)$.

At first we note that Ψ is also ε-orthogonality preserving mapping. Hence Ψ is continuous and injective, by the previous section.

Similar to the previous section, we show that for every $z \in Z$, Ψ_z is an approximately orthogonality preserving mappings.

Lemma 4.1. *Suppose that V is an open subset of Z and $x, y \in \Gamma$. If $x(z) \perp^{\delta} y(z)$ for some $\delta \geq 0$ and every $z \in V$, then $\Psi(x)(z) \perp^{\varepsilon} \Psi(y)(z)$, for all $z \in V$.*

Proof. Let z_0 be an arbitrary element of V. We can assume that $\Psi(x)(z_0) \neq 0$ and $\Psi(y)(z_0) \neq 0$. By Lemma 2.3, there are $g, h \in C_0(Z)$ such that

$$\|g\Psi(x)\| = \|g(z_0)\Psi(x)(z_0)\| = 1 \quad \& \quad \|h\Psi(y)\| = \|h(z_0)\Psi(y)(z_0)\| = 1.$$

Also, by the Urysohn's lemma there exists a $f \in C_0(Z)$ such that $\|f\| = 1$, $f(z_0) = 1$ and $f|_{V^c} = 0$.

The assumption of $x(\cdot) \perp^{\delta} y(\cdot)$ on V, yields that $fgx(z) \perp^{\delta} hy(z)$, for all $z \in Z$, i. e., $fgx \perp^{\delta} hy$ in Γ. Then we have

$$\|\langle \Psi(fgx), \Psi(hy)\rangle\| \leq \varepsilon \|\Psi(fgx)\| \|\Psi(hy)\|,$$

and so

$$\begin{aligned} &|f(z_0)||g(z_0)||h(z_0)||\langle \Psi(x)(z_0), \Psi(y)(z_0)\rangle| \\ &\qquad \leq \varepsilon \|f\| \|g\Psi(x)\| \|h\Psi(y)\| = \varepsilon |g(z_0)| \|\Psi(x)(z_0)\| |h(z_0)| \|\Psi(y)(z_0)\|. \end{aligned}$$

Consequently,

$$|\langle \Psi(x)(z_0), \Psi(y)(z_0)\rangle| \leq \varepsilon \|\Psi(x)(z_0)\| \|\Psi(y)(z_0)\|,$$

which is the desired result. ∎

Lemma 4.2. *For every $z \in Z$ and for every $\delta' < \delta$, the linear map $\Psi_z : H_z \longrightarrow K_z$ is (δ', ε)- orthogonality preserving and so continuous.*

Proof. Let $z_0 \in Z$ and $x, y \in \Gamma$. Suppose that $x(z_0) \perp^{\delta'} y(z_0)$ and also $x(z_0) \neq 0$ and $y(z_0) \neq 0$. In other words, $|\langle x(z_0), y(z_0)\rangle| \leq \delta' \|x(z_0)\| \|y(z_0)\|$. Since $\delta' < \delta$, we have

$$|\langle x(z_0), y(z_0)\rangle| < \delta \|x(z_0)\| \|y(z_0)\|.$$

Now, according to the continuity, there is some open neighborhood V of z_0 such that for every $z \in V$,

$$|\langle x(z), y(z)\rangle| < \delta \|x(z)\| \|y(z)\|.$$

Consequently, for every $z \in V$, $x(z) \perp^{\delta} y(z)$. Hence, Lemma 4.1 yields that for every $z \in V$, $\Psi(x)(z) \perp^{\varepsilon} \Psi(y)(z)$. In particular, $\Psi_{z_0}(x(z_0)) \perp^{\varepsilon} \Psi_{z_0}(y(z_0))$. Therefore, the linear map Ψ_{z_0} is (δ', ε)-orthogonality preserving and so continuous. □

PROOF OF MAIN THEOREM. As mentioned above, the map Ψ is injective and continuous. On the other hand, we have $\lim_{\delta' \to \delta^-} \theta(\delta', \varepsilon) = \theta(\delta, \varepsilon)$. Hence, all the inequalities in the main theorem hold by the above lemma and [10, Theorem 3.6]. Then, the proof is complete. □

The following result follows from Lemma 4.2 and [13, Theorem 3.4].

COROLLARY 4.3. *We have $\delta \leq \varepsilon$. Also, the following statements hold:*

1. for every $z \in Z$, $\eta \|\Psi_z\| \leq [\Psi_z]$;

2. for every $x \in \Gamma$ and $z \in Z$,

$$\eta \|\Psi_z\| \|x(z)\| \leq \|\Psi(x)(z)\| \leq \|\Psi_z\| \|x(z)\|;$$

where $\eta = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \sqrt{\frac{1+\delta}{1-\delta}}$.

ACKNOWLEDGEMENTS.

The first and second authors were partly supported by grants from IPM, No. 1400460119 and 1400460034, respectively.

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Received: 17.7.2021.
Revised: 8.12.2021.