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RECONSTRUCTION PROPERTIES OF SELECTIVE RIPS COMPLEXES

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Abstract. Selective Rips complexes associated to two parameters are certain subcomplexes of Rips complexes consisting of thin simplices. They are designed to detect more closed geodesics than their Rips counterparts. In this paper we introduce a general definition of selective Rips complexes with countably many parameters and prove basic reconstruction properties associated with them. In particular, we prove that selective Rips complexes of a closed Riemannian manifold X attain the homotopy type of X at small scales. We also completely classify the resulting persistent fundamental group and 1-dimensional persistent homology.

1. Introduction

Rips complexes, sometimes also called Vietoris-Rips complexes, are one of the most widespread constructions of simplicial complexes built upon a metric space. Originally introduced by Vietoris in [10], they have been used to approximate spaces in order to define a cohomology theory [7], study groups [6], and treat large scale structures [4]. Due to their simplicity they provide a prime construction of filtrations in the context of persistent homology and applied topology [5]. They are known to encode geometric properties of the underlying space although the treatment of the precise nature of this encoding has only recently been expedited [3, 1, 2, 11, 12, 15, 8].

Given a geodesic space X, a geodesic circle in X is a geodesic determined by an isometric embedding $(S^1, d_g) \hookrightarrow X$, where d_g is a geodesic metric. Recent results [12, 15] show that geodesic circles can be detected by persistent homology constructed via Rips complexes in dimensions 1, 2, and above.

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However, detections in dimensions 2 and above require geodesic circles to have wide neighborhoods adhering to certain geometric conditions. This technical requirement has been circumvented in [13] which demonstrated that arbitrarily small geodesically convex neighborhoods suffice to detect a geodesic circle using selective Rips complexes with two parameters. In particular, selective Rips complexes with two parameters are a modification of Rips complexes designed to detect many (in some cases all [14]) geodesic circles.

In this paper we study basic reconstruction properties of selective Rips complexes. We first provide a more general definition of selective Rips complexes, thus formalizing a construction with "thin" simplices. We then prove basic reconstruction results.

1. A functorial reconstruction theorem for closed connected Riemannian manifolds (Theorem 3.11). When restricted to Rips complexes, our approach provides a novel proof of the reconstruction result for Rips complexes [7].
2. A complete classification of 1-dimensional persistence of geodesic spaces in the last section. Using ideas of [12] we prove that 1-dimensional persistence of Rips and selective Rips complexes are isomorphic up to reparameterization. This is in sharp contrast with higher-dimensional persistence [13].

To summarize, we prove that the reconstruction properties of selective Rips complexes closely resemble those of Rips complexes.

Selective Rips complexes have a potential to act as a finer yet still easily computable version of Rips complexes. In [13] it was demonstrated that selective Rips complexes detect some geodesic circles that are undetected by Rips complexes. As our understanding of information encoded by Rips complexes (and the corresponding persistent homology) grows [11, 12, 15, 13, 14, 8, 2], we expect the variants using selective Rips complexes to encode more information of the same type. As such, the parameters of selective Rips complexes would allow us to control the level of details extracted by, for example, persistent homology. Such a control is a beneficial in theoretical and, above all, practical applications.

2. Selective Rips complex

In this section we define selective Rips complexes and prove their homotopy types are preserved by crushings of the underlying space. We first recall a definition of (open) Rips complexes that will be used here. Given metric space (X, d) and a scale $r > 0$, the Rips complex, $\mathrm{Rips}(X, r)$, is an abstract simplicial complex with vertex set X defined by the following rule: a finite $\sigma \subseteq X$ is a simplex if $\mathrm{Diam}(\sigma) < r$.

Definition 2.1. *Let X be a metric space and $r_1 \geq r_2 \geq \cdots$ positive scales forming a sequence $\tilde{r} = (r_1, r_2, \ldots)$. The* ***selective Rips complex***

$\mathrm{sRips}(X; r_1, r_2, \ldots) = \mathrm{sRips}(X; \tilde{r})$, *is an abstract simplicial complex defined by the following rule: a finite* $\sigma \subseteq X$ *is a simplex if for each positive integer* i, *the set* σ *can be expressed as a union of* i*-many sets of diameter less than* r_i.

Definition 2.1 implies that $\mathrm{sRips}(X; r_1, r_2, \ldots)$ is a subcomplex of $\mathrm{Rips}(X, r_1)$ and that $\mathrm{sRips}(X; r, r, r, \ldots) = \mathrm{Rips}(X, r)$. Furthermore, increasing the sequence of scales pointwise results in a larger selective Rips complex. The following example demonstrates a difference between selective Rips complex and Rips complex.

EXAMPLE 2.2. Let $r_1 > r_2 > 0$. Let $x, y, z \in X$ be such points that pairwise distances between points are smaller than r_1 and greater than r_2. Note that the 2-simplex $\sigma = \{x, y, z\} \in \mathrm{Rips}(X; r_1)$. On the other hand σ is not contained in $\mathrm{sRips}(X; r_1, r_2)$.

Through this paper we also use the standard notation for homotopy equivalence ($\simeq$) and isomorphism ($\cong$).

LEMMA 2.3. *Let* X *be a finite metric space and let* $r_1 \geq r_2 \geq \cdots$ *be positive scales. Then there exists* $\delta > 0$ *and a sequence of positive scales* $r'_1 \geq r'_2 \geq \cdots$ *such that:*

- *for* $i = 1, 2, \ldots, |X|$ *we have* $r'_i = r_i - 2\delta$, *and*
- id_X *induces a simplicial isomorphism*

$$\mathrm{sRips}(X; r'_1, r'_2, \ldots) \to \mathrm{sRips}(X; r_1, r_2, \ldots).$$

PROOF. Let

$$m_i = \max\{\mathrm{Diam}(A) \mid A \subseteq X \text{ and } \mathrm{Diam}(A) < r_i\}$$

for each $i = 1, 2, \ldots, |X|$. Since each point is of diameter zero, m_i are well defined non-negative numbers. Since X is finite, there exists $\delta > 0$ such that $m_i < r_i - 2\delta, \forall i$. For indices $i > |X|$ we can choose r'_i to be any decreasing sequence of positive scales with the initial term below $r'_{|X|}$.

Let K be a q-simplex in $\mathrm{sRips}(X; r_1, r_2, \ldots)$. By definition of selective Rips complex for each $i = 1, \ldots, q$ there exists $U_1, \ldots, U_i \subseteq K$ such that $U_1, \ldots, U_i = K$, where $\mathrm{Diam}(U_k) < r_i$ for each $k = 1, \ldots, i$. From definition of m_i it follows that $\mathrm{Diam}(U_k) < m_i$ for each $k = 1, \ldots, i$ and since $m_i < r_i - 2\delta$ it follows that $\mathrm{Diam}(U_k) < r_i - 2\delta$. Finally, $K \in \mathrm{sRips}(X; r'_1, r'_2, \ldots)$. ∎

Next, we define a crushing of a metric space introduced by Hausmann [7].

DEFINITION 2.4. *Let* X *be a metric space and* $A \subseteq X$. *A continuous map* $F\colon X \times [0, 1] \to X$ *satisfying*

1. $F(x, 1) = x$, $F(x, 0) \in A$ *and* $F(a, t) = a,\ \forall a \in A, t \in [0, 1]$;
2. $d(F(x, u), F(y, u)) \leq d(F(x, t), F(y, t)), \forall u \leq t, x, y \in X$.

*is called a **crushing** (or deformation contraction in* [15]*) from X onto A. We say that a metric space X is **crushable** if there is a crushing X onto a point.*

Recall that two maps $f, g : K_1 \to K_2$ between simplicial complexes are **contiguous** if for given a simplex $\sigma \in K_1$, $f(\sigma) \cup g(\sigma)$ is contained in a simplex in K_2. Note that contiguous maps are homotopic (see [9, p. 130]).

PROPOSITION 2.5. *Let X be a metric space admitting a chrushing onto a subspace $A \subseteq X$. Then for any positive scales $r_1 \geq r_2 \geq r_3 \geq \cdots$, the inclusion* $\mathrm{sRips}(A; r_1, r_2, \ldots) \to \mathrm{sRips}(X; r_1, r_2, \ldots)$ *is a homotopy equivalence.*

PROOF. The proof is an adaptation of the proof of [7, Proposition 2.2]. Let K be a finite simplicial complex and L its subcomplex. Moreover, let

$$h \colon (K, L) \to (\mathrm{sRips}(X; r_1, r_2, \ldots), \mathrm{sRips}(A; r_1, r_2, \ldots))$$

be a continuous map. By the Whitehead theorem it is enough to show that such a function is homotopic (*rel* L) to a map sending K into $\mathrm{sRips}(A; r_1, r_2, \ldots)$. By simplicial approximation we may, after replacing K and L with one of their barycentric subdivision, assume h is a simplicial map. Let $\overline{K} = h(K^0)$ and $\overline{L} = h(L^0)$ be finite subsets of X and A. Let N denote the number of points in $\overline{K}$. Since $\overline{K}$ and $\overline{L}$ are finite, Lemma 2.3 implies there exists:

- $\delta > 0$ and
- a sequence of positive scales $r'_1 \geq r'_2 \geq \cdots$ satisfying:
  - for $i = 1, 2, \ldots, |X|$ we have $r'_i = r_i - 2\delta$, and
  - id_X induces a simplicial inclusion
  $$\mathrm{sRips}(X; r'_1, r'_2, \ldots) \to \mathrm{sRips}(X; r_1, r_2, \ldots),$$
  such that

$$(\mathrm{sRips}(\overline{K}; r'_1, r'_2, \ldots), \mathrm{sRips}(\overline{L}; r'_1, r'_2, \ldots)) = (\mathrm{sRips}(\overline{K}; r_1, r_2, \ldots), \mathrm{sRips}(\overline{L}; r_1, r_2, \ldots)).$$

We have factorization $h = j \circ \overline{h}$, where

$$\overline{h} : (K, L) \to \left(\mathrm{sRips}\left(\overline{K}; r'_1, r'_2, \ldots\right), \mathrm{sRips}\left(\overline{L}; r'_1, r'_2, \ldots\right)\right)$$

is induced by h and

$$j : \left(\mathrm{sRips}\left(\overline{K}; r_1 - 2\delta, \ldots\right), \mathrm{sRips}\left(\overline{L}; r_1 - 2\delta, \ldots\right)\right) \to \left(\mathrm{sRips}\left(X; r_1, r_2, \ldots\right), \mathrm{sRips}\left(A; r_1, r_2, \ldots\right)\right)$$

is the obvious inclusion.

Let $F : X \times [0, 1] \to X$ be a crushing of X onto A. Let p be a positive integer so that for all $x \in \overline{K} \setminus \overline{L}$ and all $k = 0, 1, \ldots, p-1$ we have

$$d\left(F\left(x, \frac{k}{p}\right), F\left(x, \frac{k+1}{p}\right)\right) < \min\{\delta, r'_{2N}\}.$$

The map $x \mapsto F\left(x, \frac{k}{p}\right)$ induces then a simplicial map

$$f^k : \text{sRips}\left(\overline{K}; r_1', r_2', \ldots\right) \to \text{sRips}\left(X; r_1, r_2, \ldots\right)$$

such that $f^p\big|_{\text{sRips}\left(\overline{K}; r_1', \ldots\right)}$ and $f^k\big|_{\text{sRips}\left(\overline{L}; r_1', \ldots\right)}$ are identities on subcomplexes of sRips $(X; r_1, r_2, \ldots)$ for all k, and $f^0\left(\text{sRips}\left(\overline{K}; r_1', \ldots\right)\right) \subseteq \text{sRips}\left(A; r_1, r_2, \ldots\right)$.

Next we will prove that for all k we have $f^k \simeq f^{k+1}$ by showing the the mentioned pair of maps are contiguous. Let

$$W = \{x_0, \ldots, x_q\} \in \text{sRips}\left(\overline{K}; r_1 - 2\delta, \ldots\right).$$

From Definition 2.4 of crushing if follows that $f^k(W) \in \text{sRips}\left(\overline{K}; r_1 - 2\delta, \ldots\right)$ and $f^{k+1}(W) \in \text{sRips}\left(\overline{K}; r_1 - 2\delta, \ldots\right)$. To show that

$$U = \left\{f^k(x_1), \ldots, f^k(x_q), f^{k+1}(x_1), \ldots, f^{k+1}(x_q)\right\} \in \text{sRips}\left(X; r_1, r_2, \ldots\right)$$

we define the following clusters.

- For $i = 1, \ldots, q$ we use clustering of $f^k(W)$. In particularly let $U_1', \ldots, U_i'$ be a clustering of $f^k(W)$ with diameters less than $r_i - 2\delta$. Note that $d\left(f^{k+1}(x_n), f^k(x_n)\right) < \delta$. It is clear that we can construct $U_1, \ldots, U_i$ on the following way. For all x_n: if $f^k(x_n) \in U_j'$, then $\{f^k(x_n), f^{k+1}(x_n)\} \subseteq U_j$. It follows that diameter of each U_j is less than r_i.
- For $i = q+1, \ldots, 2q$ we define clustering on the following way:
    1. $U_n = \{f^k(x_n)\}$ for $n = 1, \ldots, i-q$,
    2. $U_n = \{f^k(x_n), f^{k+1}(x_n)\}$ for $n = i-q+1, \ldots, q$,
    3. $U_{q+n} = \{f^{k+1}(x_n)\}$ for $n = 1, \ldots, i-q$.

  Note that $\text{Diam}(U_j) < r_{2q} \leq r_i$ for all $j = 1, \ldots, i$

We have shown that $U \in \text{sRips}\left(X; r_1, r_2, \ldots\right)$ which implies that f^k and f^{k+1} are contiguous. Therefore $f^p = j$ and f^0 are in the same contiguity class $\left(rel \text{ sRips}\left(\overline{L}; r_1 - 2\delta, \ldots\right)\right)$. This implies that j and f^0 are homotopic $\left(rel \text{ sRips}\left(\overline{L}; r_1 - 2\delta, \ldots\right)\right)$. After composing with $\overline{h}$, this proves that h is homotopic $(rel\ L)$ to $\overline{h} \circ f^0$ which sends K to sRips $(A; r_1, r_2, \ldots)$. ∎

Corollary 2.6. *Let X be a crushable metric space, then for each choice of positive scales $r_1 \geq r_2 \geq r_3 \geq \cdots$, the space* sRips $(X; r_1, r_2, \ldots)$ *is contractible.*

3. Selective Rips complex of a closed Riemannian manifold

In this section we prove the main reconstruction result for selective Rips complexes built upon a geodesic space. Our argument is based on the use of the Nerve Theorem.

A cover $\mathcal{U}$ of a metric space is good if each finite intersection of elements from $\mathcal{U}$ is either empty or contractible. The nerve of $\mathcal{U}$ is the simplicial complex Nerve($\mathcal{U}$) defined by the following declarations:

- Vertices are elements of $\mathcal{U}$.
- σ is a simplex iff $\bigcap_{U\in\sigma} U \neq \emptyset$.

For our purposes we will be using the Functorial Nerve Theorem as presented in [11].

Theorem 3.1 (Functorial Nerve Theorem, an adaptation of Lemma 5.1 of [11]). *Suppose $\mathcal{U}$ is a good open cover of a metric space X. Then $X \simeq \mathrm{Nerve}(\mathcal{U})$.*

If $\mathcal{V}$ is another good open cover of $Y \subset X$ subordinated to $\mathcal{U}$ (i.e., if $\forall V \in \mathcal{V}\ \exists U_V \in \mathcal{U} : V \subseteq U_V$), then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & \mathrm{Nerve}(\mathcal{U}) \\ \uparrow & & \uparrow \\ Y & \xrightarrow{\simeq} & \mathrm{Nerve}(\mathcal{V}) \end{array}$$

commutes up to homotopy, with $\mathrm{Nerve}(\mathcal{V}) \to \mathrm{Nerve}(\mathcal{U})$ being the simplicial map mapping $V \mapsto U_V$ and the horizontal homotopy equivalences arising from partitions of unity corresponding to the involved covers as in the previous paragraph.

Given a metric space X, $x \in X$, and $q > 0$, let $N(x, q)$ denote the open q-neighborhood (equivalently, open q-ball) of x. A metric space (X, d) is geodesic, if for each $x, y \in X$ there exists a path, called **geodesic**, from x to y of length $d(x, y)$. In particular, for each $x, y \in X$ there exists an isometric embedding of $[0, d(x, y)]$ into X with $0 \mapsto x$ and $d(x, y) \mapsto y$. When necessary we will consider $\bullet \in X$ to be the basepoint of X.

Definition 3.2 (Hausmann [7]). *Let X be a geodesic space. Define $r(X) \geq 0$ as the least upper bound of the set of real numbers r satisfying the following conditions:*

1. *For all $x, y \in X$ such that $d(x, y) < 2r$ there exists a unique geodesic joining x to y of length $2r$.*
2. *Let $x, y, z, u \in X$ with $d(x, y) < r$, $d(u, x) < r$, $d(u, y) < r$ and z be a point on the shortest geodesic joining x to y. Then $d(u, z) \leq \max\{d(u, x), d(u, y)\}$.*
3. *If γ and γ' are arc-length parametrized geodesic such that $\gamma(0) = \gamma'(0)$ and if $0 \leq s,\ s' < r$ and $0 \leq t < 1$, then $d(\gamma(ts), \gamma'(ts')) \leq d(\gamma(s), \gamma'(s'))$.*

As was stated in [7], $r(X) > 0$ if X is a Riemannian manifold that admits a strictly positive radius and an upper bound on its sectional curvature. In particular, each compact Riemannian manifold has $r(X) > 0$. We will denote $r(X)$ by ρ, and we call it a **star radius**.

For the following results we fix a geodesic space X with star radius $\rho > 0$ and scales $r_1, r_2, \dots$ with $\rho/2 \geq r_1 \geq r_2 \geq r_3 \geq \cdots$. Let $A \subseteq X$. We say that A is a **star-shaped** with center at $x_0 \in A$ if for all $x \in A$ the geodesic from x_0 to x is in A.

LEMMA 3.3. *Let $x_0 \in X$ and let $A \subseteq N(x_0, \rho)$ be a star-shaped subset centered at $x_0 \in A$. Then A is crushable.*

PROOF. Since $d(x_0, x) < \rho$ Definition 3.2 (1.) implies that for each $x \in A$ the geodesic joining x and x_0 is unique. Note that the mentioned geodesic is a part of A because it is star-shaped at x_0. By Definition 3.2 (3.) the homotopy sliding each point of A towards x_0 along the unique geodesic is a crushing. □

LEMMA 3.4. *Let $z \in X$ and assume $A \subseteq N(x, \rho/2)$ is star-shaped with center at x. Then* $\mathrm{sRips}\,(A; r_1, r_2, \dots)$ *is contractible for each choice of positive scales $r_1 \geq r_2 \geq r_3 \geq \cdots$.*

PROOF. Let $a \in A$ and note that $A \subset N(a, \rho)$. By Lemma 3.3, A is crushable to a. The statement of the lemma follows from Corollary 2.6. □

LEMMA 3.5. *For each $\alpha \leq \rho/2$ the collection $\mathcal{U} = \{N(x, \alpha) \mid x \in X\}$ is a good cover of X.*

PROOF. Suppose there exists $x \in \bigcap_{i=1}^k N(x_i, \rho/2)$ for some $x_i \in X$. Observe that

$$\bigcap_{i=1}^k N(x_i, \rho/2) \subseteq N(x, \rho)$$

is geodesically convex and crushable by Lemma 3.3, thus contractible. □

LEMMA 3.6. *For each collection of subsets $A_1, A_2, \dots, A_k \subset X$ we have*

$$\bigcap_{i=1}^k \mathrm{sRips}\,(A_i; r_1, r_2, \dots) = \mathrm{sRips}\left(\bigcap_{i=1}^k A_i; r_1, r_2, \dots\right).$$

PROOF. Let σ be a simplex in $\mathrm{sRips}\,(X; r_1, r_2, \dots)$. Note that

$$\sigma \subset A_i,\ \forall i \Longleftrightarrow \sigma \subset \bigcap_{i=1}^k A_i.$$

□

LEMMA 3.7. $\mathcal{W}' = \{\mathrm{sRips}\,(N(x, \rho/2); r_1, r_2, \dots) \mid x \in X\}$ *is a cover of* $\mathrm{sRips}\,(X; r_1, r_2, \dots)$.

PROOF. Let $\sigma = \{x_0, x_1, \dots, x_q\}$ be a simplex in $\mathrm{sRips}\,(X; r_1, r_2, \dots)$. Then for each $i = 0, \dots, q$ containment $x_i \in N(x_0, r_1)$ holds and since $\rho/2 \geq r_1$, we have $x_i \in N(x_0, \rho/2)$. It follows that $\sigma \in \mathrm{sRips}\,(N(x_0, \rho/2); r_1, r_2, \dots)$. □

Remark 3.8 (Open covers). Let

$$\mathcal{W}' = \{\text{sRips}\,(N(x,\rho/2);r_1,r_2,\ldots)\ \mid x \in X\}$$

be a (closed) cover of sRips $(X;r_1,r_2,\ldots)$. We next describe how to slightly thicken the elements of $\mathcal{W}'$ to obtain an open cover $\mathcal{W}$ with the same intersection pattern, as was described in [11, Theorem 5.2]. Simplicial complex sRips $(X;r_1,r_2,\ldots)$ can be equipped with a metric d_{ℓ_1} arising from the ℓ_1 metric on the barycentric coordinates, see [11] for details. This metric simplicial complex turns out to be homotopy equivalent to the standard simplicial complex (weak) topology. In this metric each simplex of $\sigma \in$ sRips $(X;r_1,r_2,\ldots)$ is isometric to the standard simplex. Let $w'_x =$ sRips $(N(x,\rho/2);r_1,r_2,\ldots)$. We enlarge each w'_x to the open neighborhood $w_x = N(w'_x, 0.1)$ so that for each simplex $\sigma \in$ sRips $(X;r_1,r_2,\ldots)$

$$w_x \cap \sigma = N(w'_x, 0.1) \cap \sigma,$$

i.e., we thicken the sets by 0.1 in each adjacent simplex. Sets w_x are open in metric and weak topology. Note that for each finite $A \subset X$ the intersection $\bigcap_{x\in A} w_x$ deformation contracts to $\bigcap_{x\in A} w'_x$. Furthermore, $\bigcap_{x\in A} w_x = \emptyset$ if and only if $\bigcap_{x\in A} w'_x = \emptyset$. It follows that $\mathcal{W} = \{w_x \mid x \in X\}$ is an open cover of sRips $(X;r_1,r_2,\ldots)$ in d_{ℓ_1} and $\text{Nerve}(\mathcal{W}) \cong \text{Nerve}(\mathcal{W}')$.

Lemma 3.9. *An open cover $\mathcal{W}$ of* sRips $(X;r_1,r_2,\ldots)$ *described in 3.8 is a good open cover in d_{ℓ_1}.*

Proof. Previous statements imply that $\mathcal{W}$ is an open (Remark 3.8) good (Lemma 3.4, Lemma 3.6, and Remark 3.8) cover (Lemma 3.7). ◻

Proposition 3.10. *Let $\mathcal{U} = \{N(x,\rho/2) \mid x \in X\}$ and let $\mathcal{W}$ be an open cover of* sRips $(X;r_1,r_2,\ldots)$ *described in 3.8. Then*

$$\text{Nerve}(\mathcal{U}) \cong \text{Nerve}(\mathcal{W}).$$

Proof. For each $x \in X$ we map $N(x,\rho/2)$ to w_x. Let

$$\sigma = \{N(x_0,\rho/2),\ldots,N(x_q,\rho/2)\}$$

be a simplex in $\text{Nerve}(\mathcal{U})$. We map such a simplex to

$$\{\text{sRips}\,(N(x_i,\rho);r_1,\ldots)\,,\ldots,\text{sRips}\,(N(x_i,\rho/2);r_1,\ldots)\}$$

in $\text{Nerve}(\mathcal{W})$. One can easily see that such defined map is an isomorphism by Remark 3.8 and Lemma 3.6. ◻

The following result is generalization of Hausmann's Theorem [7, Theorem 3.5] to selective Rips complexes and a functorial setting.

Theorem 3.11. *Let X be a geodesic space with star radius $\rho > 0$ and let $r_1, r_2, \ldots$ be a sequence of scales where $\rho/2 \geq r_1 \geq r_2 \geq r_3 \geq \cdots$. Then $X \simeq$* sRips $(X;r_1,r_2,\ldots)$.

Furthermore, if $\hat{r}_1 \geq \hat{r}_2, \dots$ *is another sequence of scales with* $\hat{r}_i \leq r_i, \forall i$, *then the natural inclusion*

$$\mathrm{sRips}\,(X; \hat{r}_1, \hat{r}_2, \dots) \hookrightarrow \mathrm{sRips}\,(X; r_1, r_2, \dots)$$

is a homotopy equivalence.

Proof. By the Nerve theorem, Lemma 3.5, Proposition 3.10, Remark 3.8, and Lemma 3.9 we have $X \simeq \mathrm{Nerve}(\mathcal{U}) \cong \mathrm{Nerve}(\mathcal{W}) \simeq \mathrm{sRips}\,(X; r_1, r_2, \dots)$, where $\mathcal{U}$ is the open cover of X from Lemma 3.5 and $\mathcal{W}$ is the open cover of $\mathrm{sRips}\,(X; r_1, r_2, \dots)$ from Remark 3.8. The last homotopy equivalence uses the fact that the weak topology and d_{ℓ_1} metric on $\mathrm{sRips}\,(X; r_1, r_2, \dots)$ result in homotopy equivalent spaces.

The second part follows from Theorem 3.1 and the fact that the argument of the previous paragraph for the smaller sequence of scales generates covers $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{W}}$, subordinated to $\mathcal{U}$ and $\mathcal{W}$. In fact, $\widehat{\mathcal{U}} = \mathcal{U}$ and each $w_x \in \mathcal{W}$ of Remark 3.8 corresponds to analogously defined subset $\hat{w}_x$. Theorem 3.1 now implies that the following diagram commutes up to homotopy with the vertical map $\mathrm{Nerve}(\widehat{\mathcal{U}}) \to \mathrm{Nerve}(\mathcal{U})$ being identity and the vertical map $\mathrm{Nerve}(\widehat{\mathcal{W}}) \to \mathrm{Nerve}(\mathcal{W})$ mapping $w_x \mapsto \hat{w}_x$:

$$\begin{array}{ccccccc}
X & \xrightarrow{\simeq} & \mathrm{Nerve}(\mathcal{U}) & \xleftrightarrow{\cong} & \mathrm{Nerve}(\mathcal{W}) & \xleftarrow{\simeq} & \mathrm{sRips}\,(X; r_1, r_2, \dots) \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & \hookuparrow \\
X & \xrightarrow{\simeq} & \mathrm{Nerve}(\widehat{\mathcal{U}}) & \xleftrightarrow{\cong} & \mathrm{Nerve}(\widehat{\mathcal{W}}) & \xleftarrow{\simeq} & \mathrm{sRips}\,(X; \hat{r}_1, \hat{r}_2, \dots)
\end{array}$$

□

4. One-dimensional persistence

In this section we completely classify one-dimensional persistence of geodesic spaces arising from selective Rips complexes. In particular, we describe the fundamental groups of selective Rips complexes of a geodesic space along with the inclusion-induced maps corresponding to increases in the scale parameters. We then derive analogous results for first homology groups with arbitrary coefficients. The results are obtained by adapting approach of [12] and utilizing some of the results therein.

Throughout this section X is a geodesic space with basepoint $\bullet \in X$, $\tilde{r}$ is a sequence of positive scales $r_1 \geq r_2 \geq r_3 \geq \cdots$ potentially converging to 0, and we define $r = r_1$. The concatenation of loops or paths α and β is denoted by $\alpha * \beta$. We naturally require that the endpoint of α is the initial point of β. For a path $\alpha : [0, a] \to X$, the inverse path $\alpha^- : [0, a] \to X$ is defined by $\alpha^-(t) = \alpha(a - t)$.

Definition 4.1. *We define*

1. *r-**loop** L: a simplicial loop in* $\mathrm{sRips}(X; r_1, r_2, \ldots)$ *considered as a sequence of points* $(x_0, x_1, \ldots, x_k, x_{k+1} = x_0)$ *in* X *with* $d(x_i, x_{i+1}) < r$, $\forall i \in \{0, 1, \ldots, k\}$*;*
2. ***filling** of an r-loop L: any loop in X obtained from L by connecting x_i to x_{i+1} by a geodesic for all $i \in \{0, 1, \ldots, k\}$;*
3. *size(L)=$|L| = k + 1$;*
4. *r-**sample** of a loop $\alpha\colon [0, a] \to X$: a choice of* $0 \le t_0 \le t_1 \le \cdots \le t_m \le a$ *with* $\mathrm{Diam}\,(\alpha([t_i, t_{i+1}])) < r$, $\forall i \in \{0, 1, \ldots, m-1\}$ *and* $\mathrm{Diam}\,(\alpha([0, t_0] \cup [t_m, a])) < r$. *By an r-sample we usually consider the introduced r-loop $(\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_m), \alpha(t_0))$. If α is based at point, we will assume $t_0 = 0$.*

*An r-loop is $\tilde{r}$-**null** if it is contractible in* $\mathrm{sRips}(X; \tilde{r})$. *Two r-loops are $\tilde{r}$-**homotopic**, if they are homotopic in* $\mathrm{sRips}(X; \tilde{r})$. *The corresponding simplicial homotopy in* $\mathrm{sRips}(X; \tilde{r})$ *is referred as $\tilde{r}$-**homotopy**. The concatenation $L * L'$ of r-loops L and L' is defined in the obvious way by concatenating the defining sequences and note that the concatenation of fillings of r-loops is a filling of the concatenation.*

Lemma 4.2. *Let $\alpha : [0, a] \to X$ be a loop in a geodesic space X. Then any two r-samples of α are $\tilde{r}$-homotopic.*

Proof. It suffices to show that any given r-sample $0 \le t_0 \le t_1 \le \cdots \le t_m \le a$ is $\tilde{r}$-homotopic to any r_2-sample of α containing all t_i. Indeed, given two r-samples it is clear that there exists an r_2-sample containing both of the r-samples as subsequences.

We provide a formal proof by induction. Suppose an r-sample is given as $0 \le t_0 \le t_1 \le \cdots \le t_m \le a$. The inductive step is performed by adding a point $\tau \in [0, a]$ with $|\tau - t_i| < r_2$ for some t_i. The resulting r-sample is r_2-homotopic to the original r-sample by the following argument:

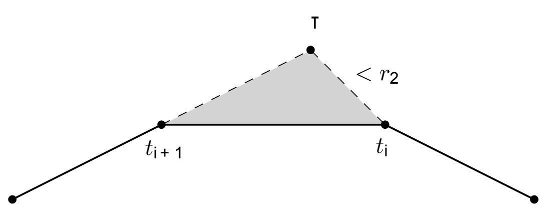

Figure 1. An excerpt of $\tilde{r}$-homotopy of Lemma 4.2.

- if $\tau \in (t_i, t_{i+1})$ then the r_2-homotopy between the changed segment $\{t_i, t_{i+1}\}$ and its new version $\{t_i, \tau, t_{i+1}\}$ is given by the triangle

$[t_i, \tau, t_{i+1}]$ in $\mathrm{sRips}(X;\tilde{r})$, see Figure 1. In a similar way we treat the case $\tau \in (t_{i-1}, t_i)$.
- If $\tau \in [0, t_0] \cup [t_m, a]$ the r_2-homotopy is given by the triangle $[t_m, \tau, t_0]$ in $\mathrm{sRips}(X;\tilde{r})$.

∎

PROPOSITION 4.3. *Let $\alpha : [0, a] \to X$ be a loop in a geodesic space X of length less than $2r_1 + r_2$. Then any r-sample of α is $\tilde{r}$-null.*

PROOF. There exists an r-sample $0 = t_0 < t_1 < t_2 \le a$ of α satisfying $\mathrm{Diam}\,(\alpha([t_0, t_1])) < r_2$, $\mathrm{Diam}\,(\alpha([t_1, t_2])) < r_1$, and $\mathrm{Diam}\,(\alpha([t_2, t_0])) < r_1$. Such an r-sample is obviously $\tilde{r}$-null as it is the boundary of a simplex in $\mathrm{sRips}(X;\tilde{r})$. Proposition 4.2 implies that each r-sample of α is $\tilde{r}$-null. ∎

PROPOSITION 4.4. *Assume $\alpha, \alpha' \colon S^1 \to X$ are loops in a geodesic space X, and let L and L' denote their r-samples. If α and α' are homotopic, then L and L' are $\tilde{r}$-homotopic.*

PROOF. Consider a homotopy $H \colon S^1 \times [0, 1] \to X$, where $\alpha = H|_{S^1 \times \{0\}}$ and $\alpha' = H|_{S^1 \times \{1\}}$. Let $\triangle$ be a triangulation of $S^1 \times [0, 1]$ subordinated to the open cover $\{H^{-1}(B(x, r_2/2))\}_{x \in X}$. Each triangle of Δ determined by points $(x_1, t_1), (x_2, t_2), (x_3, t_3)$ induces a triple

$$H(x_1, t_1), H(x_2, t_2), H(x_3, t_3),$$

which is of diameter less than r_2 and thus forms a triangle in $\mathrm{sRips}(X;\tilde{r})$. In this way the triangulation $\triangle$ induces a simplicial $\tilde{r}$-homotopy $H' \colon (S^1 \times [0, 1], \triangle) \to \mathrm{sRips}(X;\tilde{r})$ between r_2-samples of α and α', by mapping vertex $(x_i, t_i) \in \Delta$ to $H(x_i, t_i)$. The statement now follows by Proposition 4.2. ∎

COROLLARY 4.5. *Let $\alpha : [0, a] \to X$ be a contractible loop in a geodesic space X. Then any of its r-sample is $\tilde{r}$-null.*

PROOF. It follows by Proposition 4.4. ∎

4.1. *The size of holes.* In this section we introduce the subgroup generated by length of a loop. We use notation from [12].

DEFINITION 4.6. *Let $l > 0$. An l-**lasso** is a based loop of the form $\alpha * \beta * \alpha^-$, where α is a path of finite length based at the point $\bullet$ and β is a loop of length l based at the endpoint of α. The size of a lasso $\alpha * \beta * \alpha^-$ is defined as the length of β. $\mathcal{L}(X, r, \pi_1) \le \pi_1(X, \bullet)$ is generated by all l-lassos with $l < r$. We also define $\mathcal{L}(X, fin, \pi_1) = \bigcup_{n \in \mathbb{N}} \mathcal{L}(X, n, \pi_1)$, which coincides with the subgroup of all homotopy classes admitting a representative of finite length.*

REMARK 4.7. The condition in Definition 4.6 that α is of finite length is crucial in spaces, which are not semi-locally simply connected. In such spaces there are paths which are not homotopic (rel the endpoints) to a path of finite length.

The mentioned condition is inadvertently missing from Definition 4.1 of [12], which introduces the notation above. However, the proofs and results of [12] as stated hold for the definition of lassos as stated in Definition 4.6.

Proposition 4.8. *Let L be an r-loop based at $\bullet$ and let α be a filling of L. Then L is $\tilde{r}$-null if and only if $[\alpha] \in \mathcal{L}(X, 2r_1 + r_2, \pi_1)$.*

Proof. Suppose that L given by $\bullet = x_0, x_1, \ldots, x_k$ is $\tilde{r}$-null. An $\tilde{r}$-nullhomotopy can be thought of as a simplicial map $f\colon \triangle \to \mathrm{sRips}(X; \tilde{r})$ from a triangulation $\triangle$ of a closed disk D, whose restriction to the boundary coincides with L. Note that for each triangle $[z_1, z_2, z_3] \in \triangle$ the image $[f(z_1), f(z_2), f(z_3)]$ spans a triangle in $\mathrm{sRips}(X; \tilde{r})$. We define a function $\varphi\colon \triangle^{(1)} \to X$ as follows (see Figure 2):

- φ coincides with f on the vertex set V;
- edge of $\triangle$ with endpoints x, y is mapped to a geodesic between $f(x)$ and $f(y)$;
- when connecting consecutive points of L take the appropriate geodesic so that the induced filling on L is α.

One can easily see that the decomposition into triangles $\triangle$ corresponds to the decomposition of α into loops of length less than $2r_1 + r_2$ in the unbased setting and to $(2r_1 + r_2)$-lassos in the based setting (as in [12, Proposition 4.8]). If follows that $\alpha \in \mathcal{L}(X, 2r_1 + r_2, \pi_1)$.

Now suppose that $\alpha \in \mathcal{L}(X, 2r_1 + r_2, \pi_1)$. By Proposition 4.4 we may assume that α is a concatenation of $(2r_1 + r_2)$-lassos. It suffices to prove that any lasso of size less than $2r_1 + r_2$ is $\tilde{r}$-null, which follows from Proposition 4.3. ∎

Lemma 4.9. *Let $\alpha\colon [0, a] \to X$ be a loop of finite length in a geodesic space X and suppose an r-sample L of α is given as $0 = t_0 \le t_1 \le \cdots \le t_k$. Furthermore assume that for each i the length of $\alpha|_{[t_i, t_{i+1}]}$ is less than r. Then for any filling β of L we have $\alpha * \beta^- \in \mathcal{L}(X, 2r, \pi_1)$.*

Proof. According to Figure 3 loop $\alpha * \beta^-$ can be "decomposed" into $2r$-lassos, a similar decomposition was used in Proposition 4.8. ∎

Definition 4.10. *Map*

$$\lambda_{\tilde{r}} : \pi_1(\mathrm{sRips}(X; \tilde{r}), \bullet) \to \mathcal{L}(X, fin, \pi_1)/\mathcal{L}(X, 2r_1 + r_2, \pi_1)$$

is defined by mapping an r-loop to its filling.

The following is an adaptation of [12, Proposition 5.6] to selective Rips complexes.

Proposition 4.11. *Map $\lambda_{\tilde{r}}$ is a well defined isomorphism. Moreover, map $\lambda_{\tilde{r}}$ commutes with the inclusion $i_{\tilde{r},\tilde{r}'} : \mathrm{sRips}(X; \tilde{r}) \to \mathrm{sRips}(X; r_1', r_2', \ldots)$ induced maps on the fundamental groups and the quotient map*

$$\mathcal{L}(X, fin, \pi_1)/\mathcal{L}(X, 2r_1 + r_2, \pi_1) \to \mathcal{L}(X, fin, \pi_1)/\mathcal{L}(X, 2r_1' + r_2', \pi_1)$$

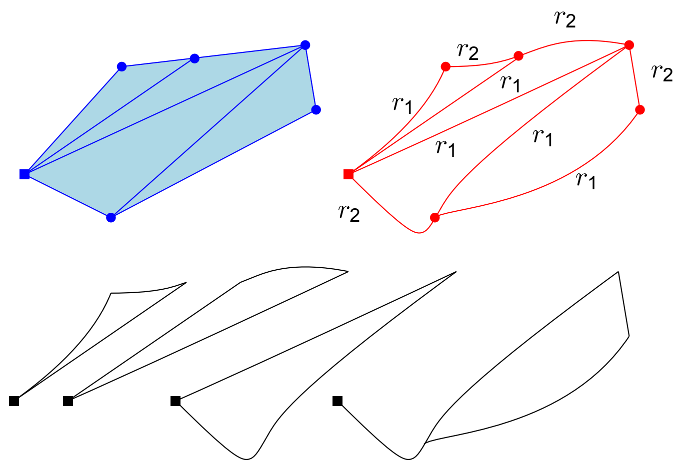

FIGURE 2. A sketch of a map φ of Proposition 4.8. Edges of a simplicial nullhomotopy in $\mathrm{sRips}(X;\tilde{r})$ (upper left) induce a system of geodesics in X (upper right, labels r_i indicate the length of the corresponding segment is less than r_i), which results in a decomposition into lassos (below). The square denotes the basepoint and all loops are oriented in the negative (clockwise) direction.

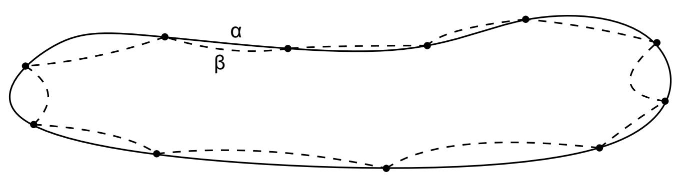

FIGURE 3. A sketch of the proof of Lemma 4.9. Solid loop represents α while dashed portions constitute β. Their difference consists of small loops of length less than $2r$.

provided $r_i \leq r'_i, \forall i$. *In particular, the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(\mathrm{sRips}(X;\tilde{r}'),\bullet) & \xrightarrow{\lambda_{\tilde{r}'}} & \mathcal{L}(X,fin,\pi_1)/\mathcal{L}(X,2r'_1+r'_2,\pi_1) \\ \uparrow & & \uparrow \\ \pi_1(\mathrm{sRips}(X;\tilde{r}),\bullet) & \xrightarrow{\lambda_{\tilde{r}}} & \mathcal{L}(X,fin,\pi_1)/\mathcal{L}(X,2r_1+r_2,\pi_1) \end{array}$$

Proof. Map $\lambda_{\tilde{r}}$ is independent from the choice of fillings by Lemma 4.9 and well defined on homotopy classes in the fundamental groups by Proposition 4.8. Map is injective by Proposition 4.8. We next show that it is also surjective.

Let $\gamma\colon [0,a] \to X$ be a lasso in X of finite length. Let L be an r-sample $0 = t_0 \le t_1 \le \cdots \le t_m = a$ of γ such that for each i the length of $\gamma|_{[t_i,t_{i+1}]}$ is less than r_1. Lemma 4.9 implies that for each filling $\tilde{\gamma}$ of L we have $[\gamma * \tilde{\gamma}^-] \in \mathcal{L}(X, 2r_1, \pi_1)$. Hence L is mapped to the equivalence class of γ in $\mathcal{L}(X, fin, \pi_1)/\mathcal{L}(X, 2r_1 + r_2, \pi_1)$ by $\lambda_{\tilde{r}}$.

Map $\lambda_{\tilde{r}}$ is apparently a homomorphism. Commutativity follows from the definitions of maps. □

Proposition 4.11 is the main technical result of this section. In the following theorem we rephrase this result using the notation of isomorphism between $\tilde{r}$-indexed groups equipped with bonding maps. Such an isomorphism consists of level-wise isomorphisms that commute with the bonding maps as was demonstrated in Proposition 4.11. This presentation is also compatible with general (including multi-parameter) filtrations and persistence modules used in the context of persistent homology, and will be used below to describe related results. In this spirit we refer to $\{\pi_1(\mathrm{sRips}(X;\tilde{r}), \bullet)\}_{r_1 \ge r_2 \ge \cdots > 0}$ as persistent fundamental group and to $\{H_1(\mathrm{sRips}(X;\tilde{r}), G)\}_{r_1 \ge r_2 \ge \cdots > 0}$ as persistent H_1. They represent different constructions of 1-dimensional persistence.

Theorem 4.12 (Persistence-circumference correspondence Theorem for selective Rips complexes). *Let X be a geodesic space. Maps $\lambda_{\tilde{r}}$ provide an isomorphism*

$$\{\pi_1(\mathrm{sRips}(X;\tilde{r}), \bullet)\}_{r_1 \ge r_2 \ge \cdots > 0} \cong \{\mathcal{L}(X, fin, \pi_1)/\mathcal{L}(X, 2r_1 + r_2, \pi_1)\}_{r_1 \ge r_2 > 0}.$$

Moreover if X is semi-locally simply-connected (see [12, Definition 2.4]*) then*

$$\{\pi_1(\mathrm{sRips}(X;\tilde{r}), \bullet)\}_{r_1 \ge r_2 \ge \cdots > 0} \cong \{\pi_1(X, \bullet)/\mathcal{L}(X, 2r_1 + r_2, \pi_1)\}_{r_1 \ge r_2 > 0}.$$

Proof. Follows from Proposition 4.11. □

Definition 4.13. *Let $r > 0$. An U_d-lasso is a based loop of the form $\alpha * \beta * \alpha^-$, where α is a path with starting at the point $\bullet$ and β is a loop in some $\tilde{B}(x, d)$. $\mathcal{S}(X, r, \pi_1) \le \pi_1(X, \bullet)$ is generated by all U_d-lassos with $d < r$. We also define $\mathcal{S}(X, \pi_1) = \bigcup_{r>0} \mathcal{S}(X, r, \pi_1)$.*

Moreover, recall that diameter of a loop α in X is defined as $\mathrm{Diam}(\alpha) = \max_{y,x\in\alpha} d(x, y)$. *$\mathcal{D}(X, r, \pi_1) \le \pi_1(X, \bullet)$ is generated by all lassos $\alpha * \beta * \alpha^-$ with* $\mathrm{Diam}(\beta) < r$.

Remark 4.14. Combining results from [12, Theorem 5.4, Theorem 5.7, Theorem 5.9] and Theorem 4.12 we get the following results:

if X is semi-locally simply connected then

$$\{\pi_1\left(\mathrm{sRips}(X;\tilde{r}),\bullet\right)\}_{r_1\geq r_2\geq\cdots>0}\cong\{\pi_1(X,\bullet)/\mathcal{S}\left(X,(2r_1+r_2)/2,\pi_1\right)\}_{r_1\geq r_2>0}$$

and

$$\{\pi_1\left(\mathrm{sRips}(X;\tilde{r}),\bullet\right)\}_{r_1\geq r_2\geq\cdots>0}\cong\{\pi_1(X,\bullet)/\mathcal{D}(X,(2r_1+r_2)/2,\pi_1)\}_{r_1\geq r_2>0}.$$

Note that similar results (as in [12, Theorem 5.4, Theorem 5.7, Theorem 5.9]) also holds for homology groups by the Hurewitcz theorem. Let G be an Abelian group and let X be G-semi locally simply connected (see [12, Definition 2.4])) then the following holds:

$$\{H_1\left(\mathrm{sRips}(X;\tilde{r}),G\right)\}_{r_1\geq r_2\geq\cdots>0}\cong\{H_1(X,G)/\mathcal{L}(X,2r_1+r_2,G)\}_{r_1\geq r_2>0},$$

$$\{H_1\left(\mathrm{sRips}(X;\tilde{r}),G\right)\}_{r_1\geq r_2\geq\cdots>0}\cong\{H_1(X,G)/\mathcal{S}\left(X,(2r_1+r_2)/2,G\right)\}_{r_1\geq r_2>0}$$

and

$$\{H_1\left(\mathrm{sRips}(X;\tilde{r}),G\right)\}_{r_1\geq r_2\geq\cdots>0}\cong\{H_1(X,G)/\mathcal{D}(X,(2r_1+r_2)/2,G)\}_{r_1\geq r_2>0}.$$

These results can be further combined with the structural results proved in [12]. For example, if $\tilde{r}(t)$ is a collection of non-decreasing continuous functions $r_i(t)\colon\mathbb{R}\to(0,\infty)$ satisfying $r_1(t)\geq r_2(r)\geq\cdots,\forall t$ and X is compact, then

- The collection of critical values of the fundamental group (i.e., the values of $t\in\mathbb{R}$ at which group $\pi_1\left(\mathrm{sRips}(X;\tilde{r}(t)),G\right)$ changes) is discrete;
- for each critical value t_c there exists a geodesic circle in X of length $r_1(t_c)+r_2(t_c)$.

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