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THE FINITE COARSE SHAPE - INVERSE SYSTEMS APPROACH AND INTRINSIC APPROACH

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Abstract. Given an arbitrary category $\mathcal{C}$, a category pro^{*^f}-$\mathcal{C}$ is constructed such that the known pro-$\mathcal{C}$ category may be considered as a subcategory of pro^{*^f}-$\mathcal{C}$ and that pro^{*^f}-$\mathcal{C}$ may be considered as a subcategory of pro^*-$\mathcal{C}$. Analogously to the construction of the shape category $Sh_{(\mathcal{C},\mathcal{D})}$ and the coarse category $Sh^*_{(\mathcal{C},\mathcal{D})}$, an (abstract) *finite coarse shape category* $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ is obtained. Between these three categories appropriate faithful functors are defined. The finite coarse shape is also defined by an intrinsic approach using the notion of the ϵ-continuity. The isomorphism of the finite coarse shape categories obtained by these two approaches is constructed. Besides, an overview of some basic properties related to the notion of the ϵ-continuity is given.

1. Introduction

The shape theory of metric compacta was founded in 1968 by K. Borsuk ([1, 2]). Later on, S. Mardešić and J. Segal ([7]) extended the shape theory to the class of all compact Hausdorff spaces using the inverse systems approach. Finally, the shape theory was extended to the class of all topological spaces by S. Mardešić ([6]) and K. Morita ([8]). In [9], J. M. R. Sanjurjo gave the reinterpretation of the shape theory of compact metric spaces. He used an intrinsic approach – the basic objects of that theory are sequences of ϵ-continuous functions. The component functions of the morphisms between metric compacta X and Y are ϵ-continuous functions from X to Y, unlike in the inverse systems approach where component functions of the morphisms are continuous and, generally, have values in the neighbourhoods of Y. Further

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generalizations were made by N. K. Bilan and N. Uglešić in [4]. They founded the coarse shape theory for all topological spaces using the inverse systems approach. The coarse shape classification of topological spaces is generally coarser then the shape classification. In the present paper the finite coarse shape category is constructed. We show that the shape category is a proper subcategory of the finite coarse shape category, which is a proper subcategory of the coarse shape category. The finite coarse shape morphisms between two topological spaces X and Y are equivalence classes of the sequences of the finite sequences between the corresponding terms of the expansions of X and Y. Furthermore, we give an intrinsic reinterpretation of the finite coarse shape category of closed subsets of the Hilbert cube Q and establish an isomorphism between the corresponding finite coarse shape categories of closed subsets of the Hilbert cube Q obtained by both inverse systems and intrinsic approach. Finally, the intrinsic finite coarse shape classification is extended to the class of metric compacta $\mathcal{MC}pt$.

2. The notion and some basic properties of ϵ-continuity

Definition 2.1. *Let X be a topological space, (Y, d) metric space and $\epsilon \in \mathbb{R}^+$. A function $f : X \to Y$ is said to be ϵ-continuous at a point $x_0 \in X$ if there exists a neighbourhood U of x_0 in X such that*

$$f(U) \subseteq B(f(x_0), \epsilon).$$

A function $f : X \to Y$ is said to be ϵ-continuous provided that it is ϵ-continuous at each point $x_0 \in X$.

It is obvious that a function f is continuous if and only if it is ϵ-continuous for every $\epsilon \in \mathbb{R}^+$. Function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \operatorname{sgn}(x)$, is an example of a $\frac{4}{3}$-continuous function that is not continuous.

If $f, g : X \to Y$ are functions, the notation $d(f, g) < \epsilon$ means that

$$d(f(x), g(x)) < \epsilon$$

for every $x \in X$.

Proposition 2.2. *Let X be a topological space and let (Y, d) be a metric space. Let $f : X \to Y$ be a continuous function and $g : X \to Y$ be a function such that $d(f, g) < \epsilon$, for some $\epsilon > 0$. Then g is 3ϵ-continuous.*

If both domain and codomain are metric spaces, ϵ-continuity can be characterized in a way that a function $f : X \to Y$ is ϵ-continuous if and only if for every point $x \in X$ there exists a $\delta_x > 0$ such that

$$f(B(x, \delta_x)) \subseteq B(f(x), \epsilon).$$

The composition of two ϵ-continuous functions $f : X \to Y$ and $g : Y \to Z$ is not, in general, an ϵ-continuous function. Moreover, there doesn't have to exist an $\epsilon' \in \mathbb{R}^+$ such that the composition $g \circ f : X \to Z$ is an ϵ'-continuous function, as it is shown in the following example.

EXAMPLE 2.3. *Let $f : \mathbb{R}_0^+ \to \mathbb{Z}_0^+$, $f(x) = \lfloor x \rfloor$, be a function that associates with every $x \in \mathbb{R}_0^+$ the greatest integer less than or equal to x, let $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $g(x) = x^2$ and let $h : \mathbb{R}_0^+ \to \mathbb{Z}_0^+$, $h(x) = g(f(x)) = \lfloor x \rfloor^2$. Functions f and g are obviously $\frac{4}{3}$-continuous. On the other hand, for every $n \in \mathbb{N}$, function h is not ϵ-continuous at point n, for any $\epsilon \leq 2n - 1$. Since* $\lim(2n-1) = +\infty$, *there doesn't exist an $\epsilon \in \mathbb{R}^+$ such that h is ϵ-continuous.*

PROPOSITION 2.4. *Let $f : X \to Y$ be a continuous function and let $g : Y \to Z$ be an ϵ-continuous function. Then $g \circ f : X \to Z$ is an ϵ-continuous function.*

PROPOSITION 2.5. *Let $X = X_1 \times \cdots \times X_n$ be a topological product and $Y = (Y_1 \times \cdots \times Y_n, d_\infty)$ be a product of metric spaces. A function $f = (f_1, \ldots, f_n) : X \to Y$ is ϵ-continuous if and only if every function $f_i : X_i \to Y_i$, $i = 1, \ldots n$, is ϵ-continuous.*

Using the inequality $d_p \leq \sqrt[p]{n} \cdot d_\infty$, $p \in \mathbb{N}$, the following proposition can easily be proved.

PROPOSITION 2.6. *Let $X = X_1 \times \cdots \times X_n$ be a topological product and $Y = (Y_1 \times \cdots \times Y_n, d_p)$, $p \in \mathbb{N}$, be a product of metric spaces. Then, for every $p \in \mathbb{N}$, the following statements hold:*

(*i*) *if $f = (f_1, \ldots, f_n) : X \to Y$ is ϵ-continuous, then every function $f_i : X_i \to Y_i$, $i = 1, \ldots, n$, is ϵ-continuous;*
(*ii*) *if every function $f_i : X_i \to Y_i$, $i = 1, \ldots, n$, is ϵ-continuous, then $f = (f_1, \ldots, f_n) : X \to Y$ is $\sqrt[p]{n} \cdot \epsilon$-continuous.*

The properties of ϵ-continuous functions are much better if considered between the compact metric spaces.

DEFINITION 2.7. *Let (X, d') and (Y, d) be compact metric spaces. A function $f : X \to Y$ is said to be uniformly ϵ-continuous if there exists a $\delta > 0$ such that for every two points $x, x' \in X$ inequality $d'(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$.*

A number δ from Definition 2.7 is called the *uniformity radius* of the function f. It is obvious that a function $f : X \to Y$ is uniformly continuous if and only if it is uniformly ϵ-continuous for every $\epsilon \in \mathbb{R}^+$. Furthermore, if a function $f : X \to Y$ is uniformly ϵ-continuous, than it is also ϵ-continuous.

The following theorem is an analogue of the Heine-Cantor theorem.

THEOREM 2.8. *Let (X, d') be a compact metric space and (Y, d) be a metric space. If $f : X \to Y$ is an ϵ-continuous function, then f is uniformly 2ϵ-continuous.*

PROOF. Since f is ϵ-continuous, for every $x \in X$ there exists a $\delta_x > 0$ such that

$$f(B(x, \delta_x)) \subseteq B(f(x), \epsilon).$$

For every $x \in X$, we define the set

$$U_x = B\left(x, \frac{\delta_x}{2}\right).$$

Then the collection $\mathcal{U} = \{U_x : x \in X\}$ is an open covering of X that, due to the compactness, admits a finite subcovering $\mathcal{U}' = \{U_{x_1}, \ldots, U_{x_n}\}$. Let

$$\delta = \min\left\{\frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2}\right\} > 0.$$

and let $x, x' \in X$ be arbitrary points such that $d'(x, x') < \delta$. Since $\mathcal{U}'$ is the covering of X, there exists $i \in \{1, \ldots, n\}$ such that $x \in U_{x_i}$. Hence,

$$d'(x, x_i) < \frac{\delta_{x_i}}{2}.$$

Now, by the triangle inequality it holds that

$$d'(x_i, x') \leq d'(x_i, x) + d'(x, x') < \frac{\delta_{x_i}}{2} + \delta \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}.$$

It means that $x, x' \in B(x_i, \delta_{x_i})$ and, by the assumption,

$$f(x), f(x') \in B(f(x_i), \epsilon).$$

Finally, it holds that

$$d(f(x), f(x')) \leq d(f(x), f(x_i)) + d(f(x_i), f(x')) < 2\epsilon,$$

i.e., f is uniformly 2ϵ-continuous. ∎

A consequence of Theorem 2.8 is the following proposition that describes the composition of ϵ-continuous functions on compact metric spaces.

PROPOSITION 2.9. *Let X, Y and Z be compact metric spaces, $g : Y \to Z$ an ϵ-continuous function and let δ be a uniformity radius of the uniformly 2ϵ-continuous function g. If $f : X \to Y$ is a δ-continuous function, then $g \circ f : X \to Z$ is 2ϵ-continuous.*

Let us now define the known relation between ϵ-continuous functions.

DEFINITION 2.10. *Let X be a topological space and let Y be a metric space. Every ϵ-continuous function $H : X \times I \to Y$ is called an ϵ-homotopy.*

Two functions $f, g : X \to Y$ are said to be ϵ-homotopic, denoted by $f \overset{\epsilon}{\simeq} g$, if there exists an ϵ-homotopy $H : X \times I \to Y$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.

On can easily prove that the relation of ϵ-homotopy is an equivalence relation on the set of all ϵ-continuous functions from X to Y. Furthermore, ϵ-continuous and near functions are mutually ϵ'-homotopic for an appropriate ϵ'. More precisely, the following proposition holds.

PROPOSITION 2.11. *Let X be a topological space and Y be a metric space. Let $f, g : X \to Y$ be ϵ_1-near functions such that f is ϵ_2-continuous and g ϵ_3-continuous. Then $f \overset{2\epsilon}{\simeq} g$ for $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.*

3. The categories of the finite coarse shape

3.1. *Categories inv^{*^f}-$\mathcal{C}$ and pro^{*^f}-$\mathcal{C}$.*

DEFINITION 3.1. *Let $\mathcal{C}$ be a category and let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{C}$. A $*^f$-morphism $(f, f_\mu^m) : \mathbf{X} \to \mathbf{Y}$ consists of a function $f : M \to \Lambda$, called the index function, and of a set of $\mathcal{C}$-morphisms $f_\mu^m : X_{f(\mu)} \to Y_\mu$, $m \in \mathbb{N}$, $\mu \in M$, such that:*

1. *for every related pair $\mu, \mu' \in M$, $\mu \leq \mu'$, there exist $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$ and $m_{\mu\mu'} \in \mathbb{N}$ such that, for every $m \geq m_{\mu\mu'}$,*
$$f_\mu^m p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^m p_{f(\mu')\lambda};$$
2. *for every $\mu \in M$ inequality $\operatorname{card}(\{f_\mu^m : m \in \mathbb{N}\}) < \aleph_0$ holds.*

If the index function f is increasing and, for every pair $\mu \leq \mu'$, one may put $\lambda = f(\mu')$, then (f, f_μ^m) is said to be a *simple* $*^f$-morphism. If, in addition, $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\Lambda, f_\mu^m)$ is said to be a *level* $*^f$-morphism.

If $*^f$-morphism $(f, f_\mu^m) : \mathbf{X} \to \mathbf{Y}$ has a property that, for every $\mu \in M$, $f_\mu^m = f_\mu$, for every $m \in \mathbb{N}$, then (f, f_μ^m) is said to be induced by the morphism $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$.

Let $(f, f_\mu^m) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu^m) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be $*^f$-morphisms. Then (h, h_ν^m), where $h = fg$ and $h_\nu^m = g_\nu^m f_{g(\nu)}^m$, for every $m \in \mathbb{N}$ and $\nu \in N$, is a $*^f$-morphism from $\mathbf{X}$ to $\mathbf{Z}$. Now we can define the *composition* of $*^f$-morphisms: if $(f, f_\mu^m) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu^m) : \mathbf{Y} \to \mathbf{Z}$, then $(h, h_\nu^m) = (g, g_\nu^m) \circ (f, f_\mu^m)$, where $h = fg$ and $h_\nu^m = g_\nu^m f_{g(\nu)}^m$. Clearly, this composition is associative.

Furthermore, for every inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, the pair $(1_\Lambda, 1_{X_\lambda}^m)$, where $1_{X_\lambda}^m = id_{X_\lambda}$, for every $m \in \mathbb{N}$, is a level $*^f$-morphism $(1_\Lambda, 1_{X_\lambda}^m) : \mathbf{X} \to \mathbf{X}$ that acts neutrally in the composition from the left and from the right side. Thus, $(1_\Lambda, 1_{X_\lambda}^m)$ may be called the *identity* $*^f$-morphism on $\mathbf{X}$. Now, given a category $\mathcal{C}$, by inv^{*^f}-$\mathcal{C}$ we may denote a category which object class consists of all the inverse systems in $\mathcal{C}$ and which morphism class consists of all the sets inv^{*^f}-$\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of all $*^f$-morphisms from $\mathbf{X}$ to $\mathbf{Y}$, together with the composition and identities described above.

We define a relation on each set inv^{*^f}-$\mathcal{C}(\mathbf{X}, \mathbf{Y})$ as follows.

DEFINITION 3.2. *A $*^f$-morphism $(f, f_\mu^m) : \mathbf{X} \to \mathbf{Y}$ is said to be equivalent to a $*^f$-morphism $(f', f_\mu^{\prime m}) : \mathbf{X} \to \mathbf{Y}$, denoted by $(f, f_\mu^m) \sim (f', f_\mu^{\prime m})$, if for every $\mu \in M$ there exist $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, and $m_\mu \in \mathbb{N}$ such that, for every $m \geq m_\mu$,*

$$f_\mu^m p_{f(\mu)\lambda} = f_\mu^{\prime m} p_{f'(\mu)\lambda}.$$

PROPOSITION 3.3. *The relation $\sim$ is a congruence on the category inv^{*^f}-$\mathcal{C}$.*

PROOF. The relation $\sim$ is obviously reflexive and symmetric. Transitivity follows from the commutative diagram and by the direction of the set Λ for $\lambda'' \geq \lambda, \lambda'$, where $\lambda \geq f(\mu), f'(\mu)$ and $\lambda' \geq f'(\mu), f''(\mu)$. □

The quotient category inv^{*^f}-$\mathcal{C}|_\sim$ is denoted by pro^{*^f}-$\mathcal{C}$ and its morphisms $[(f, f_\mu^m)]$ (the equivalence classes of $*^f$-morphisms) are denoted by $\mathbf{f}^{*^f}$. The composition in the category pro^{*^f}-$\mathcal{C}$ is defined by the representatives, i.e., if $\mathbf{f} = [(f, f_\mu^m)] : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g} = [(g, g_\nu^m)] : \mathbf{Y} \to \mathbf{Z}$ are two morphisms in pro^{*^f}-$\mathcal{C}$, then

$$\mathbf{g} \circ \mathbf{f} = [(g, g_\nu^m)] \circ [(f, f_\mu^m)] = \left[\left(f \circ g, g_\nu^m \circ f_{g(\nu)}^m\right)\right] : \mathbf{X} \to \mathbf{Z}.$$

The following Proposition 3.4 states that category *pro*-$\mathcal{C}$ may be considered as a subcategory of pro^{*^f}-$\mathcal{C}$ and that pro^{*^f}-$\mathcal{C}$ may be considered as a subcategory of pro^*-$\mathcal{C}$. Recall that category pro^*-$\mathcal{C}$ was defined in [4] as a step in the construction of the coarse shape category.

PROPOSITION 3.4. *The mapping which holds inverse systems in $\mathcal{C}$ fixed and with every morphism $\mathbf{f} = [(f, f_\mu)] : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{C}$ associates a $*^f$-morphism $\mathbf{f}^{*^f} = [(f, f_\mu^m)] : \mathbf{X} \to \mathbf{Y}$ in pro^{*^f}-$\mathcal{C}$ that is represented by the $*^f$-morphism induced by the morphism (f, f_μ), is well defined and determines a faithful functor $\mathbf{J}_\mathcal{C}^{*^f} : pro\text{-}\mathcal{C} \to pro^{*^f}\text{-}\mathcal{C}$ which, in general, is not full.*

Analogously, the mapping which holds inverse systems in $\mathcal{C}$ fixed and with every $*^f$-morphism $\mathbf{f}^{*^f} = [(f, f_\mu^m)] : \mathbf{X} \to \mathbf{Y}$ in pro^{*^f}-$\mathcal{C}$ associates a $*$-morphism $\mathbf{f}^* = [(f, f_\mu^m)] : \mathbf{X} \to \mathbf{Y}$ in pro^*-$\mathcal{C}$ that is represented by the $*$-morphism (i.e., $*^f$-morphism) (f, f_μ^m), is well defined and determines a faithful functor $\mathbf{J}_\mathcal{C}^* : pro^{*^f}\text{-}\mathcal{C} \to pro^*\text{-}\mathcal{C}$ which, in general, is not full.

PROOF. It is obvious that $\mathbf{J}_\mathcal{C}^{*^f}$ is a functor. Firstly, we prove that $\mathbf{J}_\mathcal{C}^{*^f}$ is faithful, i.e., that for every pair $\mathbf{X}$, $\mathbf{Y}$ of the inverse systems in $\mathcal{C}$ the function $\mathbf{J}_{X,Y}^{*^f} : pro\text{-}\mathcal{C}(\mathbf{X}, \mathbf{Y}) \to pro^{*^f}\text{-}\mathcal{C}(\mathbf{X}, \mathbf{Y})$ is injective.

Let $\mathbf{f}, \mathbf{f}' : \mathbf{X} \to \mathbf{Y}$ be such that $\mathbf{J}_\mathcal{C}^{*^f}(\mathbf{f}) = \mathbf{f}^{*^f} = \mathbf{J}_\mathcal{C}^{*^f}(\mathbf{f}')$, and let $(f, f_\mu), (f', f'_\mu) : \mathbf{X} \to \mathbf{Y}$ be $*$-morphisms in *inv*-$\mathcal{C}$ such that $\mathbf{f} = [(f, f_\mu)]$ and $\mathbf{f}' = [(f', f'_\mu)]$. By the assumption, $[(f, f_\mu^m)] = [(f', f_\mu^{\prime m})]$, where

$(f,f_\mu^m),(f',f_\mu^{\prime m}) : \mathbf{X} \to \mathbf{Y}$ are $*^f$-morphisms in inv^{*^f}-$\mathcal{C}$ induced by $*$-morphisms (f,f_μ) and (f',f'_μ), respectively. Hence, $f_\mu^m = f_\mu$ and $f_\mu^{\prime m} = f'_\mu$, for every $\mu \in M$, $m \in \mathbb{N}$ and, since $(f,f_\mu^m) \sim (f',f_\mu^{\prime m})$, for every $\mu \in M$ there exists $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$ such that

$$f_\mu^m p_{f(\mu)\lambda} = f_\mu^{\prime m} p_{f'(\mu)\lambda}, \ \text{ for every } m \in \mathbb{N}.$$

Previous relations mean that for every $\mu \in M$ there exists $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$ such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

So, $(f,f_\mu) \sim (f',f'_\mu)$, i.e., $\mathbf{f} = \mathbf{f}'$ and the injectivity is proved. Hence, $\mathbf{J}_\mathcal{C}^{*^f}$ is a faithful functor. The proof that $\mathbf{J}_\mathcal{C}^{*}$ is faithful is analogous.

We show by counterexamples that, in general, functors $\mathbf{J}_\mathcal{C}^{*^f}$ and $\mathbf{J}_\mathcal{C}^{*}$ are not full. Let $X,Y \in \mathcal{O}b(\mathcal{C})$ and let $g,g' : X \to Y$ be morphisms in $\mathcal{C}$ such that $g \neq g'$. The morphism

$$(f^m) : (X) \to (Y)$$
$$f^{2k} = g, \ f^{2k-1} = g', \ \text{ for every } k \in \mathbb{N},$$

in inv^{*^f}-$\mathcal{C}$, between the rudimental systems (X) and (Y), is not induced by any morphism in inv-$\mathcal{C}$ and so $[(f^m)] \notin \mathbf{J}_{X,Y}^{*^f}(pro-\mathcal{C}((X),(Y)))$, i.e., $\mathbf{J}_\mathcal{C}^{*^f}$, in general, is not full. Finally, let $X,Y \in \mathcal{O}b(\mathcal{C})$ and let (g_m) be a sequence of morphisms $g_m : X \to Y$, $m \in \mathbb{N}$, in $\mathcal{C}$ such that $g_m \neq g_{m'}$, whenever $m \neq m'$. The morphism

$$(f^m) : (X) \to (Y)$$
$$f^m = g_m, \ \text{ for every } m \in \mathbb{N},$$

in inv^*-$\mathcal{C}$, between the rudimental systems (X) and (Y), is not induced by any morphism in inv^{*^f}-$\mathcal{C}$ and so $[(f^m)] \notin \mathbf{J}_{X,Y}^{*}(pro^{*^f} - \mathcal{C}\,((X),(Y)))$, i.e., $\mathbf{J}_\mathcal{C}^{*}$, in general, is not full. □

Especially, if $\mathcal{C} = HTop$, then functors $\mathbf{J}_{HTop}^{*^f} : pro\text{-}HTop \to pro^{*^f}\text{-}HTop$ and $\mathbf{J}_{HTop}^{*} : pro^{*^f}\text{-}HTop \to pro^{*}\text{-}HTop$ are faithful and not full.

3.2. *The category and morphisms of the finite coarse shape.*

Let $\mathcal{C}$ be a category and let $\mathcal{D} \subseteq \mathcal{C}$ be a dense (pro-reflective) and full subcategory. We define a relation between pro^{*^f}-$\mathcal{D}$-morphisms as follows:

Definition 3.5. *Let $\mathcal{C}$ be a category and $\mathcal{D} \subseteq \mathcal{C}$ dense and full subcategory. Let $\mathbf{p} : (X) \to \mathbf{X}$, $\mathbf{p}' : (X) \to \mathbf{X}'$ be $\mathcal{D}$-expansions of the object $X \in \mathcal{O}b(\mathcal{C})$ and let $\mathbf{q} : (Y) \to \mathbf{Y}$, $\mathbf{q}' : (Y) \to \mathbf{Y}'$ be $\mathcal{D}$-expansions of the object $Y \in \mathcal{O}b(\mathcal{C})$. A morphism $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ is said to be equivalent to a morphism $\mathbf{f}'^{*^f} : \mathbf{X}' \to \mathbf{Y}'$ in pro^{*^f}-$\mathcal{D}$, denoted by $\mathbf{f}^{*^f} \sim \mathbf{f}'^{*^f}$, if*

$$\mathbf{f}'^{*^f} \circ \mathbf{J}_\mathcal{D}^{*^f}(\mathbf{i}) = \mathbf{J}_\mathcal{D}^{*^f}(\mathbf{j}) \circ \mathbf{f}^{*^f},$$

where $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ *and* $\mathbf{j} : \mathbf{Y} \to \mathbf{Y}'$ *are canonical isomorphisms between the expansions of the same object.*

The relation $\sim$ in pro^{*^f}-$\mathcal{D}$ is an equivalence relation on the appropriate subclass of all the pro^{*^f}-$\mathcal{D}$-morphisms between inverse systems in $\mathcal{D}$ that are expansions of the objects X and Y from $\mathcal{C}$. Moreover, if $\mathbf{f}^{*^f} \sim \mathbf{f}'^{*^f}$ and $\mathbf{g}^{*^f} \sim \mathbf{g}'^{*^f}$, then $\mathbf{g}^{*^f}\mathbf{f}^{*^f} \sim \mathbf{g}'^{*^f}\mathbf{f}'^{*^f}$ whenever it is defined. An equivalence class of the morphism $\mathbf{f}^{*^f}$ is denoted by $\left\langle \mathbf{f}^{*^f} \right\rangle$. Furthermore, given $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}$, $\mathbf{q}'$ and $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ there exists a unique $\mathbf{f}'^{*^f} : \mathbf{X}' \to \mathbf{Y}'$ such that $\mathbf{f}^{*^f} \sim \mathbf{f}'^{*^f}$.

For an arbitrary category pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D}$ is dense in $\mathcal{C}$, we now define the (abstract) *finite coarse shape category* $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ as follows: the objects of $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$ and, for any pair X, Y of objects, a morphism $F^{*^f} \in Sh^{*^f}_{(C,D)}(X,Y)$ is the pro^{*^f}-$\mathcal{D}$ equivalence class $\langle \mathbf{f}^{*^f} \rangle$ of a morphism $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ in pro^{*^f}-$\mathcal{D}$, for any choice of $\mathcal{D}$-expansions $\mathbf{p} : (X) \to \mathbf{X}$ and $\mathbf{q} : (Y) \to \mathbf{Y}$.

One can identify a finite coarse shape morphism $F^{*^f} : X \to Y$ with a morphism $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$, for any pair of fixed $\mathcal{D}$-expansions of objects X and Y. In other words, for every two objects X, Y in $\mathcal{C}$, the set $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}(X,Y)$ is bijectively correspondent with the set pro^{*^f}-$\mathcal{D}(\mathbf{X}, \mathbf{Y})$.

The *composition* of the finite coarse shape morphisms $F^{*^f} : X \to Y$, $F^{*^f} = \left\langle \mathbf{f}^{*^f} \right\rangle$, and $G^{*^f} : Y \to Z$, $G^{*^f} = \left\langle \mathbf{g}^{*^f} \right\rangle$, is defined naturally by the representatives, i.e., $G^{*^f} \circ F^{*^f} : X \to Z$, $G^{*^f} \circ F^{*^f} = \left\langle \mathbf{g}^{*^f} \circ \mathbf{f}^{*^f} \right\rangle$. Furthermore, for every object X in $\mathcal{C}$ the *identity* finite coarse shape morphism on X, $1^{*^f}_X : X \to X$, is the equivalence class $\left\langle \mathbf{1}^{*^f}_X \right\rangle$ of the identity morphism $\mathbf{1}^{*^f}_X$ in pro^{*^f}-$\mathcal{D}$. Thus, $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ is a category.

To establish the connections between the observed categories, we define the functors $J^{*^f}_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ and $J^{*}_{(\mathcal{C},\mathcal{D})} : Sh^{*^f}_{(\mathcal{C},\mathcal{D})} \to Sh^{*}_{(\mathcal{C},\mathcal{D})}$ by:

$$J^{*^f}_{(\mathcal{C},\mathcal{D})}(X) = J^{*}_{(\mathcal{C},\mathcal{D})}(X) = X, \quad \text{for every object } X \text{ in } \mathcal{C},$$
$$J^{*^f}_{(\mathcal{C},\mathcal{D})}(F) = \left\langle \mathbf{J}^{*^f}_{\mathcal{D}}(\mathbf{f}) \right\rangle = \left\langle \mathbf{f}^{*^f} \right\rangle, \text{ for every shape morphism } F = \langle \mathbf{f} \rangle,$$
$$J^{*}_{(\mathcal{C},\mathcal{D})}\left(F^{*^f}\right) = \left\langle \mathbf{J}^{*}_{\mathcal{D}}\left(\mathbf{f}^{*^f}\right) \right\rangle = \langle \mathbf{f}^{*} \rangle,$$

for every finite coarse shape morphism $F^{*^f} = \left\langle \mathbf{f}^{*^f} \right\rangle$.

PROPOSITION 3.6. *The functors* $J^{*^f}_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ *and* $J^{*}_{(\mathcal{C},\mathcal{D})} : Sh^{*^f}_{(\mathcal{C},\mathcal{D})} \to Sh^{*}_{(\mathcal{C},\mathcal{D})}$ *are faithful and, in general, not full.*

Proof. The functors $J^{*^f}_{(\mathcal{C},\mathcal{D})}$ and $J^{*}_{(\mathcal{C},\mathcal{D})}$ are faithful because $\mathbf{J}^{*^f}_{\mathcal{D}}$ are $\mathbf{J}^{*}_{\mathcal{D}}$ faithful. The counterexamples which prove that $J^{*^f}_{(\mathcal{C},\mathcal{D})}$ and $J^{*}_{(\mathcal{C},\mathcal{D})}$ are not full are analogous to the counterexamples from the proof of Proposition 3.4. ◻

By Proposition 3.6, the (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$ may be considered as a subcategory of the (abstract) finite coarse shape $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ and the (abstract) finite coarse shape category $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ may be considered as a subcategory of the (abstract) coarse shape category $Sh^{*}_{(\mathcal{C},\mathcal{D})}$.

We denote the composition of the shape functor $S_{(\mathcal{C},\mathcal{D})}$ and the functor $J^{*^f}_{(\mathcal{C},\mathcal{D})}$ by $S^{*^f}_{(\mathcal{C},\mathcal{D})}$, i.e., $S^{*^f}_{(\mathcal{C},\mathcal{D})} = J^{*^f}_{(\mathcal{C},\mathcal{D})} \circ S_{(\mathcal{C},\mathcal{D})}$.

Definition 3.7. *The functor* $S^{*^f}_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ *is called the finite coarse shape functor for pair* $(\mathcal{C},\mathcal{D})$.

The functor $S^{*^f}_{(\mathcal{C},\,\mathcal{D})}$ holds objects fixed, and associates with every $\mathcal{C}-$morphism $f : X \to Y$ a finite coarse shape morphism $F^{*^f} : X \to Y$ which is represented by a $*^f$-morphism $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ in pro^{*^f}-$\mathcal{D}$ that is induced by a morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{D}$ such that $S_{(\mathcal{C},\mathcal{D})}(f) = F = \langle \mathbf{f} \rangle$.

Definition 3.8. *We say that objects* X *and* Y *in* $\mathcal{C}$ *have the same finite coarse shape if they are isomorphic in* $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$.

It is obvious that $F^{*^f} = \left\langle \mathbf{f}^{*^f} \right\rangle : X \to Y$ is an isomorphism in $Sh^{*^f}_{(\mathcal{C},\mathcal{D})}$ if and only if $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ is an isomorphism in pro^{*^f}-$\mathcal{D}$. In other words, X and Y have the same finite coarse shape if and only if $\mathbf{X}$ and $\mathbf{Y}$ are isomorphic in pro^{*^f}-$\mathcal{D}$. Moreover, since functors preserve isomorphisms, it holds that:

(1) if X and Y are isomorphic in $\mathcal{C}$ or in $Sh_{(\mathcal{C},\mathcal{D})}$, then they have the same finite coarse shape;
(2) if X and Y have the same finite coarse shape, then they have the same coarse shape.

Thus, (1) proves $(ii) \Longrightarrow (iii)$ and (2) proves $(iii) \Longrightarrow (iv)$ from the following corollary. The remaining implications are known from [4].

Corollary 3.9. *If* P *and* Q *are objects in* $\mathcal{D}$, *then the following statements are equivalent:*

(*i*) *P and Q are isomorphic in $\mathcal{D}$;*
(*ii*) *P and Q have the same shape;*
(*iii*) *P and Q have the same finite coarse shape;*
(*iv*) *P and Q have the same coarse shape.*

Since categories $HPol$ and $HANR$ are dense and full in $HTop$, one can observe categories $Sh^{*^f}_{(HTop,HPol)}$ and $Sh^{*^f}_{(HTop,HANR)}$. For every two objects the sets of all morphisms (between these objects) of these categories are bijectively correspondent and, hence, these categories are identified, called the *topological finite coarse shape* category and denoted by Sh^{*^f}.

3.3. *The $*^f$-fundamental, $*^f$-approximative and $*^f$-proximate sequences.* For an arbitrary pair of topological spaces X and Y, let $C(X,Y)$ denote the set of all the continuous functions from X to Y.

Definition 3.10. *Let X and Y be closed subsets of the Hilbert cube Q. A function $\Phi : \mathbb{N}^2 \to C(Q,Q)$ is called a $*^f$-fundamental sequence from X to Y provided:*

(1) *for every neighbourhood V of Y in Q there exist a neighbourhood U of X in Q and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*
$$\Phi(n,m)|_U \simeq \Phi(n+1,m)|_U \quad \text{in } V, \quad \text{for every } m \geq m_n;$$
(2) *for every $n \in \mathbb{N}$ the inequality $\operatorname{card}(\{\Phi(n,m) : m \in \mathbb{N}\}) < \aleph_0$ holds.*

Remark 3.11. If Φ is a $*^f$-fundamental sequence from X to Y, the function $\Phi(n,m) : Q \to Q$ will be denoted by $\Phi_n^m : Q \to Q$, i.e., $\Phi = (\Phi_n^m) : X \to Y$.

Proposition 3.12. *A function $\Phi : \mathbb{N}^2 \to C(Q,Q)$ such that, for every $n \in \mathbb{N}$, the inequality $\operatorname{card}(\{\Phi_n^m : m \in \mathbb{N}\}) < \aleph_0$ holds, is a $*^f$-fundamental sequence from X to Y if and only if for every neighbourhood V of Y in Q there exist a neighbourhood U of X in Q and $n_0 \in \mathbb{N}$ such that for all $n, n' \geq n_0$ there exists $m_{nn'} \in \mathbb{N}$ such that*
$$\Phi_n^m|_U \simeq \Phi_{n'}^m|_U \quad \text{in } V, \quad \text{for every } m \geq m_{nn'}.$$

The *composition* of $*^f$-fundamental sequences is defined coordinatewise. Such a composition is associative and for an arbitrary closed subset X of Q the *identity* on X is a $*^f$-fundametal sequence $1_X = (1_n^m) : X \to X$ such that $1_n^m = \mathrm{id}_Q : Q \to Q$, for every $n, m \in \mathbb{N}$. Hence, all the closed subsets of Q taken as objects and all the $*^f$-fundamental sequences taken as morphisms form a category denoted by $\mathcal{C}_f^{*^f}$.

Definition 3.13. *A $*^f$-fundamental sequence $\Phi = (\Phi_n^m) : X \to Y$ is said to be homotopic to a $*^f$-fundamental sequence $\Phi' = ({\Phi'}_n^m) : X \to Y$ provided for every neighbourhood V of Y in Q there exist a neighbourhood U of X in Q and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*
$$\Phi_n^m|_U \simeq {\Phi'}_n^m|_U \quad \text{in } V, \quad \text{for every } m \geq m_n.$$

In that case, we write: $\Phi \sim \Phi'$. The relation $\sim$ is an equivalence relation on the set of all $*^f$-fundamental sequences from X to Y. An equivalence class of a $*^f$-fundamental sequence $\Phi = (\Phi_n^m) : X \to Y$ is denoted by $[\Phi] = [(\Phi_n^m)]$.

Furthermore, the *composition* of the equivalence classes of $*^f$-fundamental sequences is defined by the representatives, $[\Psi] \circ [\Phi] := [\Psi \circ \Phi]$, whenever composition $\Psi \circ \Phi$ makes sense. This composition is obviously well defined and associative, with $[1_X]$ being neutral in the composition from both sides. Thus, all the closed subsets of Q taken as objects and all the equivalence classes of $*^f$-fundamental sequences taken as morphisms form a category denoted by $Sh_f^{*^f}$.

We now introduce the notion of a $*^f$-approximative sequence which will be a crucial link between the finite coarse shape categories obtained by the inverse systems approach and the intrinsic approach.

Definition 3.14. *Let X and Y be closed subsets of Q. A function $\alpha : \mathbb{N}^2 \to C(X,Q)$ is called a $*^f$-approximative sequence from X to Y provided:*

(1) *for every neighbourhood V of Y in Q there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*

$$\alpha(n,m) \simeq \alpha(n+1,m) \quad \text{in } V, \quad \text{for every } m \geq m_n;$$

(2) *for every $n \in \mathbb{N}$ the inequality $\operatorname{card}(\{\alpha(n,m) : m \in \mathbb{N}\}) < \aleph_0$ holds.*

Remark 3.15. If α is a $*^f$-approximative sequence from X to Y, the function $\alpha(n,m) : X \to Q$ will be denoted by $\alpha_n^m : X \to Q$, i.e., $\alpha = (\alpha_n^m) : X \to Y$.

Proposition 3.16. *A function $\alpha : \mathbb{N}^2 \to C(X,Q)$ such that, for every $n \in \mathbb{N}$, the inequality $\operatorname{card}(\{\alpha_n^m : m \in \mathbb{N}\}) < \aleph_0$ holds, is a $*^f$-approximative sequence from X to Y if and only if for every neighbourhood V of Y in Q there exists $n_0 \in \mathbb{N}$ such that for all $n, n' \geq n_0$ there exists $m_{nn'} \in \mathbb{N}$ such that*

$$\alpha_n^m \simeq \alpha_{n'}^m \quad \text{in } V, \quad \text{for every } m \geq m_{nn'}.$$

Definition 3.17. *A $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$ is said to be homotopic to a $*^f$-approximative sequence $\beta = (\beta_n^m) : X \to Y$ provided for every neighbourhood V of Y in Q there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*

$$\alpha_n^m \simeq \beta_n^m \quad \text{in } V, \quad \text{for every } m \geq m_n.$$

In that case, we write: $\alpha \sim \beta$. The relation $\sim$ is an equivalence relation on the set of all $*^f$-approximative sequences from X to Y. An equivalence class of a $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$ is denoted by $[\alpha]=[(\alpha_n^m)]$.

Given a $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$, if one chooses an arbitrary subsequence (n_k) of the sequence of natural numbers, then one obtains another $*^f$-approximative sequence $\alpha' = (\alpha_{n_k}^m) : X \to Y$ such that $\alpha \sim \alpha'$.

Notice that if $\alpha = (\alpha_n^m) : X \to Y$ and $\beta = (\beta_n^m) : Y \to Z$ are $*^f$-approximative sequences, then, since $\alpha_n^m : X \to Q$ and $\beta_n^m : Y \to Q$, for every $n, m \in \mathbb{N}$, any direct composition of the components of α and β is not possible. Hence, to obtain composition $[\beta] \circ [\alpha]$, we will need to involve $*^f$-fundamental sequences. The following proposition is proved straightforwardly by the definitions.

PROPOSITION 3.18. *If* $\Phi = (\Phi_n^m) : X \to Y$ *is a* $*^f$*-fundamental sequence, then* $\Phi|_X := (\Phi_n^m|_X)$ *is a* $*^f$*-approximative sequence from* X *to* Y*.*

With each $*^f$-approximative sequence is associated, up to the equivalence, a unique $*^f$-fundamental sequence as shown in the following Theorem 3.19 and Proposition 3.20. The following theorem is a generalization of [5, Theorem 2'].

THEOREM 3.19. *Let* $\alpha : X \to Y$ *be a* $*^f$*-approximative sequence. Then there exists a* $*^f$*-fundamental sequence* $\Phi : X \to Y$ *such that* $\Phi|_X \sim \alpha$*.*

PROOF. Let $(V_k)_k$ be a decreasing sequence of open neighbourhoods of Y in Q such that $\cap V_k = V$. For every $k \in \mathbb{N}$ there exists $n_0(k) \in \mathbb{N}$ such that for every $n \geq n_0(k)$ there exists $m_n(k) \in \mathbb{N}$ such that

$$\alpha_n^m \simeq \alpha_{n+1}^m \text{ in } V_k, \text{ for every } m \geq m_n(k).$$

Notice that one can choose the indices $n_0(k)$ increasingly, i.e., $n_0(k) < n_0(k+1)$, for every $k \in \mathbb{N}$. Define $n_k := n_0(k)$, for every $k \in \mathbb{N}$. Hence we obtained an increasing sequence of indices $(n_k)_k$ and a $*^f$-approximative sequence $\alpha_0 = (\alpha_{n_k}^m) : X \to Y$ such that $\alpha_0 \sim \alpha$. By the definition of α_0 it follows that for every $k \in \mathbb{N}$ there exists $m_{n_k} \in \mathbb{N}$ such that

$$\alpha_{n_k}^m \simeq \alpha_{n_{k+1}}^m \text{ in } V_k, \text{ for every } m \geq m_{n_k}.$$

We will now construct a decreasing sequence (W_k) of closed neighbourhoods of X in Q and a $*$-fundamental sequence $\Phi = (\Phi_k^m) : X \to Y$ such that

$$\Phi_k^m|_X = \alpha_{n_k}^m, \text{ for every } k, m \in \mathbb{N}$$

and, for every $k \in \mathbb{N}$ and every $n \geq k$,

$$\Phi_k^m|_{W_k} \simeq \Phi_n^m|_{W_k} \text{ in } V_k, \text{ for almost all } m.$$

The construction will be done inductively for all $k \in \mathbb{N}$, successively extending, for almost all $m \in \mathbb{N}$, functions $\alpha_{n_k}^m$ from X, over the neighbourhoods $W_k, W_{k-1}, \ldots, W_2, W_1$, up to the functions Φ_k^m from Q to Q.

For $k = 1$ there exists $m_{n_1} \in \mathbb{N}$ such that, for every $m \geq m_{n_1}$, $\alpha_{n_1}^m \simeq \alpha_{n_2}^m$ in V_1 holds. Let, for every $m \in \mathbb{N}$, $\Phi_1^m : Q \to Q$ be an arbitrary continuous extension of $\alpha_{n_1}^m : X \to Q$ (which exists because Q is an AR). Thereat, if $m, m' \geq m_{n_1}$ and $\alpha_{n_1}^m = \alpha_{n_1}^{m'}$, the extensions are chosen in a way that $\Phi_1^m = \Phi_1^{m'}$ (the same component functions $\alpha_{n_1}^m$ are always extended by the same extension). Notice that, for every $m \geq m_{n_1}$, $\alpha_{n_1}^m : X \to V_1 \subseteq Q$ and hence, by the continuity of Φ_1^m, there exists a neighbourhood U_1^m of X in Q such that $\Phi_1^m(U_1^m) \subseteq V_1$, for every $m \geq m_{n_1}$. Thereat, if $m, m' \geq m_{n_1}$ and

$\Phi_1^m = \Phi_1^{m'}$, the neighbourhoods are chosen in a way that $U_1^m = U_1^{m'}$ (the same component functions are always associated with the same neighbourhood). Since $\mathrm{card}(\{U_1^m : m \geq m_{n_1}\}) < \aleph_0$, the intersection $\bigcap_{m \geq m_{n_1}} U_1^m$ is a neighbourhood of X. Let

$$W_1 \subseteq \bigcap_{m \geq m_{n_1}} U_1^m$$

be a closed neighbourhood of X (which exists because of the normality of Q). Then $\Phi_1^m(W_1) \subseteq V_1$, for every $m \geq m_{n_1}$.

For $k = 2$ there exists $m_{n_2} \in \mathbb{N}$ such that, for every $m \geq m_{n_2}$, $\alpha_{n_2}^m \simeq \alpha_{n_3}^m$ in V_2. Since, for every $m \geq m_{n_1}$, $\alpha_{n_1}^m \simeq \alpha_{n_2}^m$ in V_1 holds and $\alpha_{n_1}^m$ has a continuous extension $\Phi_1^m|_{W_1} : W_1 \to V_1$, by homotopy extension property for V_1 there exists an extension $\Phi_2'^m : W_1 \to V_1$ of $\alpha_{n_2}^m$ such that

$$\Phi_2'^m \simeq \Phi_1^m|_{W_1} \ \text{ in } \ V_1, \ \text{ for every } m \geq m_{n_1}.$$

Thereat, if $m, m' \geq m_{n_1}$ and $\alpha_{n_2}^m = \alpha_{n_2}^{m'}$, the extensions are chosen in a way that $\Phi_2'^m = \Phi_2'^{m'}$ (the same component functions $\alpha_{n_2}^m$ are always extended by the same extension). Let, for every $m \geq m_{n_1}$, $\Phi_2^m : Q \to Q$ be an arbitrary continuous extension of $\Phi_2'^m : W_1 \to V_1 \subseteq Q$ (paying attention that, as before, the same component functions are always extended by the same extension) and for $m < m_{n_1}$ let $\Phi_2^m : Q \to Q$ be an arbitrary continuous extension of $\alpha_{n_2}^m : X \to Q$. Since

$$\alpha_{n_2}^m : X \to V_2, \ \text{ for every } m \geq m_{n_2} \text{ and } \Phi_2^m|_X = \alpha_{n_2}^m, \ \text{ for every } m \geq m_{n_1},$$

by the continuity of Φ_2^m, for every $m \geq m_{12} = \max\{m_{n_1}, m_{n_2}\}$ there exists a neighbourhood U_2^m of X in Q such that $\Phi_2^m(U_2^m) \subseteq V_2$. Thereat, if $m, m' \geq m_{12}$ and $\Phi_2^m = \Phi_2^{m'}$, the neighbourhoods are chosen in a way that $U_2^m = U_2^{m'}$ (the same component functions are always associated with the same neighbourhood). Since $\mathrm{card}(\{U_2^m : m \geq m_{12}\}) < \aleph_0$, the intersection $\bigcap_{m \geq m_{12}} U_2^m$ is a neighbourhood of X. Let

$$W_2 \subseteq \bigcap_{m \geq m_{12}} U_2^m \cap W_1$$

be a closed neighbourhood of X (which exists because of the normality of Q). Then $\Phi_2^m(W_2) \subseteq V_2$, for every $m \geq m_{12}$. In this step we have achieved

$$\Phi_1^m|_{W_1} \simeq \Phi_2^m|_{W_1} \ \text{ in } \ V_1, \ \text{ for every } m \geq m_{12}.$$

For an arbitrary $k = n$ the construction is performed analogously.

We claim that the obtained $\Phi = (\Phi_k^m)$ is a $*^f$-fundamental sequence from X to Y. Let V be an arbitrary neighbourhood of Y in Q. Then there exists $k \in \mathbb{N}$ such that $V_k \subseteq V$. By the construction there exists a neighbourhood W_k of X in Q such that for every $n \geq k$ there exists an index m_{kn} sufficiently large such that

$$\Phi_k^m|_{W_k} \simeq \Phi_n^m|_{W_k} \ \text{ in } \ V_k \subseteq V, \ \text{ for every } m \geq m_{kn}.$$

Moreover, for every $k \in \mathbb{N}$ the inequality $\mathrm{card}(\{\Phi_k^m : m \in \mathbb{N}\}) < \aleph_0$, holds and, hence, $\Phi : X \to Y$ is a $*^f$-fundamental sequence. Finally, since

$$\Phi_k^m|_X = \alpha_{n_k}^m, \quad \text{for every } n, m \in \mathbb{N},$$

we have $\alpha_0 = (\alpha_{n_k}^m) = \Phi|_X$ and, because $\alpha_0 \sim \alpha$, $\Phi|_X \sim \alpha$ holds. This completes the proof. ∎

PROPOSITION 3.20. *Let $\Phi, \Phi' : X \to Y$ be $*^f$-fundamental sequences. Then $\Phi \sim \Phi'$ if and only if $\Phi|_X \sim \Phi'|_X$.*

PROOF. The necessity is trivial. Suppose that $\Phi|_X \sim \Phi'|_X$ and let V be an arbitrary open neighbourhood of Y in Q. Then there exist neighbourhoods U' and U'' of X in Q and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that

$$\Phi_{n_0}^m|_{U'} \simeq \Phi_n^m|_{U'} \text{ in } V, \quad \Phi_{n_0}'^m|_{U''} \simeq \Phi_n'^m|_{U''} \text{ in } V \text{ and}$$

$$\Phi_n^m|_X \simeq \Phi_n'^m|_X \text{ in } V, \text{ for every } m \geq m_n.$$

By the homotopy extension theorem, since X is closed in Q and V is an ANR, for every $m \geq m_{n_0}$ there exists a neighbourhood U^m of X in Q such that

$$\Phi_{n_0}^m|_{U^m} \simeq \Phi_{n_0}'^m|_{U^m} \text{ in } V.$$

Thereat, if $m, m' \geq m_{n_0}$ and $\left(\Phi_{n_0}^m, \Phi_{n_0}'^m\right) = \left(\Phi_{n_0}^{m'}, \Phi_{n_0}'^{m'}\right)$, the neighbourhoods are chosen in a way that $U^m = U^{m'}$. Since

$$\begin{aligned} &\mathrm{card}\left(\left\{\left(\Phi_{n_0}^m, \Phi_{n_0}'^m\right) : m \geq m_{n_0}\right\}\right) \\ &\qquad \leq \mathrm{card}\left(\left\{\Phi_{n_0}^m : m \geq m_{n_0}\right\}\right) \cdot \mathrm{card}\left(\left\{\Phi_{n_0}'^m : m \geq m_{n_0}\right\}\right) < \aleph_0, \end{aligned}$$

it holds that $\mathrm{card}(\{U^m : m \geq m_{n_0}\}) < \aleph_0$ and so $U''' := \bigcap_{m \geq m_{n_0}} U^m$ is a neighbourhood of X in Q such that

$$\Phi_{n_0}^m|_U \simeq \Phi_{n_0}'^m|_U \text{ in } V, \text{ for every } m \geq m_{n_0}.$$

For $U = U' \cap U'' \cap U'''$ and for every $n \geq n_0$ we have

$$\Phi_n^m|_U \simeq \Phi_{n_0}^m|_U \simeq \Phi_{n_0}'^m|_U \simeq \Phi_n'^m|_U \text{ in } V, \text{ for every } m \geq \max\{m_n, m_{n_0}\},$$

which means that $\Phi \sim \Phi'$. ∎

It is easy to check that the coordinatewise composition of a $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$ and a $*^f$-fundamental sequence $\Phi = (\Phi_n^m) : Y \to Z$ is a $*^f$-approximative sequence $\Phi \circ \alpha = (\Phi_n^m \circ \alpha_n^m) : X \to Z$.

PROPOSITION 3.21. *Let $\alpha, \alpha' : X \to Y$ be $*^f$-approximative sequences such that $\alpha \sim \alpha'$ and let $\Phi, \Phi' : Y \to Z$ be $*^f$-fundamental sequences such that $\Phi \sim \Phi'$. Then $\Phi \circ \alpha \sim \Phi' \circ \alpha'$.*

Hence, the composition of the equivalence class $[\alpha] : X \to Y$ of $*^f$-approximative sequences with the equivalence class $[\Psi] : Y \to Z$ of $*^f$-fundamental sequences is well defined by the representatives, i.e., $[\Psi] \circ [\alpha] := [\Psi \circ \alpha]$, and enables us to define the *composition* of the equivalence classes of $*^f$-approximative sequences. Let $[\alpha]$ and $[\beta]$ be the equivalence classes of $*^f$-approximative sequences $\alpha : X \to Y$ and $\beta : Y \to Z$. Then there exists a unique equivalence class $[\Psi] : Y \to Z$ of $*^f$-fundamental sequences such that $[\Psi|_Y] = [\beta]$. We define:

$$[\beta] \circ [\alpha] = [\Psi|_Y] \circ [\alpha] := [\Psi] \circ [\alpha] = [\Psi \circ \alpha].$$

By the previous remarks and Proposition 3.21, the composition of the equivalence classes of $*^f$-approximative sequences is well defined. It is easy to prove that the composition is associative and, for an arbitrary closed subset X of Q, the *identity* on X is an equivalence class of the $*^f$-approximative sequence $1_X = (1_n^m) : X \to X$ such that $1_n^m = \mathrm{id}_X : X \to X$, $n, m \in \mathbb{N}$. Hence, all the closed subsets of Q taken as objects and all the equivalence classes of $*^f$-approximative sequences taken as morphisms form a category denoted by $Sh_a^{*^f}$.

It remains to define a category of the equivalence classes of $*^f$-proximate sequences which will give a description of the *intrinsic* finite coarse shape.

Definition 3.22. *Let X and Y be closed subsets of Q. A function $a : \mathbb{N}^2 \to Y^X$ is called a $*^f$-proximate sequence from X to Y provided:*

(1) *for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*

$$a(n, m) \overset{\epsilon}{\simeq} a(n+1, m), \quad \textit{for every } m \geq m_n;$$

(2) *for every $n \in \mathbb{N}$ the inequality* $\mathrm{card}(\{a(n, m) : m \in \mathbb{N}\}) < \aleph_0$ *holds.*

Remark 3.23. If a is a $*^f$-proximate sequence, the function $a(n, m) : X \to Y$ will be denoted by $a_n^m : X \to Y$, i.e., $a = (a_n^m) : X \to Y$.

Proposition 3.24. *A function $a : \mathbb{N}^2 \to Y^X$ such that, for every $n \in \mathbb{N}$, the inequality* $\mathrm{card}(\{a_n^m : m \in \mathbb{N}\}) < \aleph_0$ *holds, is a $*^f$-proximate sequence from X to Y if and only if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, n' \geq n_0$ there exists $m_{nn'} \in \mathbb{N}$ such that*

$$a_n^m \overset{\epsilon}{\simeq} a_{n'}^m, \quad \textit{for every } m \geq m_{nn'}.$$

Definition 3.25. *A $*^f$-proximate sequence $a = (a_n^m) : X \to Y$ is said to be homotopic to a $*^f$-proximate sequence $b = (b_n^m) : X \to Y$ provided for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that*

$$a_n^m \overset{\epsilon}{\simeq} b_n^m, \quad \textit{for every } m \geq m_n.$$

In that case, we write: $a \sim b$. The relation $\sim$ is an equivalence relation on the set of all $*^f$-proximate sequences from X to Y. An equivalence class of a $*^f$-proximate sequence $a = (a_n^m) : X \to Y$ is denoted by $[a] = [(a_n^m)]$. Notice that, due to the poor properties of the composition of ϵ-continuous functions (as shown in Example 2.3), the composition of $*^f$-proximate sequences cannot be defined coordinatewise. Hence, the *composition* of $*^f$-proximate sequences $a = (a_k^m) : X \to Y$ and $b = (b_n^m) : Y \to Z$ is defined in the following way:

Let (ϵ_n) be a decreasing sequence of positive real numbers such that $\lim \epsilon_n = 0$ and that, for every $n_0 \in \mathbb{N}$ and every $n \geq n_0$,

$$b_{n_0}^m \overset{\frac{\epsilon_{n_0}}{2}}{\simeq} b_n^m, \quad \text{for every} \;\; m \geq m_{n_0 n}.$$

Let (δ_n) be a decreasing sequence of positive real numbers such that $\lim \delta_n = 0$ and that, for every $n \in \mathbb{N}$, for every $m \geq m_n$ and for all $y, y' \in Y$ such that $d(y, y') < \delta_n$,

$$d(b_n^m(y), b_n^m(y')) < \epsilon_n.$$

It is easy to see that such sequence (δ_n) really exists. Namely, for every $n \in \mathbb{N}$ and for every $m \geq m_n$, the function b_n^m yields a number δ_n^m (the uniformity radius of the uniformly ϵ_n-continuous function b_n^m) such that $d(y, y') < \delta_n^m$ implies $d(b_n^m(y), b_n^m(y')) < \epsilon_n$. Notice that for an arbitrary $n \in \mathbb{N}$, by choosing for all $m, m' \geq m_n$, $\delta_n^m = \delta_n^{m'}$ whenever $b_n^m = b_n^{m'}$ (the same component functions b_n^m are always associated with the same uniformity radius), one can assure that $\operatorname{card}(\{\delta_n^m : m \geq m_n\}) < \aleph_0$ and so there exists

$$\delta_n = \min(\{\delta_n^m : m \geq m_n\} \cup \{\delta_1, \ldots \delta_{n-1}\}) > 0$$

with the required property. Finally, let (k_n) be a strictly increasing sequence of indices such that, for every $k \geq k_n$,

$$a_{k_n}^m \overset{\frac{\delta_n}{2}}{\simeq} a_k^m, \quad \text{for every } m \geq m'_{k_n k}.$$

Now, for all $n, m \in \mathbb{N}$ we define $c_n^m = b_n^m \circ a_{k_n}^m$.

Proposition 3.26. *If $a = (a_k^m) : X \to Y$ and $b = (b_n^m) : Y \to Z$ are $*^f$-proximate sequences, then $c = (c_n^m)$, $c_n^m = b_n^m \circ a_{k_n}^m$, is a $*^f$-proximate sequence from X to Z.*

Proof. Given an arbitrary $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\epsilon_{n_0} < \epsilon$. Fix any $n \geq n_0$. Then there exists a sequence of $\frac{\epsilon_{n_0}}{2}$-homotopies $H_n^m : Y \times I \to Z$ such that

(1) $H_n^m(\cdot, 0) = b_{n_0}^m$ and $H_n^m(\cdot, 1) = b_n^m$, for every $m \geq m_{n_0 n}$;
(2) $\operatorname{card}(\{H_n^m : m \geq m_{n_0 n}\}) < \aleph_0$.

The property (2) is achievable because

$$\begin{aligned} &\operatorname{card}(\{(b_{n_0}^m, b_n^m) : m \geq m_{n_0 n}\}) \\ &\leq \operatorname{card}(\{b_{n_0}^m : m \geq m_{n_0 n}\}) \cdot \operatorname{card}(\{b_n^m : m \geq m_{n_0 n}\}) < \aleph_0. \end{aligned}$$

For every $m \geq m_{n_0 n}$, let δ_n^m be the uniformity radius of $\frac{\epsilon_{n_0}}{2}$-continuous function H_n^m. The property (2) allows $\mathrm{card}(\{\delta_n^m : m \geq m_{n_0 n}\}) < \aleph_0$ and so there exists $\delta = \min\{\delta_n^m : m \geq m_{n_0 n}\} > 0$. Let $p \geq n$ be an index such that $\delta_p < \delta$. Then, by Propositions 2.5 and 2.9,

$$b_{n_0}^m \circ a_{k_p}^m \overset{\epsilon_{n_0}}{\simeq} b_n^m \circ a_{k_p}^m, \quad \text{for every } m \geq \max\{m_{n_0 n}, m'_{k_p}\}$$

by the ϵ_{n_0}-homotopy $H_n^m \circ (a_{k_p}^m, id_I) : X \times I \to Z$. Since $k_p \geq k_n$, it follows that

$$a_{k_p}^m \overset{\frac{\delta_n}{2}}{\simeq} a_{k_n}^m, \quad \text{for every } m \geq m'_{k_n k_p}$$

by the $\frac{\delta_n}{2}$-homotopy $G_p^m : X \times I \to Y$. By Proposition 2.9,

$$b_n^m \circ a_{k_p}^n \overset{\epsilon_n}{\simeq} b_n^m \circ a_{k_n}^m, \quad \text{for every } m \geq \max\{m_n, m'_{k_n k_p}\}$$

by the ϵ_n-homotopy $b_n^m \circ G_p^m : X \times I \to Z$. Furthermore, since

$$a_{k_p}^m \overset{\frac{\delta_{n_0}}{2}}{\simeq} a_{k_{n_0}}^m, \quad \text{for every } m \geq m'_{k_{n_0} k_p}$$

by the $\frac{\delta_{n_0}}{2}$-homotopy $G_p^{\prime m} : X \times I \to Y$, Proposition 2.9 implies that

$$b_{n_0}^m \circ a_{k_p}^n \overset{\epsilon_{n_0}}{\simeq} b_{n_0}^m \circ a_{k_{n_0}}^m, \quad \text{for every } m \geq \max\{m_{n_0}, m'_{k_{n_0} k_p}\}$$

by the ϵ_{n_0}-homotopy $b_{n_0}^m \circ G_p^{\prime m} : X \times I \to Z$. Finally, the transitivity of the ϵ_{n_0}-homotopy gives

$$b_n^m \circ a_{k_n}^m \overset{\epsilon_{n_0}}{\simeq} b_{n_0}^m \circ a_{k_{n_0}}^m,$$

for every

$$m \geq m''_{n_0 n} = \max\{m_{n_0}, m_{n_0 n}, m'_{k_p}, m'_{k_n k_p}, m'_{k_{n_0} k_p}\}$$

and $\epsilon_{n_0} < \epsilon$ implies

$$c_{n_0}^m \overset{\epsilon}{\simeq} c_n^m, \quad \text{for every } m \geq m''_{n_0 n}.$$

Moreover, for every $n \in \mathbb{N}$, the inequality

$$\mathrm{card}(\{b_n^m \circ a_{k_n}^m : m \in \mathbb{N}\}) \leq \mathrm{card}(\{b_n^m : m \in \mathbb{N}\}) \cdot \mathrm{card}(\{a_{k_n}^m : m \in \mathbb{N}\}) < \aleph_0$$

holds and $c = (b_n^m \circ a_{k_n}^m)$ is a $*^f$-proximate sequence from X to Z. □

It is straightforward to prove that the equivalence class of the composition of $*^f$-proximate sequences does not depend either on the representatives of the equivalence classes or on the choices of the sequences (ϵ_n), (δ_n) and (k_n) made in the composition.

The *composition* of the equivalence classes $[a] = [(a_k^m)] : X \to Y$ and $[b] = [(b_n^m)] : Y \to Z$ is defined by the representatives, i.e., $[b] \circ [a] := [b \circ a] = [(b_n^m \circ a_{k_n}^m)]$. It is easy to prove that the composition is associative and, for an arbitrary closed subset X of Q, the *identity* on X is an equivalence class of the $*^f$-proximate sequence $1_X = (1_n^m) : X \to X$ such that $1_n^m = \mathrm{id}_X : X \to X$,

$n, m \in \mathbb{N}$. Hence, all the closed subsets of Q taken as objects and all the equivalence classes of $*^f$-proximate sequences taken as morphisms form a category denoted by $InSh^{*^f}$.

In the following section we will prove the second main result of this paper – the category $InSh^{*^f}$ is isomorphic to the restriction on closed subsets of Q of the topological finite coarse shape category Sh^{*^f}.

4. The isomorphisms of the finite coarse shape categories

4.1. *The isomorphism of the categories $Sh^{*^f}|_Q$ and $Sh^{*^f}_f$.*

Let $Sh^{*^f}|_Q$ denote the restriction on closed subsets of Q of the topological finite coarse shape category Sh^{*^f}. We shall associate an equivalence class of a $*^f$-fundamental sequence $\Phi = (\Phi_n^m) : X \to Y$ with a finite coarse shape morphism $F^{*^f} : X \to Y$.

Let (X_n) and (Y_n) be a decreasing basis of open neighbourhoods of X and Y in Q respectively such that $\cap X_n = X$ and $\cap Y_n = Y$. For every pair $n \leq n'$, let $p_{nn'} : X_{n'} \to X_n$ and $q_{nn'} : Y_{n'} \to Y_n$ be the inclusions and $\mathbf{X} = (X_n, p_{nn'}, \mathbb{N})$, $\mathbf{Y} = (Y_n, q_{nn'}, \mathbb{N})$ be the inverse systems of ANR-s. Hence, the inclusions $p_n : X \to X_n$ and $q_n : Y \to Y_n$ determine morphisms $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ in $pro-Top$ such that, by [7, Theorem 4, Ch. I, §4.2], morphisms $H\mathbf{p} : X \to H\mathbf{X}$ and $H\mathbf{q} : Y \to H\mathbf{Y}$ are $HPol$-expansions of X and Y, respectively.

For $n \in \mathbb{N}$ and associated neighbourhood Y_n of Y there exist a neighbourhood U_n of X in Q and $n_{Y_n} \in \mathbb{N}$ such that for every $n' \geq n_{Y_n}$ there exists $m_{n'}(n) \in \mathbb{N}$ such that

$$\Phi_{n'}^m|_{U_n} \simeq \Phi_{n_{Y_n}}^m|_{U_n} \text{ in } Y_n, \text{ for every } m \geq m_{n'}(n).$$

Let us now define a function $f : \mathbb{N} \to \mathbb{N}$ such that $X_{f(n)} \subseteq U_n$, for every $n \in \mathbb{N}$. Obviously,

$$\Phi_{n'}^m|_{X_{f(n)}} \simeq \Phi_{n_{Y_n}}^m|_{X_{f(n)}} \text{ in } Y_n, \text{ for every } n' \geq n_{Y_n} \text{ and for every } m \geq m_{n'}(n).$$

By putting

$$f_n^m = \Phi_{n_{Y_n}}^m|_{X_{f(n)}}, \text{ for every } m \in \mathbb{N},$$

we defined, for every $n \in \mathbb{N}$, a sequence of mappings $f_n^m : X_{f(n)} \to Y_n$, $m \in \mathbb{N}$, such that for every $n \in \mathbb{N}$,

$$f_n^m \simeq \Phi_{n'}^m|_{X_{f(n)}} \text{ in } Y_n, \text{ for every } n' \geq n_{Y_n} \text{ and for every } m \geq m_{n'}(n).$$

We will now prove that constructed (f, f_n^m) is a $*^f$-morphism from $\mathbf{X}$ to $\mathbf{Y}$. For an arbitrary pair $n \leq n'$ define $\lambda = \max\{f(n), f(n')\}$, $n_0 = \max\{n_{Y_n}, n_{Y_{n'}}\}$ and $m_0 = \max\{m_{n_0}(n), m_{n_0}(n')\}$. Notice that

$$q_{nn'} \circ f_{n'}^m \circ p_{f(n')\lambda} = q_{nn'} \circ f_{n'}^m|_{X_\lambda} = q_{nn'} \circ \Phi_{n_{Y_{n'}}}^m|_{X_\lambda},$$

and, by the construction,

$$\Phi^m_{n_{Y_{n'}}}|_{X_\lambda} \simeq \Phi^m_{n_0}|_{X_\lambda} \text{ in } Y_{n'}, \text{ for every } m \geq m_0$$

holds, so we have (because $Y_{n'} \subseteq Y_n$)

$$q_{nn'} \circ \Phi^m_{n_{Y_{n'}}}|_{X_\lambda} \simeq q_{nn'} \circ \Phi^m_{n_0}|_{X_\lambda} \text{ in } Y_n, \text{ za svaki } m \geq m_0.$$

Furthermore,

$$q_{nn'} \circ \Phi^m_{n_0}|_{X_\lambda} \simeq \Phi^m_{n_0}|_{X_\lambda} \text{ in } Y_n, \text{ for every } m \geq m_0,$$

and, by the construction,

$$\Phi^m_{n_0}|_{X_\lambda} \underset{\text{in } Y_n}{\simeq} \Phi^m_{n_{Y_n}}|_{X_\lambda} = f^m_n|_{X_\lambda} = f^m_n \circ p_{f(n)\lambda}, \text{ for every } m \geq m_0$$

holds. Therefore,

$$q_{nn'} \circ f^m_{n'} \circ p_{f(n')\lambda} \simeq f^m_n \circ p_{f(n)\lambda}, \text{ for every } m \geq m_0, \text{ i.e.,}$$
$$[q_{nn'}] \circ [f^m_{n'}] \circ [p_{f(n')\lambda}] = [f^m_n] \circ [p_{f(n)\lambda}], \text{ for every } m \geq m_0$$

in the category $HPol$.

Moreover, for every $n \in \mathbb{N}$

$$\text{card}(\{[f^m_n] : m \in \mathbb{N}\}) \leq \text{card}(\{\Phi^m_{n_{Y_n}} : m \in \mathbb{N}\}) < \aleph_0$$

holds and thus we proved that the homotopy classes $([f^m_n])$ together with the index function f form a $*^f$-morphism $(f, f^m_n) : \mathbf{X} \to \mathbf{Y}$ in inv^{*^f}-$HPol$. A class $\mathbf{f}^{*^f} = [(f, f^m_n)] : \mathbf{X} \to \mathbf{Y}$ of (f, f^m_n) is a morphism in pro^{*^f}-$HPol$ for which there exists a unique finite coarse shape morphism $F^{*^f} : X \to Y$. Let ω be a function which associates every $*^f$-fundamental sequence $\Phi : X \to Y$ with a $*^f$-morphism $(f, f^m_n) : \mathbf{X} \to \mathbf{Y}$ as in the previous construction. For every pair of closed subsets X, Y of Q, we define a function

$$\Omega_{X,Y} : Sh^{*^f}_f(X, Y) \to Sh^{*^f}|_Q(X, Y),$$
$$\Omega_{X,Y}([\Phi]) := \langle[\omega(\Phi)]\rangle = \langle[(f, f^m_n)]\rangle = F^{*^f}.$$

The following proposition states that $\Omega_{X,Y}$ is well defined.

Proposition 4.1. *If $\Phi, \Phi' : X \to Y$ are $*^f$-fundamental sequences such that $\Phi \sim \Phi'$, then $\Omega_{X,Y}([\Phi]) = \Omega_{X,Y}([\Phi'])$.*

Proof. Let

$$\Omega_{X,Y}([\Phi]) = F^{*k} = \langle[(f, f^m_n)]\rangle = \langle[\omega(\Phi)]\rangle \text{ and}$$
$$\Omega_{X,Y}([\Phi']) = F^{'*k} = \langle[(f', f'^m_n)]\rangle = \langle[\omega(\Phi')]\rangle.$$

To prove that $F^{*k} = F^{'*k}$ it suffices to prove that $*^f$-morphisms (f, f^m_n) and (f', f'^m_n) are equivalent in inv^{*^f}-$HPol$.

For an arbitrary $n \in \mathbb{N}$, $(\Phi_n^m) \sim (\Phi_n^{'m})$ implies that the neighbourhood Y_n of Y admits a neighbourhood U_1 of X in Q and $n^1 \in \mathbb{N}$ such that for every $n' \geq n^1$ there exists $m_{n'}^1$ such that

$$\Phi_{n'}^m|_{U_1} \simeq \Phi_{n'}^{'m}|_{U_1} \text{ in } Y_n, \text{ for every } m \geq m_{n'}^1.$$

Furthermore, there exists $n_0 \in \mathbb{N}$ such that $X_{n_0} \subseteq U_1$. Take

$$\lambda = \max\{f(n), f'(n), n_0\} \text{ and } n_1 = \max\{n^1, n_{Y_n}, n'_{Y_n}\}.$$

For every $m \in \mathbb{N}$, the relations

$$f_n^m \circ p_{f(n)\lambda} = f_n^m|_{X_\lambda} = \Phi_{n_{Y_n}}^m|_{X_\lambda} \text{ and}$$
$$f_n^{'m} \circ p_{f'(n)\lambda} = f_n^{'m}|_{X_\lambda} = \Phi_{n'_{Y_n}}^{'m}|_{X_\lambda}$$

hold. Hence,

$$\Phi_{n_{Y_n}}^m|_{X_\lambda} \simeq \Phi_{n_1}^m|_{X_\lambda} \simeq \Phi_{n_1}^{'m}|_{X_\lambda} \simeq \Phi_{n'_{Y_n}}^{'m}|_{X_\lambda} \text{ in } Y_n, \text{ i.e.,}$$
$$f_n^m \circ p_{f(n)\lambda} \simeq f_n^{'m} \circ p_{f'(n)\lambda}, \text{ for almost all } m \in \mathbb{N},$$

Thus, $(f, f_n^m) \sim (f', f_n^{'m})$ in inv^{*^f}-$HPol$ and $F^{*k} = F'^{*^f}$. ∎

One can easily prove that the associated finite coarse shape morphism F^{*^f} does not depend on the choice of the basis of neighbourhoods (X_n) and (Y_n) of X and Y in Q, respectively.

Proposition 4.2. *Let X, Y, Z be closed subsets of Q and let*

$$[\Phi] \in Sh_f^{*^f}(X, Y), [\Psi] \in Sh_f^{*^f}(Y, Z)$$

be arbitrary morphisms. Then

$$\Omega_{X,Z}([\Psi] \circ [\Phi]) = \Omega_{Y,Z}([\Psi]) \circ \Omega_{X,Y}([\Phi]).$$

Proof. Denote $\Omega_{X,Y}([\Phi]) = F^{*^f}$, $\Omega_{Y,Z}([\Psi]) = G^{*^f}$ and $\Omega_{X,Z}([\Psi] \circ [\Phi]) = H^{*^f}$. We need to prove that $H^{*^f} = G^{*^f} \circ F^{*^f}$. Let $*^f$-fundamental sequences $(\Phi_n^m) : X \to Y$ and $(\Psi_n^m) : Y \to Z$ be the representatives of the classes $[\Phi]$ and $[\Psi]$, respectively. Now (Φ_n^m), (Ψ_n^m) and $(\Theta_n^m) = (\Psi_n^m) \circ (\Phi_n^m)$ induce $*^f$-morphisms $(f, f_n^m) : \mathbf{X} \to \mathbf{Y}$, $(g, g_n^m) : \mathbf{Y} \to \mathbf{Z}$ and $(h, h_n^m) : \mathbf{X} \to \mathbf{Z}$, respectively, in inv^{*^f}-$HPol$ such that

$$\mathbf{f}^{*^f} = [(f, f_n^m)] : \mathbf{X} \to \mathbf{Y},$$
$$\mathbf{g}^{*^f} = [(g, g_n^m)] : \mathbf{Y} \to \mathbf{Z},$$
$$\mathbf{h}^{*^f} = [(h, h_n^m)] : \mathbf{X} \to \mathbf{Z},$$

where $\mathbf{f}^{*^f}$, $\mathbf{g}^{*^f}$ and $\mathbf{h}^{*^f}$ are morphisms in pro^{*^f}-$HPol$ which induce finite coarse shape morphisms F^{*^f}, G^{*^f} and H^{*^f}, respectively. Define

$$(g, g_n^m) \circ (f, f_n^m) = (h', h_n^{'m}).$$

To prove that $H^{*^f} = G^{*^f} \circ F^{*^f}$ it suffices to prove that $(h, h_n^m) \sim (h', h_n^{'m})$ in inv^{*^f}-$HPol$. For an arbitrary $n \in \mathbb{N}$ denote $\lambda = \max\{h(n), h'(n)\}$. Notice that the relations

$$\begin{aligned} h_n^{'m} \circ p_{h'(n)\lambda} &= (g_n^m \circ f_{g(n)}^m) \circ p_{f(g(n))\lambda} = g_n^m \circ (f_{g(n)}^m \circ p_{f(g(n))\lambda}) \\ &= g_n^m \circ f_{g(n)}^m|_{X_\lambda} = \Psi_{n_{Z_n}}^m \circ \Phi_{n_{Y_{g(n)}}}^m|_{X_\lambda} \text{ and} \\ h_n^m \circ p_{h(n)\lambda} &= h_n^m|_{X_\lambda} = \Theta_{n'_{Z_n}}^m|_{X_\lambda} = \Psi_{n'_{Z_n}}^m \circ \Phi_{n'_{Z_n}}^m|_{X_\lambda} \end{aligned}$$

hold for every $m \in \mathbb{N}$. For $n' = \max\{n_{Z_n}, n_{Y_{g(n)}}, n'_{Z_n}\}$,

$$\Phi_{n_{Y_{g(n)}}}^m|_{X_\lambda} \simeq \Phi_{n'}^m|_{X_\lambda} \text{ in } Y_{g(n)}, \text{ for every } m \geq m_{n'}(g(n)) \text{ and}$$

$$\Psi_{n_{Z_n}}^m|_{Y_{g(n)}} \simeq \Psi_{n'}^m|_{Y_{g(n)}} \text{ in } Z_n, \text{ for every } m \geq m'_{n'}(n)$$

hold. Thus, for every $m \geq \max\{m'_{n'}(n), m_{n'}(g(n))\}$, the compatibility of the homotopy with the composition imply

$$\Psi_{n_{Z_n}}^m \circ \Phi_{n_{Y_{g(n)}}}^m|_{X_\lambda} \simeq \Psi_{n'}^m \circ \Phi_{n'}^m|_{X_\lambda} \text{ in } Z_n.$$

Furthermore,

$$\begin{aligned} &\Theta_{n'_{Z_n}}^m|_{X_\lambda} \simeq \Theta_{n'}^m|_{X_\lambda} \text{ in } Z_n, \text{ for every } m \geq m''_{n'}(n), \text{ i.e.,} \\ &\Psi_{n'_{Z_n}}^m \circ \Phi_{n'_{Z_n}}^m|_{X_\lambda} \simeq \Psi_{n'}^m \circ \Phi_{n'}^m|_{X_\lambda} \text{ in } Z_n, \text{ for every } m \geq m''_{n'}(n). \end{aligned}$$

Finally, for $m_0 = \max\{m'_{n'}(n), m_{n'}(g(n)), m''_{n'}(n)\}$, the transitivity of the relation of homotopy implies

$$\Psi_{n_{Z_n}}^m \circ \Phi_{n_{Y_{g(n)}}}^m|_{X_\lambda} \simeq \Psi_{n'_{Z_n}}^m \circ \Phi_{n'_{Z_n}}^m|_{X_\lambda} \text{ in } Z_n, \text{ for every } m \geq m_0.$$

Hence,

$$h_n^m \circ p_{h(n)\lambda} \simeq h_n^{'m} \circ p_{h'(n)\lambda}, \text{ for every } m \geq m_0,$$

i.e., $(h, h_n^m) \sim (h', h_n^{'m})$. ∎

By Proposition 4.2,

$$\Omega : Sh_f^{*^f} \to Sh^{*^f}|_Q$$

$$\Omega(X) = X, \quad \Omega([\Phi]) := \Omega_{X,Y}([\Phi]) = F^{*^f},$$

is a functor.

Theorem 4.3. *The functor $\Omega : Sh_f^{*^f} \to Sh^{*^f}|_Q$ is an isomorphism.*

Proof. Let X and Y be closed subsets of Q. We shall prove that $\Omega_{|(X,Y)} : Sh_f^{*^f}(X, Y) \to Sh^{*^f}|_Q(X, Y)$ is a bijection.

Injectivity: Let $[\Phi], [\Phi'] \in Sh_f^{*^f}(X, Y)$ be such that

$$F^{*^f} = \Omega([\Phi]) = \Omega([\Phi']) = F^{'*^f}$$

and let $(\Phi_n^m), (\Phi_n^{'m}) : X \to Y$ be $*^f$-fundamental sequences such that $[\Phi] = [(\Phi_n^m)]$ and $[\Phi'] = [(\Phi_n^{'m})]$. There exists a unique $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ in

pro^{*^f}-$HPol$ such that $F^{*^f} = \langle \mathbf{f}^{*^f} \rangle = F'^{*^f}$ and $[(f, f_n^m)] = \mathbf{f}^{*^f} = [(f', f_n^{'m})]$, where $(f, f_n^m), (f', f_n^{'m}) : \mathbf{X} \to \mathbf{Y}$ are $*^f$-morphisms in inv^{*^f}-$HPol$ such that $\omega(\Phi_n^m) = (f, f_n^m)$ and $\omega(\Phi_n^{'m}) = (f', f_n^{'m})$. We claim that $[\Phi] = [\Phi']$, i.e., $(\Phi_n^m) \sim (\Phi_n^{'m})$. For an arbitrary $n \in \mathbb{N}$, the relations

$$f_n^m \simeq \Phi_{n'}^m|_{X_{f(n)}} \text{ in } Y_n, \text{ for every } n' \geq n_{Y_n} \text{ and for every } m \geq m_{n'}(n) \text{ and}$$

$$f_n^{'m} \simeq \Phi_{n'}^{'m}|_{X_{f'(n)}} \text{ in } Y_n, \text{ for every } n' \geq n'_{Y_n} \text{ and for every } m \geq m'_{n'}(n)$$

hold. Define $n_1 = \max\{f(n), f'(n)\}$, $n''_{Y_n} = \max\{n_{Y_n}, n'_{Y_n}\}$ and $m_n^1 = \max\{m_{n'}(n), m'_{n'}(n)\}$. Now

$$f_n^m|_{X_{n_1}} \simeq \Phi_{n'}^m|_{X_{n_1}} \text{ in } Y_n, \text{ for every } n' \geq n''_{Y_n} \text{ and for every } m \geq m_n^1 \text{ and}$$

$$f_n^{'m}|_{X_{n_1}} \simeq \Phi_{n'}^{'m}|_{X_{n_1}} \text{ in } Y_n, \text{ for every } n' \geq n''_{Y_n} \text{ and for every } m \geq m_n^1$$

hold. Furthermore, by the assumption $(f, f_n^m) \sim (f', f_n^{'m})$ and so for every $n \in \mathbb{N}$ there exist $\lambda_n \geq \max\{f(n), f'(n)\} = n_1$ and $m_n^2 \in \mathbb{N}$ such that

$$f_n^m \circ p_{f(n)\lambda} \simeq f_n^{'m} \circ p_{f'(n)\lambda}, \text{ for every } m \geq m_n^2, \text{ i.e.,}$$

$$f_n^m|_{X_{\lambda_n}} \simeq f_n^{'m}|_{X_{\lambda_n}} \text{ in } Y_n, \text{ for every } m \geq m_n^2.$$

For any neighbourhood V of Y in Q there exist $n \in \mathbb{N}$ such $Y_n \subseteq V$. Define $U = X_{\lambda_n}$ and $m_n = \max\{m_n^1, m_n^2\}$ and let $n' \geq n''_{Y_n}$ be arbitrary. Then

$$\Phi_{n'}^m|_U \simeq f_n^m|_U \simeq f_n^{'m}|_U \simeq \Phi_{n'}^{'m}|_U \text{ in } V, \text{ for every } m \geq m_n,$$

i.e., $(\Phi_n^m) \sim (\Phi_n^{'m})$ and so $[\Phi] = [\Phi']$.

Surjectivity: Let $F^{*^f} : X \to Y$ be an arbitrary finite coarse shape morphism. Then there exist $\mathbf{f}^{*^f} : \mathbf{X} \to \mathbf{Y}$ in pro^{*^f}-$HPol$ and $(f, f_n^m) : \mathbf{X} \to \mathbf{Y}$ in inv^{*^f}-$HPol$ such that $\langle \mathbf{f}^{*^f} \rangle = \langle [(f, f_n^m)] \rangle = F^{*^f}$. Since the index set $\mathbb{N}$ is cofinite, one may assume that (f, f_n^m) is simple and so the index function f is increasing. Let, for every $n \in \mathbb{N}$, $f'(n) \geq f(n)$ be such that $\mathrm{Cl}(X_{f'(n)}) \subseteq X_{f(n)}$. Define $X'_n = \mathrm{Cl}(X_{f'(n)})$. Hence the index function $f' : \mathbb{N} \to \mathbb{N}$ is defined and one may assume that f is increasing, i.e., that $X'_{n+1} \subseteq X'_n$, for every $n \in \mathbb{N}$. It is obvious that (X'_n) is a decreasing sequence of closed neighbourhood of X in Q (which exists due to the normality of Q) such that $\cap X'_n = X$. Let

$$f_n^{'m} := f_n^m \circ p_{f(n)f'(n)} = f_n^m|_{X_{f'(n)}} : X_{f'(n)} \to Y_n$$

It is easy to see that $(f', f_n^{'m}) \sim (f, f_n^m)$ in inv^{*^f}-$HPol$ and so these two $*^f$-morfisms induce the same finite coarse shape morphism

$$\langle [(f, f_n^m)] \rangle = F^{*^f} = \langle [(f', f_n^{'m})] \rangle.$$

Using techniques demonstrated in the proof of Theorem 3.19, we obtain a $*^f$-fundamental sequence $\Phi = (\Phi_n^m) : X \to Y$ associated with the $*^f$-morphism

$(f', f_n'^m)$, i.e., $\omega(\Phi) = (f, f_n'^m)$. The construction is carried out successively by extending, for every $n \in \mathbb{N}$, functions $f_n^m|_{X_n'}$ over the closed neighbourhoods

$$X_n' \subseteq \cdots \subseteq X_2' \subseteq X_1' \subseteq Q$$

of X up to the functions $\Phi_n^m : Q \to Q$. Hence, $\Omega([\Phi]) = \langle [(f', f_n'^m)] \rangle = F^{*^f}$ and the proof is completed. □

4.2. *The isomorphism of the categories $Sh_f^{*^f}$ and $Sh_a^{*^f}$.*

Let π be a function which associates every $*^f$-approximative sequence $\alpha : X \to Y$ with some $*^f$-fundamental sequence $\Phi : X \to Y$ such $\Phi|_X \sim \alpha$. By Theorem 3.19 such a $*^f$-fundamental sequence Φ exists and by Proposition 3.20 the equivalence class $[\Phi]$ of $\Phi = \pi(\alpha)$ does not depend on the choice of the function π.

Therefore it makes sense to define, for every pair of closed subsets X, Y of Q, the mapping

$$\Pi_{X,Y} : Sh_a^{*^f}(X, Y) \to Sh_f^{*^f}(X, Y),$$

$$\Pi_{X,Y}([\alpha]) = [\pi(\alpha)].$$

THEOREM 4.4. *The function $\Pi_{X,Y}$ is a bijection, for every pair of closed subsets X, Y of Q.*

PROOF. Firstly, we prove that $\Pi_{X,Y}$ is well defined. Let $\alpha, \alpha' : X \to Y$ be $*^f$-approximative sequences such that $\alpha \sim \alpha'$ and $\Phi = \pi(\alpha)$, $\Phi' = \pi(\alpha')$. Since $\Phi|_X \sim \alpha \sim \alpha' \sim \Phi'|_X$, by Proposition 3.20 it follows that $\Phi \sim \Phi'$.

Injectivity: Let $\alpha, \alpha' : X \to Y$ be $*^f$-approximative sequences such that

$$\pi(\alpha) = \Phi \sim \Phi' = \pi(\alpha').$$

Since $\Phi|_X \sim \alpha$ and $\Phi'|_X \sim \alpha'$, Proposition 3.20 implies $\alpha \sim \alpha'$.

Surjectivity: Let $[\Phi] \in Sh_f^{*^f}(X, Y)$ be an arbitrary equivalence class of $*^f$-fundamental sequences and let $\Phi = (\Phi_n^m) : X \to Y$ be its representative. By Proposition 3.18, putting $\alpha = \Phi|_X : X \to Y$ one obtains a $*^f$-approximative sequence α such that $\Pi_{X,Y}([\alpha]) = [\pi(\alpha)] = [\Phi]$. □

LEMMA 4.5. *Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be $*^f$-approximative sequences and let $\Phi : X \to Y$, $\Theta : Y \to Z$ and $\Psi : X \to Z$ be $*^f$-fundamental sequences such that*

$$[\Phi|_X] = [\alpha], [\Theta|_Y] = [\beta] \ \ \textit{and} \ \ [\Psi|_X] = [\beta] \circ [\alpha] = [\Theta \circ \alpha].$$

Then $[\Psi] = [\Theta \circ \Phi]$.

PROOF. The proof is straightforward using the definitions of the compositions between equivalence classes of $*^f$-approximative and $*^f$-fundamental sequences. □

We shall now define the mapping $\Pi : Sh_a^{*^f} \to Sh_f^{*^f}$ between the objects and between the morphisms of the categories $Sh_a^{*^f}$ and $Sh_f^{*^f}$ as follows. Let

- for every closed subset $X \subseteq Q$, $\Pi(X) = X$;
- for every pair of closed subsets $X, Y \subseteq Q$ and for every $[\alpha] \in Sh_a^{*^f}(X, Y)$,

$$\Pi([\alpha]) := \Pi_{X,Y}([\alpha]) = [\pi(\alpha)].$$

THEOREM 4.6. *The mapping* $\Pi : Sh_a^{*^f} \to Sh_f^{*^f}$ *is a functor.*

PROOF. The theorem follows from Lemma 4.5. □

COROLLARY 4.7. *The functor* $\Pi : Sh_a^{*^f} \to Sh_f^{*^f}$ *is an isomorphism.*

PROOF. This is the direct consequence of Theorems 4.4 and 4.6. □

4.3. *The isomorphism of the categories* $Sh_a^{*^f}$ *and* $InSh^{*^f}$.

LEMMA 4.8 ([3], Ho). *If X is a paracompact topological space, then every ϵ-continuous function $f : X \to Q$ admits a continuous 2ϵ-near approximation $f' : X \to Q$.*

By the virtue of Lemma 4.8, it is possible to associate every $*^f$-proximate sequence with a $*^f$-approximative sequence such that the distance between the corresponding component functions tends to 0 as indices n tend to $+\infty$, for all the indices m sufficiently large.

DEFINITION 4.9. *A $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$ is said to be a continuous approximation of a $*^f$-proximate sequence $a = (a_n^m) : X \to Y$ provided for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that $d(a_n^m, \alpha_n^m) < \epsilon$, for every $m \geq m_n$.*

THEOREM 4.10. *Let $a = (a_n^m) : X \to Y$ be a $*^f$-proximate sequence. Then the following statements hold:*

(i) *there exists a $*^f$-approximative sequence $\alpha = (\alpha_n^m) : X \to Y$ which is a continuous approximation of a,*
(ii) *every two continuous approximations $\alpha, \alpha' : X \to Y$ of a are homotopic, i.e., $\alpha \sim \alpha'$.*

PROOF. (i) Let $a = (a_n^m) : X \to Y$ be a $*^f$-proximate sequence and let (ϵ_n) be a decreasing sequence of positive real numbers such that $\lim \epsilon_n = 0$ and that, for every $n_0 \in \mathbb{N}$ and for every $n \geq n_0$,

$$a_{n_0}^m \overset{\frac{\epsilon_{n_0}}{2}}{\simeq} a_n^m, \text{ for every } m \geq m_{n_0 n}.$$

For every $n \in \mathbb{N}$ define

$$\alpha_n^m = \begin{cases} f_n^m : X \to Q, & m < m_{nn} \\ a_n^{'m} : X \to Q, & m \geq m_{nn} \end{cases},$$

where every f_n^m is an arbitrary continuous function and every $a_n^{'m}$ is a continuous ϵ_n-near approximation of $\frac{\epsilon_n}{2}$-continuous function a_n^m. The existance of the functions $a_n^{'m}$ follows from Lemma 4.8. Thereat, if $m, m' \geq m_{nn}$ and $a_n^m = a_n^{m'}$, continuous approximations are chosen in a way that $a_n^{'m} = a_n^{'m'}$ (the same component functions a_n^m are always approximated by the same continuous approximation). One can easily check that $\alpha = (\alpha_n^m)$ is a $*^f$-approximative sequence from X to Y which is a continuous approximation of a.

(ii) Let $\alpha = (\alpha_n^m), \alpha' = (\alpha_n^{'m}) : X \to Y$ be continuous approximations of $a : X \to Y$. Let V be an arbitrary open neighbourhood of Y in Q and let $\epsilon > 0$ be such that $B(Y, \epsilon) \subseteq V$ and that every two ϵ-near mappings in V are homotopic (V is an ANR). Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that

$$d(a_n^m, \alpha_n^m) < \frac{\epsilon}{2} \text{ and } d(a_n^m, \alpha_n^{'m}) < \frac{\epsilon}{2} \quad \text{for every } m \geq m_n.$$

Now, for every $n \geq n_0$ and for every $m \geq m_n$

$$d(\alpha_n^m, \alpha_n^{'m}) \leq d(\alpha_n^m, a_n^m) + d(a_n^m, \alpha_n^{'m}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

hold. Since $d(a_n^m, \alpha_n^m), d(a_n^m, \alpha_n^{'m}) < \frac{\epsilon}{2}$, the inclusions

$$\alpha_n^m(X), \alpha_n^{'m}(X) \subseteq B\left(Y, \frac{\epsilon}{2}\right) \subseteq V$$

hold and so $\alpha_n^m \simeq \alpha_n^{'m}$ in V, for every $n \geq n_0$ and every $m \geq m_n$. Hence, $\alpha \sim \alpha'$. □

Let λ be a function which associates every $*^f$-proximate sequence $a : X \to Y$ with some continuous approximation $\alpha : X \to Y$ of a. By the claim (ii) of Theorem 4.10, the equivalence class $[\alpha]$ of the $*^f$-approximative sequence $\alpha = \lambda(f)$ does not depend on the choice of the function λ. Hence, for an arbitrary pair of closed subsets X, Y of Q, we define a function

$$\Lambda_{X,Y} : InSh^{*^f}(X, Y) \to Sh_a^{*^f}(X, Y),$$
$$\Lambda_{X,Y}([a]) = [\lambda(a)].$$

THEOREM 4.11. *The function $\Lambda_{X,Y}$ is a bijection, for every pair of closed subsets X, Y of Q.*

PROOF. Firstly, we prove that $\Lambda_{X,Y}$ is well defined. Let $a, a' : X \to Y$ be $*^f$-proximate sequences such that $a \sim a'$ and $\alpha = \lambda(a)$, $\alpha' = \lambda(a')$. Let V be an arbitrary open neighbourhood of Y in Q and let $\epsilon > 0$ be a number such that $B(Y, \epsilon) \subseteq V$ and that every two ϵ-near mappings in V are homotopic.

Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that

$$d(a_n^m, \alpha_n^m) < \frac{\epsilon}{2}, \quad d(a_n^{'m}, \alpha_n^{'m}) < \frac{\epsilon}{2} \text{ and } a_n^m \overset{\frac{\epsilon}{4}}{\simeq} a_n^{'m}, \text{ for every } m \geq m_n.$$

For arbitrary $n \geq n_0$ and $m \geq m_n$ there exists $\frac{\epsilon}{4}$-homotopy $H_n^m : X \times I \to Y$ such that $H_n^m(\cdot, 0) = a_n^m$ i $H_n^m(\cdot, 1) = a_n^{'m}$. By Lemma 4.8, there exists a continuous function $H_n^{'m} : X \times I \to Q$ such that $d(H_n^m, H_n^{'m}) < \frac{\epsilon}{2}$. Notice that $H_n^{'m}(X \times I) \subseteq B(Y, \frac{\epsilon}{2}) \subseteq V$. Furthermore,

$$\begin{aligned} d(\alpha_n^m, H_n^{'m}(\cdot,0)) &\leq d(\alpha_n^m, H_n^m(\cdot,0)) + d(H_n^m(\cdot,0), H_n^{'m}(\cdot,0)) \\ &= d(\alpha_n^m, a_n^m) + d(H_n^m(\cdot,0), H_n^{'m}(\cdot,0)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \\ d(\alpha_n^{'m}, H_n^{'m}(\cdot,1)) &\leq d(\alpha_n^{'m}, H_n^m(\cdot,1)) + d(H_n^m(\cdot,1), H_n^{'m}(\cdot,1)) \\ &= d(\alpha_n^{'m}, a_n^{'m}) + d(H_n^m(\cdot,1), H_n^{'m}(\cdot,1)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

hold and so $\alpha_n^m \simeq H_n^{'m}(\cdot,0) \simeq H_n^{'m}(\cdot,1) \simeq \alpha_n^{'m}$ in V. Hence, $\alpha \sim \alpha'$.

Injectivity: Let $a, a' : X \to Y$ be $*^f$-proximate sequences such that

$$\lambda(f) = \alpha \sim \alpha' = \lambda(f').$$

For an arbitrary $\epsilon > 0$ put $V = B(Y, \frac{\epsilon}{3})$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists $m_n \in \mathbb{N}$ such that

$$\alpha_n^m \simeq \alpha_n^{'m} \text{ in } V, \quad d(a_n^m, \alpha_n^m) < \frac{\epsilon}{3}$$

and

$$(a_n^{'m}, \alpha_n^{'m}) < \frac{\epsilon}{3}, \text{ for every } m \geq m_n.$$

For arbitrary $n \geq n_0$ and $m \geq m_n$ there exists a homotopy $H_n^m : X \times I \to V$ such that $H_n^m(\cdot,0) = \alpha_n^m$ and $H_n^m(\cdot,1) = \alpha_n^{'m}$. By Proposition 2.2, functions a_n^m and $a_n^{'m}$ are ϵ-continuous. Let $G_n^m : X \times I \to Y$ be a function such that

$$d(G_n^m, H_n^m) < \frac{\epsilon}{3}, \ G_n^m(\cdot,0) = f_n^m \text{ and } G_n^m(\cdot,1) = f_n^{'m}.$$

By Proposition 2.2, function G_n^m is ϵ-continuous and so $a_n^m \overset{\epsilon}{\simeq} a_n^{'m}$. Hence, $a \sim a'$.

Surjectivity: Let $[\alpha] \in Sh_a^{*^f}(X,Y)$ be an arbitrary equivalence class of $*^f$-approximative sequences and let $\alpha = (\alpha_n^m) : X \to Y$ be its representative. For every $k \in \mathbb{N}$ define $V_k = B(Y, \frac{1}{k})$. For every $k \in \mathbb{N}$ there exists $n_0(k) \in \mathbb{N}$ such that for every $n \geq n_0(k)$ there exists $m_n(k)$ such that

$$\alpha_n^m \simeq \alpha_{n+1}^m \text{ in } V_k, \text{ for every } m \geq m_n(k).$$

It means that $\alpha_{n_0(k)}^m : X \to V_k$, for every $m \geq m'_k := m_{n_0(k)}(k)$.

For all $k, m \in \mathbb{N}$ define $\alpha_k^{'m} = \alpha_{n_0(k)}^m$. Hence, we obtained a $*^f$-approximative sequence $\alpha' = (\alpha_k^{'m}) : X \to Y$ such that $[\alpha'] = [\alpha]$.

For every $k \in \mathbb{N}$ let:

- $a_k^m : X \to Y$ be a $\frac{3}{k}$-continuous function such that $d(a_k^m, \alpha_k^{'m}) < \frac{1}{k}$, for every $m \geq m'_k$. Thereat, if $m, m' \geq m'_k$ and $\alpha_k^{'m} = \alpha_k^{'m'}$, $\frac{3}{k}$-continuous functions are chosen in a way that $a_k^m = a_k^{m'}$ (the same component functions are always associated with the same $\frac{3}{k}$-continuous function);
- $a_k^m : X \to Y$ be an arbitrary function, for every $m < m'_k$.

It is straightforward to prove that $a = (a_k^m) : X \to Y$ is a $*^f$-proximate sequence for which α' is a continuous approximation, i.e., $\lambda(a) = \alpha'$. Hence, $\Lambda_{X,Y}([a]) = [\lambda(a)] = [\alpha'] = [\alpha]$ and the proof is completed. ∎

We shall now define the mapping $\Lambda : InSh^{*^f} \to Sh_a^{*^f}$ between the objects and between the morphisms of the categories $InSh^{*^f}$ and $Sh_a^{*^f}$ as follows. Let

- for every closed subset $X \subseteq Q$, $\Lambda(X) = X$;
- for every pair of closed subsets $X, Y \subseteq Q$ and for every

$$[a] \in InSh^{*^f}(X,Y), \Lambda([a]) := \Lambda_{X,Y}([a]) = [\lambda(a)].$$

THEOREM 4.12. *The mapping $\Lambda : InSh^{*^f} \to Sh_a^{*^f}$ is a functor.*

PROOF. Let $a : X \to Y$ and $b : Y \to Z$ be $*^f$-proximate sequences, the representatives of the equivalence classes $[a]$ and $[b]$, respectively, and let $\beta : X \to Y$ be a continuous approximation of b. Then there exists a $*^f$-fundamental sequence $\Psi : Y \to Z$ such that $\Psi|_Y \sim \beta$. Let $b' : Y \to Z$ be a $*^f$-proximate sequence such that $\Psi|_Y$ is its continuous approximation. The injectivity of $\Lambda_{Y,Z}$ and $\Psi|_Y \sim \beta$ implies $b \sim b'$.

Let (ϵ_n) be a decreasing sequence of positive real numbers such that $\lim \epsilon_n = 0$ and that

(1) for every $n_0 \in \mathbb{N}$ and for every $n \geq n_0$,

$$b_{n_0}^m \overset{\frac{\epsilon_{n_0}}{2}}{\simeq} b_n^m, \text{ for every } \ m \geq m_{n_0 n};$$

(2) for every $n \in \mathbb{N}$, for every $m \geq m_{nn}$ and for every $y \in Y$,

$$d(\Psi_n^m(y), b_n^{'m}(y)) < \frac{\epsilon_n}{2}.$$

Let (δ_n) be a decreasing sequence of positive real numbers such that $\lim \delta_n = 0$ and that

(1′) for every $n \in \mathbb{N}$, for every $m \geq m_{nn}$ and for all $y, y' \in Y$ such that $d(y, y') < \delta_n$,

$$d(b_n^m(y), b_n^m(y')) < \epsilon_n,$$

(2′) for every $n \in \mathbb{N}$, for all $m \geq m_{nn}$ and for all $y, y' \in Q$ such that $d(y, y') < \delta_n$,

$$d(\Psi_n^m(y), \Psi_n^m(y')) < \frac{\epsilon_n}{2}.$$

Notice that the conditions (2′) and (3′) can be fulfilled because

$$\mathrm{card}(\{b_n^m : m \geq m_n\}) < \aleph_0, \quad \text{for every } n \in \mathbb{N},$$

$$\mathrm{card}(\{\Psi_n^m : m \geq m_n\}) < \aleph_0, \quad \text{for every } n \in \mathbb{N}$$

and due to the compactness of Y. Finally, let (k_n) be a strictly increasing sequence of indices such that, for every $k \geq k_n$,

$$a_{k_n}^m \overset{\frac{\delta_n}{2}}{\simeq} a_k^m, \quad \text{for every } m \geq m'_{k_n k}.$$

Lemma 4.8 yields mappings $\alpha_n^{'m} : X \to Q$ such that, for every $n \in \mathbb{N}$,

$$d(\alpha_n^{'m}, a_{k_n}^m) < \delta_n, \quad \text{for every } m \geq m'_{k_n k_n}.$$

Thereat, if $m, m' \geq m'_{k_n k_n}$ and $a_{k_n}^m = a_{k_n}^{m'}$, the approximations are chosen in a way that $\alpha_n^{'m} = \alpha_n^{'m'}$.

For every $n \in \mathbb{N}$ and for every $m < m'_{k_n k_n}$, let $\alpha_n^{'m} : X \to Q$ be an arbitrary mappings. It is obvious that $\alpha' = (\alpha_n^{'m})$ is a $*^f$-approximative sequence from X to Y and a continuous approximation of $a' = (a_{k_n}^m)$. Now $a \sim a'$ implies

$$\Lambda([a]) = \Lambda([a']) = [\alpha'].$$

Moreover, for a $*^f$-approximative sequence $\Psi \circ \alpha'$ and for every $x \in X$, $n \in \mathbb{N}$ and $m \geq \max\{m'_{k_n k_n}, m_{nn}\}$, the inequality

$$\begin{aligned} &d(\Psi_n^m \alpha_n^{'m}(x), b_n^{'m} a_{k_n}^m(x)) \\ &\qquad \leq d(\Psi_n^m \alpha_n^{'m}(x), \Psi_n^m a_{k_n}^m(x)) + d(\Psi_n^m a_{k_n}^m(x), b_n^{'m} a_{k_n}^m(x)) < \epsilon_n \end{aligned}$$

holds and thus $\Psi \circ \alpha'$ is a continuous approximation of $b' \circ a'$. Hence,

$$\begin{aligned} \Lambda([b] \circ [a]) &= \Lambda([b'] \circ [a']) = \Lambda([b' \circ a']) = [\Psi \circ \alpha'] \\ &= [\Psi|_Y] \circ [\alpha'] = \Lambda([b']) \circ \Lambda([a']) = \Lambda([b]) \circ \Lambda([a]), \end{aligned}$$

and the theorem is proved. ∎

Corollary 4.13. *The functor* $\Lambda : InSh^{*^f} \to Sh_a^{*^f}$ *is an isomorphism.*

Proof. This is the consequence of Theorems 4.11 and 4.12. ∎

At last, let Υ denote the composition of the functors Λ, Π and Ω, i.e.,

$$\Upsilon = \Omega \circ \Pi \circ \Lambda.$$

Theorem 4.14. *The functor* $\Upsilon : InSh^{*^f} \to Sh^{*^f}|_Q$ *is an isomorphism.*

Proof. The composition of isomorphisms is an isomorphism. ∎

By Theorem 4.14. the finite coarse shape category of closed subsets of Q is described using an intrinsic approach through the category $InSh^{*^f}$. It can be shown that every two embeddings in Q of a compact metric space M are isomorphic in $InSh^{*^f}$. Since every compact metric space can be embedded in the Hilbert cube as a closed subset, the classification by the intrinsic finite coarse shape is actually given for all compact metric spaces.

The question is: can the coarse shape category Sh^* be described using an intrinsic approach in the compact metric case? So far, the authors of this article have encountered some severe technical difficulties while solving that problem, which still remains open.

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