GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 129 – 147

EXPONENTIAL MOMENTS OF SIMULTANEOUS HITTING TIME FOR NON-ATOMIC MARKOV CHAINS

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Abstract. This paper is devoted to studying the first simultaneous hitting time of a given set by two discrete-time, inhomogeneous Markov chains with values in general phase space. Established conditions for the existence of the hitting time's exponential moment. Computable bounds for the exponential moment are obtained under the condition of stochastic dominance.

1. Introduction

The general theory of homogeneous Markov chains is well studied and has its applications in many areas. This theory is well covered in the classical books [20] and [23] while [5] includes recent advances in addition to classical results. However, the theory of inhomogeneous Markov chains is not well developed, while plays an important role in theoretical and practical considerations and attracts modern scientists' interest. For example [1, 2] and [16] include applications of inhomogeneous Markov chains, and [6, 13, 15, 17, 18] present new results in the theory. Papers [4, 19] include classical results in the ergodic theory of inhomogeneous Markov chains.

This paper is devoted to the evaluation of an exponential moment for the simultaneous hitting of a specific set C by two time-inhomogeneous Markov chains defined on the general state space. Simultaneous hitting times play a crucial role in the investigation of the stability of Markov chains, especially in time-inhomogeneous case (see [12, 13, 15, 16, 17, 18]). Typically, stability results could be obtained by coupling techniques in which simultaneous hitting

2020 *Mathematics Subject Classification.* 60J05.

Key words and phrases. Markov chains, coupling method, inhomogeneous renewal theory.

plays a key role. The expectation of a simultaneous hitting time studied in papers [7, 9, 10, 11, 14].

In this paper, we established conditions of existence of the exponential moment of simultaneous hitting time for two time-inhomogeneous Markov chains and found an estimate for that moment that can be computed in practical applications. We expand the development originated in [8], where similar results for atomic chains where developed. We adapted a technique of splitting chain, well known in the homogeneous theory, to this particular situation, and applied results from [8] to the general non-atomic case. The splitting technique was introduced in the pioneering work of Nummelin [22] and later used by many authors (see [5, 20]).

This paper organized as follows. Section 2 introduces notation, the minorization condition, and the construction of a probability space used across the paper. In Section 3, we construct a splitting chain. In Section 4, we present the main results of the paper. Section 5 includes auxiliary results used in the proof of the main theorems.

2. Notation and Conditions

In this paper we consider a pair of time-inhomogeneous Markov chains $X_t^{(1)}$ and $X_t^{(2)}$, $t \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ on a general state space $(E, \mathcal{E})$. Denote by $\mathbb{M}_1(\mathcal{E})$ a set of all probability measures on $\mathcal{E}$. We will denote by $a \vee b = \max\{a, b\}$, $a^+ = \max\{a, 0\}$ and $\lfloor a \rfloor$ an integer part of a.

The critical role in the subsequent development plays a minorization condition defined below.

Condition (A). We say that a sequence of Markov transition kernels $(P_t, t \in \mathbb{N}_0)$ satisfies Condition (A) if there exists a set $C \in \mathcal{E}$, a sequence of probability measures $\nu_t \in \mathbb{M}_1(\mathcal{E})$ and a sequence of constants $\alpha_t \in (0, 1)$ such that $\forall x \in C,\ A \in \mathcal{E}$ and all $t \in \mathbb{N}_0$

$$P_t(x, A) \geq \alpha_t \nu_t(A), \text{ and } \alpha := \inf_t \alpha_t > 0.$$

The set C from Condition (A) is called a small set.

Minorization condition is an essential tool in the study of homogeneous Markov chains. In our definition we added additional requirement $\inf_t \alpha_t > 0$. This additional condition guarantees that minorization holds uniformly over time and does not vanish as $t \to \infty$, and it is specific for time-inhomogeneity.

In this paper we obtained computable bounds for the first hitting time $\sigma_{C\times C}$ of the set $C \times C$ by the pair of chains $\left(X^{(1)}, X^{(2)}\right)$ defined by a pair of sequences of Makrov kernels $(P_{t,1}, P_{t,2}),\ t \in \mathbb{N}_0$, and assume both $X^{(1)}$ and $X^{(2)}$ satisfy Condition (A) with the same $\alpha_t, \nu_t,\ t \in \mathbb{N}_0$.

It is well known (see [21]) that a sequence of Markov kernels $P_t\colon E \times \mathcal{E} \to [0, 1]$, where $t \in \mathbb{N}_0$ induces canonical Markov chain $(X_t,\ t \in \mathbb{N}_0)$ defined on the measurable space $(\Omega, \mathcal{F})$ and a family of probability measures $\mathbb{P}_\mu$,

$\mu \in \mathbb{M}_1(\mathcal{E})$ such that

$$\mathbb{P}_\mu\{X_0 \in A_0, \dots X_k \in A_k\} = \int_{A_0} \dots \int_{A_k} \mu(dx_0) P_0(x_0, dx_1) \dots P_{k-1}(x_{k-1}, dx_k).$$

Since we deal with two sequences of Markov kernels $P_{t,1}$ and $P_{t,2}$ we consider measurable space $(\Omega, \mathcal{F})$ and a family of probability measures $\mathbb{P}_{\mu,\lambda},\ \mu, \lambda \in \mathbb{M}_1(\mathcal{E})$ such that

$$\Omega = (E \times E)^\infty = \{\omega = ((\omega_0^{(1)}, \omega_0^{(2)}), (\omega_1^{(1)}, \omega_1^{(2)}), \dots, (\omega_n^{(1)}, \omega_n^{(2)}), \dots)\},$$

and $\mathcal{F}$ is a σ-field induced by the infinite product of $\mathcal{E} \otimes \mathcal{E}$. In such a construction we define random variables

$$X_t^{(1)}(\omega) = \omega_t^{(1)},\ X_t^{(2)}(\omega) = \omega_t^{(2)},$$

such that $X_t^{(1)}$ is independent of $X_t^{(2)}$ and each of $X^{(i)}$ is a time-inhomogeneous Markov chain with transition probabilities $P_{t,i},\ i \in \{1, 2\}$. Moreover, for any $t \in \mathbb{N}_0$, $x \in E$, $A \in \mathcal{E}$ and $i \in \{0, 1\}$ we have

$$\mathbb{P}\left\{X_{t+1}^{(i)} \in A | X_t^{(i)} = x\right\} = P_{t,i}(x, A).$$

Let $\theta \colon \Omega \to \Omega$ be a shift operator, such that for any $\omega = ((\omega_0^{(1)}, \omega_0^{(2)}), (\omega_1^{(1)}, \omega_1^{(2)}), \dots, (\omega_n^{(1)}, \omega_n^{(2)}), \dots) \in \Omega$

$$\theta(\omega) = ((\omega_1^{(1)}, \omega_1^{(2)}), (\omega_2^{(1)}, \omega_2^{(2)}), \dots, (\omega_{n+1}^{(1)}, \omega_{n+1}^{(2)}), \dots),$$

and $\theta_n = \theta \circ \dots \circ \theta$ - is the n-th iteration of θ.

We find it convenient to define a chain started at fixed moment t in the following way. Let $t \in \mathbb{N}_0$ be fixed. Let $(\Omega_t, \mathcal{F}_t)$ be a copy of $(\Omega, \mathcal{F})$. We set $X_{t,n}^{(i)}(\omega) = \omega_n^{(i)}$ for $\omega \in \Omega_t$ and for any $x, y \in E$ define a probability measure

$$\begin{aligned} &\mathbb{P}_{x,y}^t\{X_{t,k}^{(i)} \in A_{ik},\ i \in \{1, 2\}, k \in \{0, \dots, n\}\} \\ &\qquad = \mathbb{P}\{X_{t+k}^{(i)} \in A_{ik},\ i \in \{1, 2\}, k \in \{0, \dots, n\} | X_t^{(1)} = x, X_t^{(2)} = y\}, \end{aligned}$$

$A_{ik} \in \mathcal{E}$. Note that $X_{t,k}^{(i)}$ and $X_t^{(i)}$ are defined on a different probability spaces. In the following, we will omit index t in $X_{t,k}^{(i)}$ in the context of measure $\mathbb{P}_{x,y}^t$, i.e. $\mathbb{P}_{x,y}^t\{X_n^{(1)} \in A\}$ should be understood as $\mathbb{P}_{x,y}^t\{X_{t,n}^{(1)} \in A\}$. We can define $\mathbb{P}_{\mu,\lambda}^t$ for $\mu, \lambda \in \mathbb{M}_1(\mathcal{E})$ as

$$\mathbb{P}_{\mu,\lambda}^t = \int_{E \times E} \mu(dx)\lambda(dy)\mathbb{P}_{x,y}^t.$$

Finally, we may write $\mathbb{P}_\mu^t$ for the probability of the event that only depends on either one of $X^{(1)}$ or $X^{(2)}$ when it is clear from the context which chain is under consideration.

3. Splitting construction

Assuming that minorization condition (A) holds for both chains with the same set of α_t, ν_t we can define split chains $\check{X}_t^{(i)}$ and $\check{X}_{t,n}^{(i)}$ such that $\check{X}_t^{(i)}$ and $\check{X}_{t+n}^{(i)}$ are atomic chains.

Let $\check{E} = E \times \{0,1\}$ and $\check{\mathcal{E}} = \sigma[A \times \{0\}, A \times \{1\} \mid A \in \mathcal{E}]$. For every set $A \in \mathcal{E}$ we will write $\check{A}_0 = A \times \{0\}$, $\check{A}_1 = A \times \{1\}$, $\check{A} = A \times \{0,1\}$.

Let $\varepsilon_t > 0$ be a sequence of positive values. For every $P_{t,i}$ we can now define $\check{P}_{t,i} \colon E \times \{0,1\} \to \check{\mathcal{E}}$.

For any $x \in E \setminus C$, $A \in \mathcal{E}$ and $d \in \{0,1\}$ we define:

$$\begin{aligned} \check{P}_{t,i}((x,d), A \times \{0\}) &= (1-\alpha_t) P_{t,i}(x,A), \\ \check{P}_{t,i}((x,d), A \times \{1\}) &= \alpha_t P_{t,i}(x,A). \end{aligned} \tag{3.1}$$

For any $x \in C$

$$\begin{aligned} \check{P}_{t,i}((x,0), A \times \{0\}) &= (1-\alpha_t) \frac{P_{t,i}(x,A) - \alpha_t \nu_t(A)}{1-\alpha_t}, \\ \check{P}_{t,i}((x,0), A \times \{1\}) &= \alpha_t \frac{P_{t,i}(x,A) - \alpha_t \nu_t(A)}{1-\alpha_t}, \\ \check{P}_{t,i}((x,1), A \times \{0\}) &= (1-\alpha_t) \nu_t(A), \\ \check{P}_{t,i}((x,1), A \times \{1\}) &= \alpha_t \nu_t(A). \end{aligned} \tag{3.2}$$

From definitions (3.1) and (3.2) we see

$$\check{P}_{t,i}((x,d), A \times \{0,1\}) = P_{t,i}(x,A), \tag{3.3}$$

for all $x \in E \setminus C,\ A \in \mathcal{E}$, and

$$(1-\alpha_t)\check{P}_{t,i}((x,0), A \times \{0,1\}) + \alpha_t \check{P}_{t,i}((x,1), A \times \{0,1\}) = P_{t,i}(x,A), \tag{3.4}$$

for all $x \in C,\ A \in \mathcal{E}$.

Another important identity that follows from definitions (3.2) is

$$\check{P}_{t,i}((x,d), A \times \{0\}) = (1-\alpha_t) \check{P}_{t,i}((x,d), A \times \{0,1\}), \tag{3.5}$$

for all $x \in E$, $d \in \{0,1\}$.

Let μ_1, μ_2 be two probability measures from $\mathbb{M}_1$, $d_1, d_2 \in \{0,1\}$, and δ_j is a Dirac measure, $j \in \{0,1\}$. With transition probabilities $\check{P}_{t,i}$, we can construct a canonical space in the same way as we did in the previous section. So, we assume that random variables $\check{X}_t^{(i)}$ with $\check{X}_0^{(i)} \sim \mu_i \times \delta_{d_i}$, $d_i \in \{0,1\}$ and probabilities $\check{\mathbb{P}}^t_{\mu_1 \times \delta_{d_1}, \mu_2 \times \delta_{d_2}}$ are properly defined on the aforementioned canonical space. To simplify notation, we will denote

$$\mu \times \{d\} := \mu \times \delta_d,\ \mu \in \mathbb{M}_1,\ d \in \{0,1\}. \tag{3.6}$$

The chains $\check{X}_t^{(i)}$ now are atomic chains with the atom $C \times \{1\}$. Let us denote

$$\sigma_C^{(i)} = \sigma_C^{(1,i)} = \inf\{t > 0 : \ X_t^{(i)} \in C\},$$
$$\sigma_C^{(n,i)} = \sigma_C^{(i)} \circ \theta_{\sigma_C^{(n-1,i)}}. \tag{3.7}$$

$$\check{\sigma}_{\check{C}}^{(i)} = \check{\sigma}_{\check{C}}^{(1,i)} = \inf\{t > 0 : \ \check{X}_t^{(i)} \in \check{C} = C \times \{0,1\}\},$$
$$\check{\sigma}_{\check{C}}^{(n,i)} = \check{\sigma}_C^{(i)} \circ \theta_{\check{\sigma}_{\check{C}}^{(n-1,i)}}, \ n \geq 2. \tag{3.8}$$

For fixed $k \in \{0,1\}$ we define

$$\check{\sigma}_{C_k}^{(i)} = \check{\sigma}_{C_k}^{(1,i)} = \inf\{t > 0 : \ \check{X}_t^{(i)} \in C \times \{k\}\},$$
$$\check{\sigma}_{C_k}^{(n,i)} = \check{\sigma}_{C_k}^{(i)} \circ \theta_{\check{\sigma}_{C_k}^{(n-1,i)}}, \ n \geq 2. \tag{3.9}$$

Let μ be a probability measure in $\mathbb{M}_1(\mathcal{E})$, and event A only depends on one of the chains $\check{X}^{(1)}$ or $\check{X}^{(2)}$, such that exact index $i \in \{1,2\}$ is clear from the context. We will use the symbol $\check{\mathbb{P}}_\mu^t\{A\}$ whose meaning is the following

$$\check{\mathbb{P}}_\mu^t\{A\} = (1-\alpha_t)\check{\mathbb{P}}_{\mu\times\{0\}}^t\{A\} + \alpha_t\check{\mathbb{P}}_{\mu\times\{1\}}^t\{A\},$$

where $\mu \times \{d\}$ is defined in (3.6). We use the same notation for the expectation $\check{\mathbb{E}}_\mu^t\left[Z(X^{(i)})\right]$. Finally, note that C_1 is an atom for each of the chains $\{X_n^{(i)}, \ n \geq 0\}$, so for all $x \in C$ probability $\check{\mathbb{P}}_{x\times\{1\}}^t$ does not depend on x. We will denote it by $\check{\mathbb{P}}_{C_1}^t$ and corresponding expectation by $\mathbb{E}_{C_1}^t$.

4. Main results

The first result of the paper shows that the existence of the exponential moment for the chain $X^{(i)}$ is sufficient for the existence of the exponential moment of the variable $\check{\sigma}_{C_1}^{(i)}$. Next theorem, however, does not provide a good computable bound since it involves two constants γ and δ that are difficult to estimate in practical applications. We will show how to evaluate these values under some additional conditions in Theorem 4.3.

THEOREM 4.1. *Let $i \in \{1,2\}$ be fixed, the chains $X^{(i)}$ and $\check{X}^{(i)}$ are defined in Sections 2 and 3 (i.e. chain $X^{(i)}$ satisfies* CONDITION (A)*), random variables $\sigma_C^{(i)}$ and $\check{\sigma}_{C_1}^{(i)}$ are defined at (3.7) and (3.9) respectively and $\mu \in \mathbb{M}_1(\mathcal{E})$. Assume that there exists $\beta > 1$ such that*

$$\mathbb{E}_\mu^t\left[\beta^{\sigma_C^{(i)}}\right] < \infty,$$

and

$$\sup_{t\in\mathbb{N}_0, x\in C}\left\{\frac{1}{1-\alpha_t}\left(\mathbb{E}_x^t\left[\beta^{\sigma_C^{(i)}}\right] - \alpha_t\mathbb{E}_{\nu_t}^t\left[\beta^{\sigma_C^{(i)}}\right]\right)\right\} < \infty.$$

Then there exist constants $\delta > 1$ and $\gamma > 0$ such that

$$\check{\mathbb{E}}^t_\mu \left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right] \le \frac{1+\gamma}{1-\gamma} M_0 \mathbb{E}^t_\mu \left[\delta^{\sigma^{(i)}_C}\right], \tag{4.1}$$

$$\check{\mathbb{E}}^t_{C_1} \left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right] \le \frac{1+\gamma}{1-\gamma} M_0 \mathbb{E}^t_{\nu_t} \left[\delta^{\sigma^{(i)}_C}\right] \tag{4.2}$$

and

$$\check{\mathbb{E}}^t_\mu \left[\sum_{j=0}^{\check{\sigma}^{(i)}_{C_1}-1} \delta^j\right] \le \frac{M_1}{1-\gamma} \mathbb{E}^t_\mu \left[\delta^{\sigma^{(i)}_C}\right], \tag{4.3}$$

where

$$\begin{gathered} M_0 = \sup_{t\in\mathbb{N}_0, x\in C} \left\{ \frac{1}{1-\alpha_t} \left(\mathbb{E}^t_x \left[\delta^{\sigma^{(i)}_C}\right] - \alpha_t \mathbb{E}^t_{\nu_t} \left[\delta^{\sigma^{(i)}_C}\right] \right) \right\}, \\ M_1 = \sup_{t\in\mathbb{N}_0, x\in C} \left\{ \frac{1}{1-\alpha_t} \left(\mathbb{E}^t_x \left[\sum_{l=0}^{\sigma^{(i)}_C-1} \delta^l\right] - \alpha_t \mathbb{E}^t_{\nu_t} \left[\sum_{l=0}^{\sigma^{(i)}_C-1} \delta^l\right] \right) \right\}. \end{gathered} \tag{4.4}$$

In equations (4.1)-(4.4) the constant δ could be any real number greater than one, such that

$$\sup_{t\in\mathbb{N}_0, x\in C} \check{\mathbb{E}}^t_{x\times\{0\}} \left[\delta^{2\check{\sigma}^{(i)}_{\check{C}}}\right] = \sup_{t\in\mathbb{N}_0, x\in C} \left(\frac{\mathbb{E}^t_x \left[\delta^{2\sigma^{(i)}_C}\right] - \alpha_t \mathbb{E}^t_{\nu_t} \left[\delta^{2\sigma^{(i)}_C}\right]}{1-\alpha_t} \right) < \frac{1}{1-\alpha}, \tag{4.5}$$

where α is defined in Condition (A), *and*

$$\gamma = \left((1-\alpha) \sup_{t\in\mathbb{N}_0, x\in C} \left(\check{\mathbb{E}}^t_{x\times\{0\}} \left[\delta^{2\sigma^{(i)}_C}\right] \right) \right)^{1/2}. \tag{4.6}$$

Proof. From Corollary 5.3 we know that for any $\delta > 1$

$$\check{\mathbb{E}}^t_\mu \left[\delta^{\check{\sigma}^{(i)}_{\check{C}}}\right] = \mathbb{E}^t_\mu \left[\delta^{\sigma^{(i)}_C}\right]. \tag{4.7}$$

Let $d \in \{0,1\}$ and denote $\mu \times \{d\}$ by μ_d, where $\mu \in \mathbb{M}_1$. Let us consider an arbitrary $\delta > 1$ and use the trivial relation

$$\check{\mathbb{E}}^t_{\mu_d} \left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right] = 1 + (\delta-1) \check{\mathbb{E}}^t_{\mu_d} \left[\sum_{j=0}^{\check{\sigma}^{(i)}_{C_1}-1} \delta^j\right].$$

We will decompose $\check{\sigma}^{(i)}_{C_1}$ in the following way

$$\check{\mathbb{E}}^t_{\mu_d} \left[\sum_{j=0}^{\check{\sigma}^{(i)}_{C_1}-1} \delta^j\right] = \sum_{k=1}^{\infty} \check{\mathbb{E}}^t_{\mu_d} \left[\mathbb{1}_{\check{\sigma}^{(i)}_{C_1} = \check{\sigma}^{(k,i)}_{\check{C}}} \sum_{j=0}^{\check{\sigma}^{(k,i)}_{\check{C}}-1} \delta^j\right]. \tag{4.8}$$

To simplify further derivation let us denote

$$\bar\sigma = \check\sigma^{(i)}_{C_1},$$

$$\bar\sigma_j = \check\sigma^{(j,i)}_{\check C},$$

$$B_j = \sum_{k=\check\sigma^{(j,i)}_{\check C}}^{\check\sigma^{(j+1,i)}_{\check C}-1} \delta^k = \sum_{k=\bar\sigma_j}^{\bar\sigma_{j+1}} \delta^k.$$

Assuming that $\check\sigma^{(0,i)}_{\check C} = 0$ equation (4.8) then can be rewritten as

$$\begin{aligned}\check{\mathbb{E}}^t_{\mu_d}\left[\sum_{j=0}^{\bar\sigma-1}\delta^j\right] &= \sum_{k=1}^{\infty}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma=\bar\sigma_k}\sum_{j=0}^{k-1}B_j\right] = \sum_{j=0}^{\infty}\sum_{k>j}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma=\bar\sigma_k}B_j\right]\\ &= \sum_{j=0}^{\infty}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}B_j\right] = \sum_{j=0}^{\infty}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}\sum_{l=\bar\sigma_j}^{\bar\sigma_{j+1}-1}\delta^l\right]\\ &= \sum_{j=0}^{\infty}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}\check{\mathbb{E}}^{t+\bar\sigma_j}_{\check X^{(i)}_{\bar\sigma_j}}\left[\sum_{k=0}^{\bar\sigma_1-1}\delta^k\right]\delta^{\bar\sigma_j}\right]\\ &\le \sup_{x\in C,t\in\mathbb{N}_0}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\sum_{l=0}^{\bar\sigma_1-1}\delta^l\right]\sum_{j=0}^{\infty}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}\delta^{\bar\sigma_j}\right].\end{aligned} \tag{4.9}$$

We now consider each term in the product on the right-hand side of inequality (4.9). For any $x \in C$ we have from (5.7) that

$$\check{\mathbb{E}}^t_{x\times\{0\}}\left[\sum_{l=0}^{\bar\sigma_1-1}\delta^l\right] = \frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\sum_{l=0}^{\sigma^{(i)}_C-1}\delta^l\right] - \alpha_t\mathbb{E}^t_\nu\left[\sum_{l=0}^{\sigma^{(i)}_C-1}\delta^l\right]\right). \tag{4.10}$$

Let us now consider the second term in (4.9).

$$\begin{aligned}\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}\delta^{\bar\sigma_j}\right] &= \check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_{j-1}+\bar\sigma_1\circ\theta_{\bar\sigma_{j-1}}}\delta^{\bar\sigma_{j-1}+\bar\sigma_1\circ\theta_{\bar\sigma_{j-1}}}\right]\\ &= \check{\mathbb{E}}^t_{\mu_d}\left[\check{\mathbb{E}}^t\left[\mathbb{1}_{\bar\sigma>\bar\sigma_{j-1}+\bar\sigma_1\circ\theta_{\bar\sigma_{j-1}}}\delta^{\bar\sigma_{j-1}+\bar\sigma_1\circ\theta_{\bar\sigma_{j-1}}}|\mathcal{F}_{\bar\sigma_{j-1}}\right]\right]\\ &= \check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_{j-1}}\delta^{\bar\sigma_{j-1}}\check{\mathbb{E}}^{t+\bar\sigma_{j-1}}_{\check X^{(i)}_{\bar\sigma_{j-1}}}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_1}\delta^{\bar\sigma_1}\right]\right].\end{aligned}$$

Now we apply Cauchy-Schwarz inequality to the internal expectation and equality (3.5) to get

$$\begin{aligned}\check{\mathbb{E}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_1}\delta^{\bar\sigma_1}\right] &\le \left(\check{\mathbb{P}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\{\bar\sigma>\bar\sigma_1\}\right)^{1/2}\left(\check{\mathbb{E}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\left[\delta^{2\bar\sigma_1}\right]\right)^{1/2}\\ &= \left(\check{\mathbb{P}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\left\{X^{(i)}_{\bar\sigma_1\in C_0}\right\}\right)^{1/2}\left(\check{\mathbb{E}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\left[\delta^{2\bar\sigma_1}\right]\right)^{1/2}\\ &\le (1-\alpha_t)^{1/2}\left(\check{\mathbb{E}}^t_{\check{X}^{(i)}_{\bar\sigma_{j-1}}}\left[\delta^{2\bar\sigma_1}\right]\right)^{1/2}\\ &\le (1-\alpha)^{1/2}\left\{\sup_{t,x\in C}\left(\check{\mathbb{E}}^t_{x\times\{0\}}\left[\delta^{2\bar\sigma_1}\right]\right)\right\}^{1/2}.\end{aligned} \tag{4.11}$$

Last inequality is true for all ω such that $\check{X}^{(i)}_{\bar\sigma_{j-1}}\in C\times\{0\}$, and $\alpha=\inf_t\alpha_t>0$ is defined in CONDITION (A).

From the condition of the Theorem and (5.7) we know that

$$\sup_{t\in\mathbb{N}_0,x\in C}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\beta^{\check\sigma_{\check C}}\right]<\infty,$$

so for any $n>0$

$$\sup_{t\in\mathbb{N}_0,x\in C}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\beta^{\frac{\check\sigma_{\check C}}{n}}\right]\le\left(\sup_{t\in\mathbb{N}_0,x\in C}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\beta^{\check\sigma_{\check C}}\right]\right)^{1/n}\to 1,\ n\to\infty$$

which means

$$\sup_{t\in\mathbb{N}_0,x\in C}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\delta^{\check\sigma_{\check C}}\right]\to 1,\ \delta\to 1.$$

So, we have to select δ such that

$$\gamma^2 := (1-\alpha)\sup_{t\in\mathbb{N}_0,x\in C}\left(\check{\mathbb{E}}^t_{x\times\{0\}}\left[\delta^{2\bar\sigma_1}\right]\right)<1. \tag{4.12}$$

Substituting these δ and γ into (4.11) we get

$$\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_j}\delta^{\bar\sigma_j}\right]\le\gamma\check{\mathbb{E}}^t_{\mu_d}\left[\mathbb{1}_{\bar\sigma>\bar\sigma_{j-1}}\beta^{\bar\sigma_{j-1}}\right]\le\gamma^j\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\bar\sigma_1}\right]. \tag{4.13}$$

Plugging (4.13) and (4.10) into (4.9) we derive

$$\begin{aligned}\check{\mathbb{E}}^t_{\mu_d}\left[\sum_{j=0}^{\check\sigma^{(i)}_{C_1}-1}\delta^j\right] &\le \sup_{x\in C}\check{\mathbb{E}}^t_{x\times\{0\}}\left[\sum_{l=0}^{\check\sigma^{(i)}_{\check C}-1}\delta^l\right]\sum_{j=0}^{\infty}\gamma^j\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\bar\sigma_1}\right]\\ &\le \frac{\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\bar\sigma_1}\right]}{(1-\gamma)}\sup_{t\in\mathbb{N}_0,x\in C}\left\{\frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\sum_{l=0}^{\sigma^{(i)}_C-1}\delta^l\right]-\alpha_t\mathbb{E}^t_{\nu_t}\left[\sum_{l=0}^{\sigma^{(i)}_C-1}\delta^l\right]\right)\right\}.\end{aligned}$$

Note that the previous inequality is valid for $\mu\times\{0\}$ and $\mu\times\{1\}$, which means we can safely change μ_d to μ (which means initial measure $(1-\alpha_t)\mu\times\{0\}+$

$\alpha_t\mu\times\{1\})$. From (4.7) we know that $\check{\mathbb{E}}^t_\mu\left[\delta^{\bar{\sigma}_1}\right] = \mathbb{E}^t_\mu\left[\delta^{\sigma^{(i)}_C}\right]$ which proves (4.3). Finally

$$\begin{aligned}
\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right] &= 1+(\delta-1)\check{\mathbb{E}}^t_{\mu_d}\left[\sum_{l=0}^{\check{\sigma}^{(i)}_{\check{C}}}\delta^l\right]\\
&\le \frac{1+\gamma}{1-\gamma}\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\check{\sigma}^{(i)}_{\check{C}}}\right]\sup_{t\in\mathbb{N}_0,x\in C}\left\{\frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\delta^{\sigma^{(i)}_C}\right]-\alpha_t\mathbb{E}^t_{\nu_t}\left[\delta^{\sigma^{(i)}_C}\right]\right)\right\}\\
&= \frac{1+\gamma}{1-\gamma}M_0\check{\mathbb{E}}^t_{\mu_d}\left[\delta^{\check{\sigma}^{(i)}_{\check{C}}}\right],
\end{aligned}$$

where M_0 is defined in (4.4). Setting $\mu_d = \delta_x\times\{1\}$, $x\in C$ immediately proves (4.2), since definitions (3.1) and (3.2) imply $\check{\mathbb{P}}^t_{\delta_x\times\{1\}}\left\{\check{\sigma}^{(i)}_{\check{C}}>n\right\} = \mathbb{P}^t_{\nu_t}\left\{\sigma^{(i)}_C>n\right\}$. Setting $\mu_d=\mu\times\{0\}$ and then $\mu_d=\mu\times\{1\}$ we get

$$\check{\mathbb{E}}^t_\mu\left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right]\le\frac{1+\gamma}{1-\gamma}M_0\check{\mathbb{E}}^t_\mu\left[\delta^{\check{\sigma}^{(i)}_{\check{C}}}\right]=\frac{1+\gamma}{1-\gamma}M_0\mathbb{E}^t_\mu\left[\delta^{\sigma^{(i)}_C}\right],$$

where last equality follows from (4.7). So we have proven (4.1). Statement (4.5) now follows from the selection of δ in (4.12). ∎

The proven theorem establishes a relationship between $\mathbb{E}^t_\mu\left[\beta^{\sigma^{(i)}_C}\right]$ and $\check{\mathbb{E}}^t_\mu\left[\delta^{\check{\sigma}^{(i)}_{C_1}}\right]$, where C_1 is an atom of the split chain $\check{X}^{(i)}$. Our end goal, however, is to prove the existence of the exponential moment of the simultaneous hitting the set C by both chains $X^{(1)}$ and $X^{(2)}$. This is addressed in the next theorem.

THEOREM 4.2. *Let $X^{(1)}$ and $X^{(2)}$ be two Markov chains that are defined in Section 2 (which means they both satisfy* CONDITION (A)*) and $\check{X}^{(1)}$, $\check{X}^{(2)}$ are corresponding split chains defined in Section 3, random variables $\sigma^{(1)}_C$, $\sigma^{(2)}_C$ are defined at (3.7) and $\mu_1,\mu_2\in\mathbb{M}_1(\mathcal{E})$. Denote $\check{\mu}_i=(1-\alpha_t)\mu_i\times\{0\}+\alpha_t\mu_i\times\{1\}$. Assume that there exists $\beta>1$ such that for all $i\in\{1,2\}$*

$$\mathbb{E}^t_{\mu_i}\left[\beta^{\sigma^{(i)}_C}\right]<\infty,$$

and

$$S_i(\beta)=\sup_{t,x\in C}\left\{\frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\beta^{\sigma^{(i)}_C}\right]-\alpha_t\mathbb{E}^t_{\nu_t}\left[\beta^{\sigma^{(i)}_C}\right]\right)\right\}<\infty.$$

Then there exist constants $\delta_0,\delta_1>1$, $\varepsilon>0$ and $\gamma_0,\gamma_1>0$ such that

$$\mathbb{E}^t_{\mu_1,\mu_2}\left[\delta_1^{\sigma_{C\times C}}\right]\le M\left(\mathbb{E}^t_{\mu_1}\left[\delta_0^{\sigma^{(1)}_C}\right]S_1(\delta_0)+\mathbb{E}^t_{\mu_2}\left[\delta_0^{\sigma^{(2)}_C}\right]S_2(\delta_0)\right)<\infty, \tag{4.14}$$

where

$$M=\left(1+\frac{1}{1-\sqrt{(1+\varepsilon)(1-\gamma_1)}}\right)\left(\frac{1+\gamma_0}{1-\gamma_0}\right).$$

PROOF. From Theorem 4.1 it follows that there exist $\delta_0 > 1$ and $\gamma_0 > 0$ such that for both $i \in \{1, 2\}$

$$\check{\mathbb{E}}^t_{\mu_i}\left[\delta_0^{\check{\sigma}^{(i)}_{C_1}}\right] \le \frac{1+\gamma_0}{1-\gamma_0}\mathbb{E}^t_{\mu_i}\left[\delta_0^{\sigma^{(i)}_C}\right] S_i(\delta_0) < \infty. \tag{4.15}$$

The set C_1 is an atom for each of the chains $\check{X}^{(1)}_t$ and $\check{X}^{(2)}_t$. Then from [8], Theorem 2 it follows that there exist $\delta_1 > 1$, $\varepsilon > 0$ and $\gamma_1 > 0$ such that

$$\check{\mathbb{E}}^t_{\check{\mu}_1,\check{\mu}_2}\left[\delta_1^{\check{\sigma}_{C_1\times C_1}}\right] \le \left(1 + \frac{1}{1-\sqrt{(1+\varepsilon)(1-\gamma_1)}}\right)\left(\check{\mathbb{E}}^t_{\check{\mu}_1}\left[\delta_0^{\check{\sigma}^{(1)}_{C_1}}\right] + \check{\mathbb{E}}^t_{\check{\mu}_2}\left[\delta_0^{\check{\sigma}^{(2)}_{C_1}}\right]\right). \tag{4.16}$$

Combining (4.15) and (4.16) together we obtain

$$\check{\mathbb{E}}^t_{\check{\mu}_1,\check{\mu}_2}\left[\delta_1^{\check{\sigma}_{C_1\times C_1}}\right] \le M\left(\mathbb{E}^t_{\mu_1}\left[\delta_0^{\sigma^{(1)}_C}\right] S_1(\delta_0) + \mathbb{E}^t_{\mu_2}\left[\delta_0^{\sigma^{(2)}_C}\right] S_2(\delta_0)\right) < \infty, \tag{4.17}$$

where M is defined in (4.14).

Denote for positive integers m and $k_1 < k_2 < \ldots < k_m$

$$\begin{aligned}
\check{A}_{k_1,\ldots,k_m} &= \left\{\check{\sigma}^{(1,1)}_{\check{C}} = k_1, \check{\sigma}^{(1,2)}_{\check{C}} = k_2, \ldots, \check{\sigma}^{(1,m)}_{\check{C}} = k_m\right\},\\
\check{B}_{k_1,\ldots,k_m} &= \left\{\check{X}^{(2)}_{k_1} \notin \check{C}, \check{X}^{(2)}_{k_2} \notin \check{C}, \ldots, \check{X}^{(2)}_{k_m} \notin \check{C}\right\},\\
A_{k_1,\ldots,k_m} &= \left\{\sigma^{(1,1)}_{C} = k_1, \sigma^{(1,2)}_{C} = k_2, \ldots, \sigma^{(1,m)}_{C} = k_m\right\},\\
B_{k_1,\ldots,k_m} &= \left\{X^{(2)}_{k_1} \notin C, X^{(2)}_{k_2} \notin C, \ldots, X^{(2)}_{k_m} \notin C\right\}.
\end{aligned}$$

Now we have

$$\begin{aligned}
\check{\mathbb{P}}^t_{\check{\mu}_1,\check{\mu}_2}\left\{\check{\sigma}_{\check{C}\times\check{C}} > n\right\} &= \sum_{j=1}^{n}\sum_{k_1<\ldots<k_j\le n}\check{\mathbb{P}}^t_{\check{\mu}_1,\check{\mu}_2}\left\{\check{A}_{k_1,\ldots,k_j}, \check{B}_{k_1,\ldots,k_j}\right\}\\
&= \sum_{j=1}^{n}\sum_{k_1<\ldots<k_j\le n}\check{\mathbb{P}}^t_{\check{\mu}_1}\left\{\check{A}_{k_1,\ldots,k_j}\right\}\check{\mathbb{P}}^t_{\check{\mu}_2}\left\{\check{B}_{k_1,\ldots,k_j}\right\}\\
&= \sum_{j=1}^{n}\sum_{k_1<\ldots<k_j\le n}\mathbb{P}^t_{\mu_1}\left\{A_{k_1,\ldots,k_j}\right\}\mathbb{P}^t_{\mu_2}\left\{B_{k_1,\ldots,k_j}\right\}\\
&= \mathbb{P}^t_{\mu_1,\mu_2}\left\{\sigma_{C\times C} > n\right\},
\end{aligned} \tag{4.18}$$

here second equality follows from independence of $\check{X}^{(1)}$ and $\check{X}^{(2)}$, and third equality follows from definitions (3.1), (3.2) using the fact that both events $\check{A}_{k_1,\ldots,k_m}$ and $\check{B}_{k_1,\ldots,k_m}$ do not depend on the second coordinate d of the process $\check{X}^{(i)}_n = (x^{(i)}_n, d)$.

Finally, it follows from (4.18) and relation $C_1 \subset \check{C}$ that

$$\mathbb{E}^t_{\mu_1,\mu_2}\left[\delta_1^{\sigma_{C\times C}}\right] = \check{\mathbb{E}}^t_{\check{\mu}_1,\check{\mu}_2}\left[\delta_1^{\check{\sigma}_{\check{C}\times\check{C}}}\right] \le \check{\mathbb{E}}^t_{\check{\mu}_1,\check{\mu}_2}\left[\delta_1^{\check{\sigma}_{C_1\times C_1}}\right],$$

which concludes the proof of the theorem. ∎

Now, we will work on estimating parameters $\delta_0, \delta_1, \varepsilon, \gamma_0$ and γ_1 from Theorem 4.2. The key role in constructing computable bounds plays a dominating sequence.

We will say that decreasing sequence of positive numbers $\{G_n,\ n \ge 0\}$ is a dominating sequence for $\sigma_C^{(i)}$ if

$$\sup_{t\in\mathbb{N}_0, x\in C} \mathbb{P}^t_x\left\{\sigma_C^{(i)} > k\right\} \le G_k,\ \forall k \ge 0. \tag{4.19}$$

We will say that dominating sequence $\{G_n,\ n \ge 0\}$ is exponentially dominating with constants $C > 0$ and $\beta > 1$ if

$$G_k \le C\beta^{-k},\ \forall k \ge 0. \tag{4.20}$$

Note that condition

$$\sup_{t\in\mathbb{N}_0} \mathbb{E}^t_\mu\left[\beta^{\sigma_C^{(i)}}\right] < \infty$$

implies the existence of exponentially dominating sequence with constants $C = \sup_{t\in\mathbb{N}_0} \mathbb{E}^t_\mu\left[\beta^{\sigma_C^{(i)}}\right]$ and any $\delta \in (1, \beta]$ due to the Chernoff inequality

$$\mathbb{P}^t_\mu\left\{\sigma_C^{(i)} > k\right\} \le e^{-uk}\mathbb{E}^t_\mu\left[e^{u\sigma_C}\right], \forall u > 0.$$

In particular, we may put $u = \ln\beta$ and get

$$\mathbb{P}^t_\mu\left\{\sigma_C^{(i)} > k\right\} \le \beta^{-k}\mathbb{E}^t_\mu\left[\beta^{\sigma_C}\right] \le \beta^{-k}\sup_{t\in\mathbb{N}_0}\mathbb{E}^t_\mu\left[\beta^{\sigma_C}\right].$$

However, it could be difficult to find the exact value of $\mathbb{E}^t_\mu\left[\beta^{\sigma_C^{(i)}}\right]$, while it is often feasible to get an estimate of it. That is why an assumption of the existence of exponentially dominating sequence is reasonable and not restrictive. In the next theorem we show how to estimate parameters in Theorem 4.2.

THEOREM 4.3. *Let $X^{(i)}$, $i \in \{1, 2\}$ be Markov chains defined in Section 2 (which means they satisfy* CONDITION (A)*), and $\check{X}^{(i)}$ its split chain. Let $\sigma_C^{(i)}, \check{\sigma}_{\check{C}}^{(i)}$ and $\check{\sigma}_{C_1}^{(i)}$ be the hitting times defined in (3.7) and (3.8).*

1. *Assume there exist exponential dominating sequences with constants $D > 0$, $\beta_0 > 1$ and measurable functions $V_t^{(i)}\colon E \to \mathbb{R}_+$ such that*

$$\sup_{t\in\mathbb{N}_0, x\in C} \mathbb{P}^t_x\left\{\sigma_C^{(i)} > n\right\} \le G_n = D\beta_0^{-n},$$

and for all $x \in E \setminus C$

$$\mathbb{P}_x^t \left\{ \sigma_C^{(i)} > n \right\} \le G_n(x) = V_t^{(i)}(x) \beta_0^{-n}.$$

2. *For every* $t \in \mathbb{N}_0, x \in C, i \in \{0,1\}$ *denote a probability measure* $q_{t,x}^{(i)}(\cdot) \in \mathbb{M}_1$ *such that*

$$q_{t,x}^{(i)}(A) = \frac{P_{t,i}(x,A) - \alpha_t \nu_t(A)}{1-\alpha_t}.$$

Assume that

$$\hat{Q} = \sup_{t\in\mathbb{N}_0, x\in C, i\in\{1,2\}} q_{t,x}^{(i)}(V_t^{(i)}) = \sup_{t\in\mathbb{N}_0, x\in C,\ i\in\{1,2\}} \int_E q_{t,x}^{(i)}(dy) V_t^{(i)}(y) < \infty.$$

3. *Assume that*

$$\inf_{t\in\mathbb{N}_0} \nu_t(C) > 0.$$

If conditions 1.-3. hold true, then constants $\delta_0 > 1, \delta_1 > 1, \gamma_0 > 0, \gamma_1 > 0$ *and* $\varepsilon > 0$ *could be selected in the following way:*

$$\delta_0 < \sqrt{1 + \frac{\alpha(\beta_0 - 1)}{(1-\alpha)\beta_0 \hat{Q} + \alpha}},$$

$$\gamma_0 = \left\{ (1-\alpha) \left(1 + \frac{(\delta_0^2 - 1)\beta_0}{\beta_0 - \delta_0^2} \hat{Q} \right) \right\}^{\frac{1}{2}},$$

ε *- an arbitrary small constant,*

$$\gamma_1 = \left(\alpha \inf_t \nu_t(C) \right)^m \exp\left(\ln\left(1 - \check{D}\delta^{-m}\right) \left(\frac{\delta_0^{m+1}}{\delta_0 - 1} - 1 \right) \right),$$

$$\delta_1 = (1 + \varepsilon/2)^{\frac{1}{m+n_0}},$$

where

$$\check{D} = D\frac{1+\gamma_0}{1-\gamma_0} \left(1 + \frac{(\delta_0 - 1)\beta_0}{\beta_0 - \delta_0} \hat{Q} \right) \left(1 + \frac{\delta_0(\beta_0 - 1)}{\beta_0 - \delta_0} \right),$$

$$m = \min\left\{ n \ge 1 | \check{D}\delta_0^{-n} < 1 \right\},$$

$$n_0 = \left\lfloor \ln\left(\frac{\varepsilon(\delta_0 - \beta_0)}{2\check{D}\beta_0^{m+1}} \right) / \ln\left(\frac{\beta_0}{\delta_0} \right) \right\rfloor + 3,$$

and α *defined in* CONDITION (A).

PROOF. First, we show that conditions of Theorem 4.1 are satisfied.

It is clear that for every $x \in C$ and $\beta \in (1, \beta_0)$

$$\mathbb{E}_x^t \left[\beta^{\sigma_C^{(i)}} \right] - 1 \le D(\beta - 1) \sum_{k\ge 0} (\beta/\beta_0)^k = D\frac{(\beta-1)\beta_0}{\beta_0 - \beta} < \infty.$$

Similarly, for any $\mu \in \mathbb{M}_1$, such that $\mu(V) < \infty$

$$\mathbb{E}^t_\mu \left[\beta^{\sigma^{(i)}_C}\right] - 1 \leq \mu(V)\frac{(\beta-1)\beta_0}{\beta_0 - \beta} < \infty.$$

Next, we note the following relation

$$\frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\beta^{\sigma^{(i)}_C}\right] - \alpha_t \mathbb{E}^t_{\nu_t}\left[\beta^{\sigma^{(i)}_C}\right]\right) = \mathbb{E}^t_{q^{(i)}_{t,x}}\left[\beta^{\sigma^{(i)}_C}\right].$$

So, the condition

$$\sup_{t\in\mathbb{N}_0, x\in C}\left\{\frac{1}{1-\alpha_t}\left(\mathbb{E}^t_x\left[\beta^{\sigma^{(i)}_C}\right] - \alpha_t \mathbb{E}^t_{\nu_t}\left[\beta^{\sigma^{(i)}_C}\right]\right)\right\} < \infty$$

is equivalent to

$$\sup_{t\in\mathbb{N}_0, x\in C} \mathbb{E}^t_{q^{(i)}_{t,x}}\left[\beta^{\sigma^{(i)}}\right] < \infty,$$

which is true since conditions 1 and 2 imply

$$\sup_{t\in\mathbb{N}_0, x\in C} \mathbb{E}^t_{q^{(i)}_{t,x}}\left[\beta^{\sigma^{(i)}_C}\right] \leq 1 + \frac{(\beta-1)\beta_0}{\beta_0-\beta}\hat{Q} < \infty. \tag{4.21}$$

Now, according to Theorem 4.1 we select δ_0 such that

$$\sup_{t\in\mathbb{N}_0, x\in C} \mathbb{E}^t_{q^{(i)}_{t,x}}\left[\delta_0^{2\sigma^{(i)}_C}\right] < \frac{1}{1-\alpha},$$

where α is defined in Condition (A). Inequality (4.21) implies it is sufficient to select δ_0 such that

$$1 + \frac{(\delta_0^2-1)\beta_0}{\beta_0-\delta_0^2}\hat{Q} < \frac{1}{1-\alpha},$$

which means

$$\delta_0 < \sqrt{1 + \frac{\alpha(\beta_0-1)}{(1-\alpha)\beta_0\hat{Q}+\alpha}}.$$

From Theorem 4.1 it follows that γ_0 can be defined as

$$\gamma_0 = \left\{(1-\alpha)\left(1+\frac{(\delta_0^2-1)\beta_0}{\beta_0-\delta_0^2}\hat{Q}\right)\right\}^{\frac{1}{2}}.$$

To obtain expressions for other constants, we use results from [8]. First we note that condition 3 implies

$$\check{\mathbb{P}}^t_{C_1}\left\{\check{X}^{(i)}_1 \in C_1\right\} \geq \alpha \inf_{t\in\mathbb{N}_0} \nu_t(C) > 0,$$

which means conditions of Lemmas 2,3 and 4 from [8] hold true for every $m \geq 1$. Using Chernoff inequality, formulas (4.2) and (4.21) we can build exponentially dominating sequence for

$$\check{\mathbb{P}}^t_{C_1}\left\{\check{\sigma}^{(i)}_{C_1} > n\right\} \le \check{\mathbb{E}}^t_{C_1}\left[\delta_0^{\check{\sigma}^{(i)}_{C_1}}\right]\delta_0^{-n} \le \frac{1+\gamma_0}{1-\gamma_0}M_0\mathbb{E}^t_{\nu_t}\left[\delta_0^{\sigma_C}\right]\delta_0^{-n}$$
$$= D_1\mathbb{E}^t_{\nu_t}\left[\delta_0^{\sigma_C}\right]\delta_0^{-n} \le D_1 D\left(1+\frac{\delta_0(\beta_0-1)}{\beta_0-\delta_0}\right)\delta_0^{-n},$$

where M_0 is defined in (4.4) and

$$D_1 = \frac{1+\gamma_0}{1-\gamma_0}\left(1+\frac{(\delta_0-1)\beta_0}{\beta_0-\delta_0}\hat{Q}\right).$$

So we can define exponential dominating sequence for $\check{\mathbb{P}}^t_{C_1}\left\{\check{\sigma}^{(i)}_{C_1} > n\right\}$ as

$$\check{G}_n = \check{D}\delta_0^{-n},$$

where

$$\check{D} = D\frac{1+\gamma_0}{1-\gamma_0}\left(1+\frac{(\delta_0-1)\beta_0}{\beta_0-\delta_0}\hat{Q}\right)\left(1+\frac{\delta_0(\beta_0-1)}{\beta_0-\delta_0}\right).$$

Now we can select

$$m = \min\left\{n \ge 1 | \check{G}_n < 1\right\}.$$

Lemmas 3 and 4 from [8] render an expression for γ_1 and δ_1:

$$\gamma_1 = \left(\alpha\inf_t \nu_t(C)\right)^m \exp\left(\ln\left(1-\check{D}\delta^{-m}\right)\left(\frac{\delta_0^{m+1}}{\delta_0-1}-1\right)\right),$$
$$\delta_1 = (1+\varepsilon/2)^{\frac{1}{m+n_0}},$$

where ε is an arbitrary small positive constant, and

$$n_0 = \left\lfloor \ln\left(\frac{\varepsilon(\delta_0-\beta_0)}{2\check{D}\beta_0^{m+1}}\right)/\ln\left(\frac{\beta_0}{\delta_0}\right)\right\rfloor + 3.$$

□

We conclude this section with a small toy example that illustrates how results, in particular, Theorem 4.3 can be applied to some inhomogeneous Markov chains. We will see that despite complicated formulas, we can employ standard techniques from the homogeneous theory.

Inhomogeneous ARCH(1) model. Consider, first, the Markov chain

$$X_k = \sqrt{a_k + b_k X_{k-1}^2}Z_k,\ a_k \in (a_{min}, a_{max}),\ b_k \in (b_{min}, b_{max}),$$

where $a_{min}, a_{max}, b_{min}, b_{max} > 0$ - real numbers, $\{Z_k,\ k \ge 1\}$ independent sequence of real-valued random variables such that:

1. Each Z_k has a density g_k with respect to a Lebesgue measure μ.
2. There is a constant $c > 0$ and interval $[-c_0, c_0] \in \mathbb{R}$ such that, for all $k \ge 1$

$$g_k(x) \ge c\mathbb{1}_{[-c_0,c_0]}(x),\ x \in \mathbb{R}.$$

3. There exists $s \in (0,1]$ such that

$$\lambda_0 := \sup_k \left\{ b_k^s \mathbb{E}[Z_k^{2s}] \right\} = \sup_k \left\{ b_k^s \int_{-\infty}^{+\infty} z^{2s} g_k(z) dz \right\} < 1.$$

We denote $\mathbb{E}[Z_k^{2s}]$ by m_k. It is well known from the homogeneous theory that the function $V(x) = 1 + x^{2s}$ is a test function, i.e

$$P_k V(x) \leq 1 + a_k^s m_k + b_k^s m_k x^{2s} \leq \lambda_k V(x) + (1 - \lambda_k + a_k^s m_k),$$

where $\lambda_k = b_k^s m_k \leq \lambda_0 < 1$. It is also well known that each closed interval $[-c_1, c_1] \in \mathbb{R}, c_1 > 0$ is small in the sense

$$P_k(x, A) \geq \alpha_k \nu_k(A), \; x \in [-c_1, c_1],$$

where

$$\alpha_k = 2c_0 c\sqrt{a_k}(a_k + b_k(1 - \lambda_k + a_k^s m_k)^2)^{-1/2},$$

$$\nu_k(A) = \frac{1}{2c_0\sqrt{a_k}} \mu(A \cap [-c_0\sqrt{a_k}, c_0\sqrt{a_k}]).$$

Clearly $\inf_k \alpha_k > 0$, since both a_k and b_k are bounded away from 0 and from ∞. Finally, we select

$$\lambda = (1 + \lambda_0)/2 \in (\lambda_0, 1),$$

and set $C = [-c_2, c_2]$ such that for all $x \in \mathbb{R}$ with $|x| > c_2$ and $k > 0$

$$(\lambda_k - \lambda)(1 + x^{2s}) + (1 - \lambda_k + a_k^s m_k) < 0,$$

it is clear that such selection is always possible provided conditions 1.-3. hold. So we get for all $k \geq 1$

$$P_k V(x) \leq \lambda V(x) + \tilde{b} \mathbb{1}_C(x), \tag{4.22}$$

where $\tilde{b} = \sup_{x \in C} \left\{ (\lambda_k - \lambda)(1 + x^{2s}) + (1 - \lambda_k + a_k^s m_k) \right\}$. Using Theorem 1 from [8] we get

$$\mathbb{E}_x^k \left[\lambda^{-\sigma_C} \right] \leq V(x) + \frac{\tilde{b}}{\lambda} \mathbb{1}_C(x), \tag{4.23}$$

which is an analogue of a very well-known classical result from the homogeneous theory. Now, we can verify that Theorem 4.3 is valid for the chain $(X_n, \; n \geq 0)$. Condition 1 of Theorem 4.3 is valid with $\beta_0 = \lambda$ and $D = (1 + c_2^{2s}) + \tilde{b}/\lambda$. Conditions 2 and 3 of Theorem 4.3 are obvious.

Now we can consider two inhomogeneous chains of the form

$$X_k^{(i)} = \sqrt{a_k^{(i)} + b_k^{(i)} (X_{k-1}^{(i)})^2} Z_k^{(i)}, \; a_k^{(i)} \in (a_{min}, a_{max}), \; b_k^{(i)} \in (b_{min}, b_{max}),$$

where $i \in \{1, 2\}$. Assuming conditions 1-3 from this example hold true for each of them (possibly with a different set of constants), we derive that function $V(x) = 1 + 2x^{2s}$ is a valid test function for both chains, with different $\lambda^{(i)} < 1$ and there exists set C that satisfies CONDITION (A) for both chains as well inequality (4.23), possibly with different $\lambda^{(i)}$ and $\tilde{b}^i$, $i \in \{1, 2\}$. Thus we can employ Theorem 4.2 and Theorem 4.3 to obtain the direct formula for the bound of the exponential moment of the simultaneous hitting time $\sigma_{C\times C}$.

5. Auxiliary results

Lemma 5.1. *Let $i \in \{1,2\}$ be fixed, $C \in \mathcal{E}$ be some set, and let random variables $\check{\sigma}^{(i)}_{\check{C}}$, $\sigma^{(i)}_C$ be defined in (3.7) and (3.8). Then the following equalities hold true.*

1. *For any $x \in E \setminus C$, $d \in \{0,1\}$*

$$\mathbb{P}^t_x\left\{\sigma^{(i)}_C > k\right\} = \check{\mathbb{P}}^t_{x\times\{d\}}\left\{\check{\sigma}^{(i)}_{\check{C}} > k\right\}, \tag{5.1}$$

2. *For any $x \in C$*

$$\check{\mathbb{P}}^t_{x\times\{0\}}\left\{\check{\sigma}^{(i)}_{\check{C}} > k\right\} = \frac{1}{1-\alpha_t}\left(\mathbb{P}^t_x\left\{\sigma^{(i)}_C > k\right\} - \alpha_t\mathbb{P}^t_{\nu_t}\left\{\sigma^{(i)}_C > k\right\}\right). \tag{5.2}$$

Proof. Let x be a fixed element from E, then

$$\begin{aligned}\mathbb{P}^t_x\{\sigma^{(i)}_C > k\} &= \mathbb{P}^t_x\{X^{(i)}_l \notin C, 1 \le l \le k\}\\ &= \int_{C^c}\dots\int_{C^c} P_{t,i}(x,dx_1)\dots P_{t+k-1,i}(x_{t+k-1},dx_{t+k}).\end{aligned} \tag{5.3}$$

Equalities (3.1) imply for $x \in E \setminus C$

$$\begin{aligned}P_{t+j,i}(x,dy) &= \check{P}_{t+j,i}((x,d),dy\times\{0,1\})\\ &= \check{\mathbb{P}}^t_{x\times\{d\}}\{\check{X}^{(i)}_1 = dy\times\{0,1\}\}.\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) we get for $x \in E \setminus C$

$$\mathbb{P}^t_x\left\{\sigma^{(i)}_C > k\right\} = \check{\mathbb{P}}^t_{x\times\{d\}}\left\{\check{X}^{(i)}_l \notin \check{C},\ 1 \le l \le k\right\} = \check{\mathbb{P}}^t_{x\times\{d\}}\left\{\check{\sigma}^{(i)}_{\check{C}} > k\right\},$$

which proves statement 1.

Now we apply equality (5.3) to the right-hand side of formula (5.2).

$$\begin{aligned}&\frac{\mathbb{P}^t_x\left\{\sigma^{(i)}_C > k\right\} - \alpha_t\mathbb{P}^t_{\nu_t}\left\{\sigma^{(i)}_C > k\right\}}{1-\alpha_t}\\ &\qquad= \int_{C^c}\left(\frac{P_{t,i}(x,dx_1) - \alpha_t\nu(dx_1)}{1-\alpha_t}\right)\times\mathbb{P}^{t+1}_{x_1}\left\{\sigma^{(i)}_C > k-1\right\}.\end{aligned}$$

since $x_1 \notin C$ we can use proven formula (5.1) and definition (3.2) to obtain

$$\begin{aligned}&\frac{\mathbb{P}^t_x\left\{\sigma^{(i)}_C > k\right\} - \alpha_t\mathbb{P}^t_{\nu_t}\left\{\sigma^{(i)}_C > k\right\}}{1-\alpha_t}\\ &\qquad= \int_{C^c}\check{P}((x,0),(dx_1\times\{0,1\})\times\check{\mathbb{P}}^t_{x_1}\left\{\check{\sigma}^{(i)}_{\check{C}} > k-1\right\}\\ &\qquad= \check{\mathbb{P}}^t_{x\times\{0\}}\left\{\check{\sigma}^{(i)}_{\check{C}} > k\right\}.\end{aligned}$$

which completes the proof of statement 2. ∎

COROLLARY 5.2. *Assume* $\mu \in \mathbb{M}_1(\mathcal{E})$ *is some probability measure. Then*

$$\mathbb{P}^t_\mu \left\{ \sigma^{(i)}_C > k \right\} = \check{\mathbb{P}}^t_\mu \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\}.$$

PROOF. First, we show that for all $x \in E$

$$\mathbb{P}^t_x \left\{ \sigma^{(i)}_C > k \right\} = \check{\mathbb{P}}^t_x \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\}. \tag{5.5}$$

Statement 1 from Lemma 5.1 implies that for any $x \in E \setminus C$

$$\begin{aligned} \check{\mathbb{P}}^t_x \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} &= (1-\alpha_t)\check{\mathbb{P}}^t_{x\times\{0\}} \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} + \alpha_t \check{\mathbb{P}}^t_{x\times\{1\}} \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} \\ &= (1-\alpha_t)\mathbb{P}^t_x \left\{ \sigma^{(i)}_C > k \right\} + \alpha_t \mathbb{P}^t_x \left\{ \sigma^{(i)}_C > k \right\} = \mathbb{P}^t_x \left\{ \sigma^{(i)}_C > k \right\}. \end{aligned}$$

In case $x \in C$ we have

$$\check{\mathbb{P}}^t_{x\times\{1\}} \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} = \mathbb{P}^t_{\nu_t} \left\{ \sigma^{(i)}_C > k \right\}$$

by construction (see formulas (3.1) and (3.2)), combining this with statement 2 from Lemma 5.1 we get for $x \in C$

$$\begin{aligned} \check{\mathbb{P}}^t_x \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} &= (1-\alpha_t)\check{\mathbb{P}}^t_{x\times\{0\}} \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} + \alpha_t \check{\mathbb{P}}^t_{x\times\{1\}} \left\{ \check{\sigma}^{(i)}_{\check{C}} > k \right\} \\ &= (1-\alpha_t)\frac{1}{1-\alpha_t} \left(\mathbb{P}^t_x \left\{ \sigma^{(i)}_C > k \right\} - \alpha_t \mathbb{P}^t_{\nu_t} \left\{ \sigma^{(i)}_C > k \right\} \right) \\ &\quad + \alpha_t \mathbb{P}^t_{\nu_t} \left\{ \sigma^{(i)}_C > k \right\} \\ &= \check{\mathbb{P}}^t_x \left\{ \sigma^{(i)}_C > k \right\}. \end{aligned}$$

Now we integrate (5.5) with respect to μ and receive the statement of the Corollary. ∎

COROLLARY 5.3. *Let* $\beta > 1$ *be a constant. Then*

$$\check{\mathbb{E}}^t_\mu \left[\beta^{\check{\sigma}_{\check{C}}} \right] = \mathbb{E}^t_\mu \left[\beta^{\sigma_C} \right], \tag{5.6}$$

and

$$\check{\mathbb{E}}^t_{x\times\{0\}} \left[\beta^{\check{\sigma}_{\check{C}}} \right] = \frac{1}{1-\alpha_t} \left(\mathbb{E}^t_x \left[\beta^{\sigma_C} \right] - \alpha_t \mathbb{E}^t_{\nu_t} \left[\beta^{\sigma_C} \right] \right). \tag{5.7}$$

PROOF. Taking into account Corollary 5.2 we can write

$$\begin{aligned} (\beta-1)^{-1} \left(\mathbb{E}^t_\mu \left[\beta^{\sigma_C} \right] - 1 \right) &= \mathbb{E}^t_\mu \left[\sum_{k=0}^{\sigma_C - 1} \beta^k \right] = \sum_{k=0}^{\infty} \mathbb{E}^t_\mu \left[\beta^k \mathbb{1}_{\sigma_C > k} \right] \\ &= \sum_{k=0}^{\infty} \check{\mathbb{E}}^t_\mu \left[\beta^k \mathbb{1}_{\check{\sigma}_{\check{C}} > k} \right] = \check{\mathbb{E}}^t_\mu \left[\sum_{k=0}^{\check{\sigma}_c - 1} \beta^k \right] \\ &= (\beta-1)^{-1} \left(\check{\mathbb{E}}^t_\mu \left[\beta^{\check{\sigma}_{\check{C}}} \right] - 1 \right). \end{aligned}$$

So formula (5.6) is proven. The proof of equality (5.7) follows the same arguments with application of formula (5.2) instead of Corollary 5.2. □

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Received: 7.4.2021.
Revised: 15.5.2021.