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Free Vibration of a Cantilever Euler-Bernoulli Beam Carrying a Point Mass by Using SEM

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Abstract: The objective of this research is to study the free vibration of a cantilever Euler-Bernoulli beam carrying a point mass with moment of inertia at the free end using the spectral element method (SEM). Typically, the shape (or interpolation) functions used in the Spectral element method are derived from exact solutions of the governing differential equations of motion in the frequency domain. The beam was discretized by a single spectral element which was connected by a point mass at the free end. The dynamic stiffness matrix of the beam is formulated in frequency domain by considering compatibility conditions at the additional mass position. Then, the first three natural frequencies of the cantilever beam are determined. After the validation of the spectral element method, the influence of the non-dimensional mass parameter and the non-dimensional mass moment of inertia on the first three natural frequencies and shape mode are examined.

Keywords: cantilever beam; free vibration; point mass; spectral element method

1 INTRODUCTION

Beams structures are used as one of the fundamental structural components in civil and mechanical engineering. Free vibration analysis of beams is of great importance in design and fabrication of structures and machines such as highway bridges, railroads, tall building, wind turbines and huge cranes.

There have been extensive research published in the area about the vibration characteristics of a uniform beam with different boundary conditions and with/without various concentrated elements (such as intermediate point masses, rotary inertias, linear springs, rotational springs, spring-mass systems, etc.). Analytical and numerical methods were used to obtain the natural frequencies and different aspects have been considered [1-8]. For the vibration analysis of beams carrying concentrated masses at arbitrary locations, a lot of studies have been published. Hong et.al [9] investigated the transverse vibration of clamped-pinned-free Euler-Bernoulli beam with mass at free end; obtained analytical eigenvalues of the system were compared to experimental data. Natural frequencies and model shapes of a clumped beam with mass at free end have been determined [10]. Wang et al. [11] examined the transverse vibration of a cantilever Euler-Bernoulli beam that has a mass with moment of inertia on its free end. Wu and Lin [12] analysed the frequency equation of flexural vibrating cantilever beam with masses attached at multiple points by using an analytical-numerical combined method. They derived the eigenvalue equation analytically by using an expansion theorem and frequencies and mode shapes were calculated numerically. Chang [13] performed the free vibration analysis of a simply supported beam carrying a rigid mass at the middle. Low [14] used both the methods of frequency determinant and the method of Laplace transform to determine the Eigenvalues of a beam with any number of point masses. Naguleswaran [15, 16] studied free vibration of an Euler-Bernoulli beam with point masses and negligible inertia moment of the point mass. Gürgöze [17-19] has carried out several studies on the frequency equation of flexural vibrating beam carrying a rigid mass. Reference [20] dealt with the determination of the eigenvalues of an Euler-Bernoulli beam with one end spring-mass system attached and the other fixed.

The purpose of this study is to use the spectral element method to establish the dynamic equation of a cantilever Euler–Bernoulli beam with tip mass at the free end. The spectral formulation requires that the equation of motion is solved in the frequency domain and the fast Fourier transform (FFT) is utilized to convert the time domain responses to the wave domain and back. The accuracy and the validation of the proposed approach are confirmed by the comparison with existing results in literature. Finally, the effects of non-dimensional mass parameter and the non-dimensional mass moment of inertia on the natural frequencies and shape mode are investigated.

2 MATHEMATICAL MODEL AND FORMULATIONS

Consider a cantilever beam with an additional mass M_0 having a moment of inertia J_0 at the free end of length L as shown in Fig.1. The beam has flexural stiffness EI and mass per unit length M. Based on the Euler-Bernoulli beam theory, the governing equation of motion can be written:

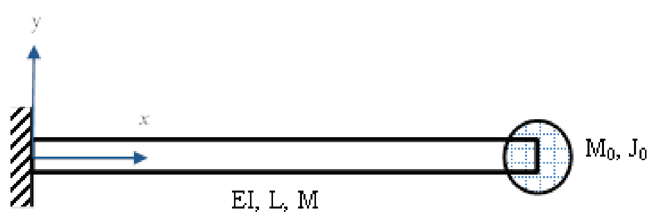


Figure 1 Schematic of a cantilever beam with an additional mass at free end

$$\frac{\partial^4 w(x,t)}{\partial x^4}+\frac{M}{EI}\frac{\partial^2 w(x,t)}{\partial t^2}=0, \tag{1}$$

The solution of Eq. (1) can be assumed in the spectral form as:

$$w(x,\, t)=\frac{1}{N}\sum_{n=0}^{N-1}W_n(x)\cdot \mathrm{e}^{i\omega_n t}, \tag{2}$$

Where $w(x, t)$ is the transverse displacement, E is Young's modulus, I is the area moment of inertia about the neutral axis.

Inserting Eq. (2) into Eq. (1) and after rearrangement, the following equation is derived:

$$\frac{\mathrm{d}^4 W(x)}{\mathrm{d}x^4} - \alpha^4 W(x) = 0, \tag{3}$$

In which $\alpha^4 = \frac{M\omega^2}{EI}$, and ω is the circular frequency.

Then the solution of Eq. (3) can be written as:

$$W(x, \omega) = \bar{A}_1 \cos\alpha x + \bar{A}_2 \sin\alpha x + \bar{A}_3 \cosh\alpha x + \bar{A}_4 \sinh\alpha x, \tag{4}$$

Writing displacements and rotations at nodes:

$w_1 = w(0, t)$, $\vartheta_1 = \frac{\partial w(0, t)}{\partial x}$, $w_2 = w(L, t)$, $\vartheta_2 = \frac{\partial w(L, t)}{\partial x}$.

The nodal displacement and slope at both ends can be expressed as:

$$\{q_e\} = [\boldsymbol{D}] \mathrm{e}^{i\omega t} \{\bar{A}\}. \tag{5}$$

Where $\{\bar{A}\} = \{\bar{A}_1 \quad \bar{A}_2 \quad \bar{A}_3 \quad \bar{A}_4\}^{\mathrm{T}}$ is the constant

Vector and the matrix $[\boldsymbol{D}]$ has the form

$$[\boldsymbol{D}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \alpha \\ \cos\alpha L & \sin\alpha L & \cosh\alpha L & \sinh\alpha L \\ -\alpha\cdot\sin\alpha L & \alpha\cdot\cos\alpha L & \alpha\cdot\sinh\alpha L & \alpha\cdot\cosh\alpha L \end{bmatrix}. \tag{6}$$

Corresponding to the general problem Fig. 1, the vector of forces at the ends of the beam can be established by using boundary conditions of the structural system, which are:

$$\text{At } x = 0 \begin{cases} V_1 = EI\dfrac{\partial^3 w(0, t)}{\partial x^3} \\ M_1 = -EI\dfrac{\partial^2 w(0, t)}{\partial x^2} \end{cases}, \tag{7}$$

$$\text{At } x = L \begin{cases} V_2 = EI\dfrac{\partial^3 w(L, t)}{\partial x^3} + M_0\omega^2 \cdot w(L, t) \\ M_2 = EI\dfrac{\partial^2 w(L, t)}{\partial x^2} + J_0\omega^2 \dfrac{\partial w(L, t)}{\partial x} \end{cases}, \tag{8}$$

In matrix form, the forces at the ends of beam are

$$\{F_e\} = \begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = [\boldsymbol{F}] \cdot \mathrm{e}^{i\omega t} \{\bar{A}\}. \tag{9}$$

With,

$$[\boldsymbol{F}] = \begin{bmatrix} 0 & -EI\alpha^3 & 0 & EI\alpha^3 \\ EI\alpha^2 & 0 & -EI\alpha^2 & 0 \\ C_1 & C_2 & C_3 & C_4 \\ R_1 & R_2 & R_3 & R_4 \end{bmatrix}. \tag{10}$$

$$\begin{aligned} C_1 &= EI\alpha^3 \sin\alpha L + M_0\omega^2 \cos\alpha L, \\ C_2 &= -EI\alpha^3 \cos\alpha L + M_0\omega^2 \sin\alpha L, \\ C_3 &= EI\alpha^3 \sinh\alpha L + M_0\omega^2 \cosh\alpha L, \\ C_4 &= EI\alpha^3 \cosh\alpha L + M_0\omega^2 \sinh\alpha L, \\ R_1 &= -EI\alpha^2 \cos\alpha L - J_0\omega^2 \sin\alpha L, \\ R_2 &= -EI\alpha^2 \sin\alpha L + J_0\omega^2 \cos\alpha L, \\ R_3 &= EI\alpha^2 \cosh\alpha L + J_0\omega^2 \sinh\alpha L, \\ R_4 &= EI\alpha^2 \sinh\alpha L + J_0\omega^2 \cosh\alpha L. \end{aligned}$$

The relationship between nodal force and degree of freedom vectors is expressed by

$$\{F_e\} = [\boldsymbol{F}][\boldsymbol{D}]^{-1} \{q_e\}. \tag{11}$$

Where $[\boldsymbol{F}][\boldsymbol{D}]^{-1}$ is the spectral stiffness matrix of Euler-Bernoulli beam with mass M attached at end ($x = L$).

3 NUMERIC APPLICATIONS

The applications reported in this section are provided by applying the proposed approach on a cantilever beam with a mass attached at the free end with the following non-dimensional parameters; β_M, β_J, and λ denoting the non-dimensional mass parameter , non-dimensional rotary mass moment of inertia and non-dimensional natural frequencies of the beam, respectively.

$$\beta_M = \frac{M_0}{M \cdot L} \tag{12}$$

$$\beta_J = \frac{J_0}{M \cdot L^3} \tag{13}$$

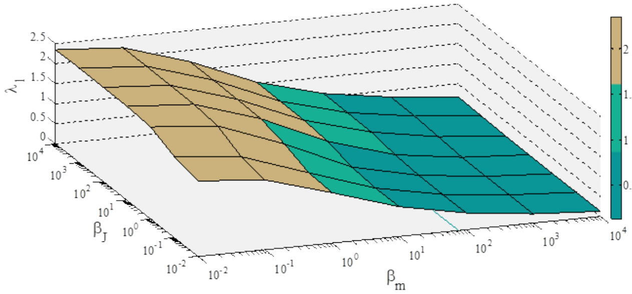
$$\lambda = \alpha L \tag{14}$$

To check the numerical model, the first three eigenvalues for the cantilever beam are determined by finding the nontrivial solutions of the determinant in Eq. (9). Computer programs based on SEM have been developed in MATLAB software to calculate numerical results.

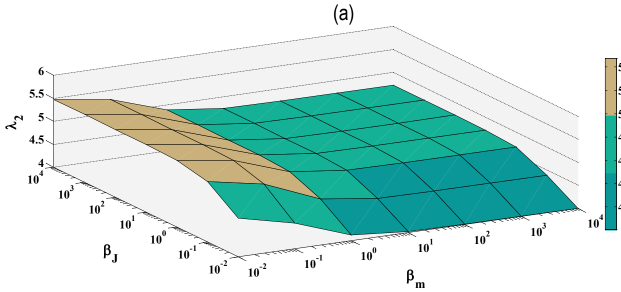
In order to check the effectiveness of the proposed method, the first three dimensionless frequencies of a cantilever beam carrying a tip mass beam are shown in Tab. 1. Without considering the moment of inertia of the attached mass, it is clear from Tab. 1 that the obtained values are in good agreement with those obtained by Rao [1] and Gürgöze [16].

Table 1 The effect of the variation of β_M and β_J on the first three dimensionless frequencies λ of cantilever beam carrying a tip mass is presented in Fig. 2

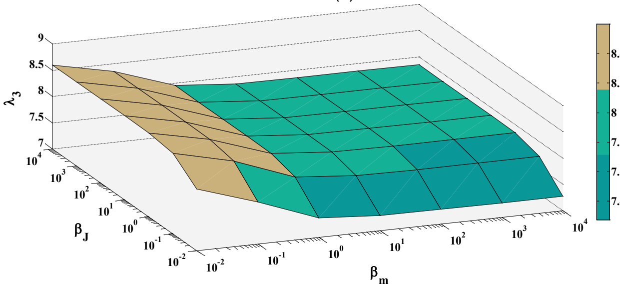
β_M, β_J	Method	λ_1	λ_2	λ_3
$\beta_M = 0.01$ $\beta_J = 0$	SEM	1.857	4.650	7.783
	Rao [1]	1.852	4.650	-
	Gürgöze [16]	1.857	-	-
$\beta_M = 1$ $\beta_J = 0$	SEM	1.248	4.031	7.134
	Rao [1]	1.248	4.031	-
	Gürgöze [16]	1.248	-	-
$\beta_M = 100$ $\beta_J = 0$	SEM	0.416	3.928	7.069
	Rao [1]	0.416	3.928	-
	Gürgöze [16]	0.416	-	-
$\beta_M = 0.01$ $\beta_J = 0.01$	SEM	1.875	4.843	8.168
$\beta_M = 0.01$ $\beta_J = 0$	SEM	2.286	5.454	8.591
$\beta_M = 1$ $\beta_J = 0.01$	SEM	1.252	4.128	7.367
$\beta_M = 1$ $\beta_J = 1$	SEM	1.540	4.854	7.949



(a)



(b)

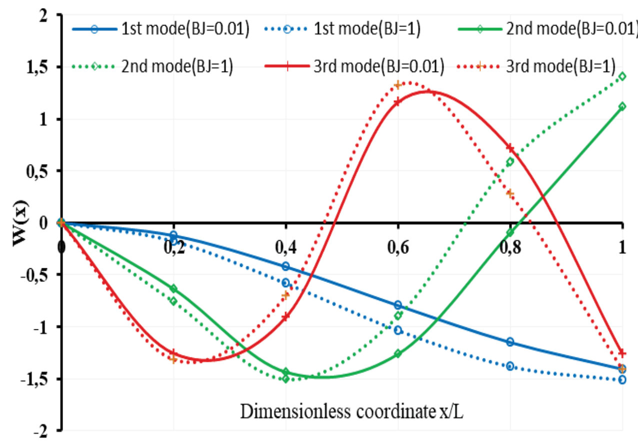


(c)

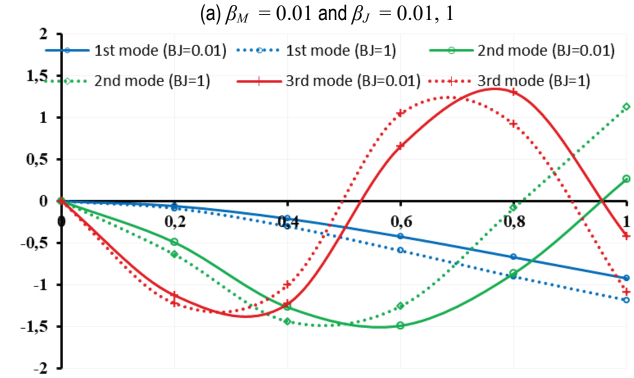
Figure 2 Variation of the natural frequency parameter (λ_1) as β_M and β_J varies from 10^{-2} to 10^4. (a) λ_1, (b) λ_2, (c) λ_3.

It observed from Fig. 2a that surface has a region of lowest dimensionless frequencies λ_1 corresponds to both β_M and β_J ($\beta_M \geq 100$, $\beta_J \geq 100$) being very extremely large values which represents the clumped-simply supported system. For negligible effect of both β_M and β_J ($\beta_M \leq 0.01$, $\beta_J \leq 0.01$), the values of dimensionless frequencies λ_1 represents the clumped-free beam. The Fig. 2 shows that the vibration dimensionless frequencies are significantly influenced by the increases in the non-dimensional mass parameter and non-dimensional rotary mass moment of inertia. The dimensionless frequencies will decrease by any increase of non-dimensional mass parameter (β_M) and also increase by any increase of non-dimensional rotary mass moment of inertia (β_J).

Moreover, the first three transverse mode shapes of the cantilever beam carrying a point mass with moment of inertia are illustrated in Fig. 3 for different values of β_M and β_J. As seen in this Fig. 3, β_M and β_J effects on fundamental mode shapes are significantly observed .Note that β_M have great effects on higher mode shapes.



(a) $\beta_M = 0.01$ and $\beta_J = 0.01, 1$



(b) $\beta_M = 1$ and $\beta_J = 0.01, 1$

Figure 3 First three mode shapes for a cantilever Euler-Bernoulli beam considering the Rotary Inertial Moment of an Attached Mass at right end with (a) $\beta_M = 0.01$ and $\beta_J = 0.01, 1$ (b) $\beta_M = 1$ and $\beta_J = 0.01, 1$

4 CONCLUSION

This paper presents the free vibration analysis of a cantilever beam with a rigid body exciting flexural vibration using SEM. The spectral stiffness matrix of the problem was derived and the first three eigenvalues are determined. For special cases results compared with existing results in literature and very good agreement was achieved. The proposed approach yields high accuracy and rapid convergence. Also, the effect of non-dimensional mass parameter and non-dimensional mass moment of inertia on the dimensionless frequency parameter and mode shapes of the system was investigated. The results show that the values of non-dimensional mass parameter and non-dimensional rotary mass moment of inertia had significant effects on the on the dimensionless frequency parameter and mode shapes of the cantilever beam with attached mass at the free end.

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