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Vol. 26 = 551 (2022): 21-43
DOI: https://doi.org/10.21857/ygjwrcpgoy

THE PROBLEM OF THE EXTENSION OF $D(4)$-TRIPLE $\{1, b, c\}$

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Abstract. In this paper, we study the extensibility of the $D(4)$-triple $\{1, b, c\}$, where $1 < b < c$, by proving that such a set cannot be extended to an irregular $D(4)$-quadruple only for some values of c. For this study, we will use the classical methods based on the resolution of the binary recurrence sequences with new approaches in order to confirm a conjecture of uniqueness of such an extension.

1. Introduction

Diophantus raised the problem of finding four (positive rational) numbers a_1, a_2, a_3, a_4 such that $a_ia_j + 1$ is a square for each $1 \leq i < j \leq 4$ and gave a solution $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$. The first set of four positive integers $\{1, 3, 8, 120\}$ with this property above was found by Fermat. Replacing " $+1$" by " $+n$" leads to the following definition.

Definition 1.1. *Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-m-tuple (or a Diophantine m-tuple with the property $D(n)$, or a P_n-set of size m) if $a_ia_j + n$ is a square for each $1 \leq i < j \leq m$.*

One of the most interesting and most studied questions is how large those sets can be. In the classical case, first studied by Diophantus, i.e. when $n = 1$, Dujella [7] proved that $D(1)$-sextuple does not exist and that there are at most finitely many quintuples. Over the years many authors improved the upper bound for the number of $D(1)$-quintuples and finally He, the third author and Ziegler [14] gave the proof of the nonexistence of $D(1)$-quintuples. To see details of the history of the problem with all references, one can visit the webpage of Dujella [6].

Variants of the problem when $n = 4$ or $n = -1$ are also studied frequently. In the case $n = 4$, similar conjectures and observations can be made as in the

2020 *Mathematics Subject Classification.* 11D09, 11B37, 11J68, 11J86, 11Y65.
Key words and phrases. Diophantine m-tuples, system of Pellian equations, reduction method, linear form in logarithms.

$D(1)$ case. In the light of that observation, the second author and Trebješanin have proven in [5] that $D(4)$-quintuple also doesn't exist.

A $D(4)$-pair can be extended with a larger element c to form a $D(4)$-triple. The smallest such c is $c = a + b + 2r$, where $r = \sqrt{ab+4}$ and such triple is often called a regular triple, or in the $D(1)$ case it is also called an Euler triple. There are infinitely many extensions of a pair to a triple and they can be studied by finding solutions of a Pellian equation

$$at^2 - bs^2 = 4(a-b), \tag{1.1}$$

where s and t are positive integers defined by $ac + 4 = s^2$ and $bc + 4 = t^2$. Then, for a $D(4)$-triple $\{a, b, c\}$, $a < b < c$, we define

$$d = d_{\pm} = a + b + c + \frac{1}{2}\left(abc \pm \sqrt{(ab+4)(ac+4)(bc+4)}\right),$$

and it is easy to check that $\{a, b, c, d_+\}$ is a $D(4)$-quadruple, which we will call a regular $D(4)$-quadruple, and if $d_- \neq 0$ then $\{a, b, c, d_-\}$ is also a regular $D(4)$-quadruple with $d_- < c$. A famous and still open conjecture related to the regularity problem is as follows.

CONJECTURE 1.2. *If $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then $d = d_+$.*

Results which support this conjecture in some special cases can be found for example in [9], [11], [12]. But, in the framework of the proof of this conjecture, Trebješanin [4] recently proved the following result.

THEOREM 1.3. *(See [4, Corollary 1.7]). Any $D(4)$-triple can be extended to a $D(4)$-quadruple with $d > \max\{a, b, c\}$ in at most 8 ways. A regular $D(4)$-triple $\{a, b, c\}$ can be extended to a $D(4)$-quadruple with $d > \max\{a, b, c\}$ in at most 4 ways.*

It is interesting to observe that the problem of extending a $D(4)$-pair $\{1, b\}$ to a $D(4)$-triple $\{1, b, c\}$ can be reduced to the solving of the Pellian equation (1.1). Notice that in this particular case where $a = 1$, we have to solve the following equation

$$t^2 - bs^2 = 4(1-b), \tag{1.2}$$

which has infinitely many solutions divided into classes of solutions. By the arguments of Nagell [17, Theorem 108a], equation (1.2) has finitely many fundamental solutions (so also classes of solutions). Also, there are at most $2^{\omega(4(b-1))}$ classes of solutions with $\gcd(t, s) = 1$, where $\omega(4(b-1))$ denotes the number of distinct prime factors of $4(b-1)$. All elements of one class of solutions of equation (1.2) can be obtained from a fundamental solution by multiplication with a power of the minimal solution in positive rationals for the associated Pellian equation. Therefore, all positive solutions (t, s) to equation (1.2) which belong to the same class and induced by the fundamental solutions $(\pm 2, 2)$ can be expressed as

$$t+s\sqrt{b}=(\pm 2+2\sqrt{b})\left(\frac{r+\sqrt{b}}{2}\right)^k=(\pm 2+2\sqrt{b})(T_k+U_k\sqrt{b}),\quad k\geq 0, \tag{1.3}$$

where (T_k, U_k) is the k-th positive rationals solutions to the Pellian equation

$$T^2-bU^2=T^2-(r^2-4)U^2=1.$$

Note that, for some choices of b there are other fundamental solutions of the equation (1.2). It is easy to show by induction that

$$T_0=1,\quad T_1=\tfrac{r}{2},\quad T_{k+2}=rT_{k+1}-T_k,\quad k\geq 0 \tag{1.4}$$

$$U_0=0,\quad U_1=\tfrac{1}{2},\quad U_{k+2}=rU_{k+1}-U_k,\quad k\geq 0. \tag{1.5}$$

So, by (1.3) we observe that

$$(s,t)=(s_k^{(\pm)},t_k^{(\pm)})=(2T_k\pm 2U_k,\pm 2T_k+2bU_k) \tag{1.6}$$

and

$$c_k^{(\pm)}=c=s^2-4=4r^2U_k^2-12U_k^2\pm 8T_kU_k. \tag{1.7}$$

Thus, we will study the extensibility of the $D(4)$-triples

$$\{1,b,c_k^{(\pm)}\},\quad \text{for}\quad k=1,2,\ldots$$

To effectively study the extensibility of our previous $D(4)$-triple, we will focus our attention on the following theorem which is the main result obtained by Trebješanin [4]. In reality, this result will help us to have an idea of the element $c=c_k^{(\pm)}$ of the extension of $D(4)$-triple $\{1,b,c\}$ to an irregular $D(4)$-quadruple. So, for a fixed $D(4)$-triple $\{a,b,c\}$, denote by N the number of positive integers $d>d_+$ such that $\{a,b,c,d\}$ is a $D(4)$-quadruple. We have the next result

Theorem 1.4. *([4, Theorem 1.6]). Let $\{a,b,c\}$ be a $D(4)$-triple with $a<b<c$.*

i) If $c=a+b+2r$, then $N\leq 3$.
ii) If $a+b+2r\neq c<b^2$, then $N\leq 7$.
iii) If $b^2<c<39247b^4$, then $N\leq 6$.
iv) If $c\geq 39247b^4$, then $N=0$.

Using the item *iv)* of Theorem 1.4, equation (1.3) and the inequality $b>10^5$ of [4, Lemma 2.2], we get

$$s^2\geq(\sqrt{b}-1)^2\cdot b^{k-1},$$

and for $k \geq 5$ we obtain $c = s^2 - 4 > 39247b^4$. Consequently, we only need to consider $1 \leq k \leq 4$. Therefore, there are the following 8 cases to take into account:

$$\begin{aligned}
c_1^{(\pm)} &= r^2 - 3 \pm 2r, \\
c_2^{(\pm)} &= r^4 - 3r^2 \pm (2r^3 - 4r), \\
c_3^{(\pm)} &= r^6 - 5r^4 + 7r^2 - 3 \pm (2r^5 - 8r^3 + 6r), \\
c_4^{(\pm)} &= r^8 - 7r^6 + 16r^4 - 12r^2 \pm (2r^7 - 12r^5 + 20r^3 - 8r).
\end{aligned}$$

Since $bc_1^- + 4 = b^2 - 2br + r^2 = (b-r)^2 < b^2$, it follows that $c_1^- < b$. We set $r' = r-1$ in the case c_1^-. Then, the triple $\{1, c, b\}$ is $\{1, r'^2-4, r'^2+2r'-3\}$. This corresponds to $\{1, b', (c')_1^{\pm}\}$. Notice that in all other cases we have $1 < b < c_k^{\pm}$. Also, it is easy to see that the $D(4)$-quadruple $\{1, b, c_k^{(\pm)}, c_{k+1}^{(\pm)}\}$ is regular.

The preceding observations allow us to state our main result, which is to prove the following theorem.

Theorem 1.5. *If $\{1, b, c_k^{(\pm)}, d\}$ is a $D(4)$-quadruple with $d > c_k^{(\pm)}$, then $d = c_{k+1}^{(\pm)}$. More precisely, the $D(4)$-triple $\{1, b, c_k^{(\pm)}\}$ cannot be extended to an irregular $D(4)$-quadruple.*

Taking into account the observations mentioned above, Theorem 1.5 allows us to deduce the following statement.

Corollary 1.6. *If $4(b-1)$ is a prime power, then any $D(4)$-quadruple which contains the pair $\{1, b\}$ is regular.*

Remark 1.7. However, $4(b-1)$ can be prime power only if $b = 5$. But it should be noted that it is not only the case when we will have such sequences of c's that are extending our pair $\{1, b\}$. But we can also have some additional c's that extend our $D(4)$-pair $\{1, b\}$. For example, for $b = 96$, we have the sequences:

$$\begin{aligned}
c_\nu^- &= 77,\ 7740, ... \\
c_\nu^+ &= 117,\ 11660, ...
\end{aligned}$$

while, for example $c = 672$ also extends the pair $\{1, 96\}$. That extension comes from the fundamental solution $(t_0, s_0) = (22, 3)$ of the equation $t^2 - 96s^2 = -380$.

In order to prove Theorem 1.5, we will follow the methods described in [13] by He, Pintér, the third author, and Yang. In Section 2 of this paper, we will recall some useful lemmas and then we will transform the problem of extending a $D(4)$-triple $\{1, b, c\}$ to a $D(4)$-quadruple $\{1, b, c, d\}$ into solving a system of simultaneous Pellian equations, which furthermore transforms to finding intersections of binary recurrent sequences. In Section 3, we will use a linear form in three logarithms to obtain a lower bound for one of the index of the sequences. In Section 4, we will now give an upper bound to the bounded index of the previous section through the application of a result due to Laurent [15]. Section 5 of this paper will be devoted to the proof of our main theorem. For this, we will apply a Matveev [16] result and then we will end by applying the Baker-Davenport reduction method.

2. Some useful Lemmas and system of Pellian equations

In this section, we will recall or prove some useful lemmas that will be used to prove Theorem 1.5. So, let us consider a $D(4)$-triple $\{1, b, c\}$. When trying to extend $D(4)$-triple $\{1, b, c\}$ to a $D(4)$-quadruple $\{1, b, c, d\}$, we have to solve the system

$$d + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2,$$

where x, y, z are positive integers. Eliminating d, we obtain the following system of Pellian equations

$$z^2 - cx^2 = 4(1 - c), \tag{2.1}$$

$$bz^2 - cy^2 = 4(b - c), \tag{2.2}$$

$$y^2 - bx^2 = 4(1 - b). \tag{2.3}$$

Each of equations (2.1), (2.2) and (2.3) has finitely many fundamental solutions (z_0, x_0), (z_1, y_1) and (y_2, x_2), respectively. From these solutions, all solutions (z, x), (z, y) and (y, x), of (2.1), (2.2) and (2.3), respectively, are, by ([7, Lemma 1] or [17, Theorem 108a]), given with

$$z + x\sqrt{c} = (z_0 + x_0\sqrt{c})\left(\frac{s + \sqrt{c}}{2}\right)^m, \quad m \geq 0, \tag{2.4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})\left(\frac{t + \sqrt{bc}}{2}\right)^n, \quad n \geq 0, \tag{2.5}$$

$$y + x\sqrt{b} = (y_2 + x_2\sqrt{b})\left(\frac{r + \sqrt{b}}{2}\right)^l, \quad l \geq 0. \tag{2.6}$$

For any solution (x, y, z) of the system (2.1)-(2.2)-(2.3), we have $z = v_m = w_n$, for some non-negative integers m and n, where the sequences $(v_m)_{m\geq 0}$ and $(w_n)_{n\geq 0}$ are obtained using (2.4) and (2.5) and given by

$$v_0 = z_0, \quad v_1 = \frac{1}{2}(sz_0 + cx_0), \quad v_{m+2} = sv_{m+1} - v_m, \tag{2.7}$$

$$w_0 = z_1, \quad w_1 = \frac{1}{2}(tz_1 + cy_1), \quad w_{n+2} = tw_{n+1} - w_n. \tag{2.8}$$

Hence, we are solving the equation

$$v_m = w_n, \tag{2.9}$$

in $n, m \geq 0$. The initial terms of these equations were determined by the second author in [10, Lemma 9] and recently improved by Trebješanin in the following Lemmas:

Lemma 2.1. *([4, Theorem 1.3]). Suppose that $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$ and that w_m and v_n are defined as before.*

i) If equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$ and $|z_0| = 2$ or $|z_0| = \frac{1}{2}(cr - st)$.
ii) If equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$ and $z_0 z_1 < 0$.
iii) If equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$ and $z_0 z_1 < 0$.
iv) If equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$ and $z_0 z_1 > 0$.

Moreover, if $d > d_+$, case ii) cannot occur.

Lemma 2.2. *([4, Lemma 2.13]). Let $\{v_{z_0,m}\}$ denote a sequence $\{v_m\}$ with an initial value z_0 and $\{w_{z_1,n}\}$ denote a sequence $\{w_n\}$ with an initial value z_1. It holds that $v_{\frac{1}{2}(cr-st),m} = v_{-t,m+1}$, $v_{-\frac{1}{2}(cr-st),m+1} = v_{t,m}$ for each $m \geq 0$ and $w_{\frac{1}{2}(cr-st),n} = w_{-s,n+1}$, $w_{-\frac{1}{2}(cr-st),n+1} = w_{s,n}$ for each $n \geq 0$.*

By Lemma 2.1 and Lemma 2.2, we have to consider the following result.

Lemma 2.3. *Assume that $\{1, b, c', c\}$ is not a $D(4)$-quadruple for any c' with $0 < c' < c^{\pm}_{k-1}$ and $k \geq 2$. Then, neither $v_{2m+1} = w_{2n}$ nor $v_{2m} = w_{2n+1}$. Moreover,*

(i) If $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1 = \pm 2$ and $x_0 = y_1 = 2$.
(ii) If $v_{2m+1} = w_{2n+1}$ has a solution, then $z_0 = \pm t$, $z_1 = \pm s$, $x_0 = y_1 = r$ and $z_0 z_1 > 0$.

Proof. The proof of this lemma is similar to that made by Baćić and the second author in [1, Lemma 3] and also in [2, Lemma 2]. For the first part of the lemma, if $v_{2m+1} = w_{2n}$, define $d_0 := (z_1^2 - 4)/c$. Then the proof of part (ii) in [10, Lemma 9] implies that $d_0 > 0$ and that $\{1, b, c, d_0\}$ is $D(4)$-quadruple. On the other hand from the estimate for $|z_1|$ in [4, Lemma 2.5], it is easy to conclude

$$d_0 = \frac{z_1^2 - 4}{c} = \frac{1}{c}\left(z_1^2 - 4\right) < \frac{1}{c}\left(\frac{c\sqrt{c}}{\sqrt{b}} - 4\right) < \frac{\sqrt{c}}{\sqrt{b}}. \tag{2.10}$$

Moreover, from (1.7) for $k \geq 2$, we have $U_k \leq rU_{k-1}$ and $T_k < rT_{k-1}$ and therefore

$$\begin{aligned}\left(c_{k-1}^{(\pm)}\right)^2 &= 16\left(r^2U_{k-1}^4(r^2-6) + U_{k-1}^2(4T_{k-1}^2 + 9U_{k-1}^2) \pm T_{k-1}U_{k-1}^3(4r^2-12)\right)\\ &> \frac{16}{r^2}\left[U_k^4(r^2-6) + U_k^2(4T_{k-1}^2 + 9U_{k-1}^2) \pm T_{k-1}U_{k-1}^3(4r^4-12r^2)\right] > 0\\ &> \frac{16}{r^2}c_k^{(\pm)}.\end{aligned}$$

So we have

$$\left(c_{k-1}^{(\pm)}\right)^2 > \frac{16}{r^2}c_k^{(\pm)}$$

and

$$\sqrt{c_k^{(\pm)}} < \frac{\sqrt{b+4}}{4}c_{k-1}^{(\pm)}. \tag{2.11}$$

From (2.10) and (2.11) we conclude

$$d_0 < \sqrt{\frac{1}{16} + \frac{1}{4b}} \cdot c_{k-1}^{(\pm)} < c_{k-1}^{(\pm)}$$

which contradicts the assumption of the lemma. Therefore, $v_{2m+1} = w_{2n}$ has no solution. The case $v_{2m} = w_{2n+1}$ can be proven in the exactly same way. For the second part of the lemma, (ii) is just Lemma 2.1, part (iv). As for (i), if $v_{2m} = w_{2n}$, then we know by [10, Lemma 9], part (i) that $z_0 = z_1$. Suppose that $|z_1| > 2$ and define $d_0 := (z_1^2 - 4)/c$. Then we see that $d_0 > 0$ and that $\{1, b, c, d_0\}$ is $D(4)$-quadruple. On the other hand, as we have already seen we have $d_0 < c_{k-1}$, which contradicts the assumption of the lemma. This completes the proof of the lemma. □

In other cases, we get the same intersections of sequences. For any solution (x, y, z) of the system (2.1)-(2.2)-(2.3), we also have $y = A_n = B_l$, for some non-negative integers n and l, where the sequences $(A_n)_{n\geq 0}$ and $(B_l)_{l\geq 0}$ are obtained using (2.5) and (2.6) and given by

$$A_0 = y_1, \quad A_1 = \frac{1}{2}(ty_1 + bz_1), \quad A_{n+2} = tA_{n+1} - A_n, \tag{2.12}$$

$$B_0 = y_2, \quad B_1 = \frac{1}{2}(ry_2 + bx_2), \quad B_{l+2} = rB_{l+1} - B_l. \tag{2.13}$$

Thus, we have to solve the following equation

$$A_n = B_l, \tag{2.14}$$

in $n, l \geq 0$. Using Lemma 2.3 and [4, Lemma 2.2], we have the following result.

Lemma 2.4. *Assume that $\{1, b, c', c\}$ is not a $D(4)$-quadruple for any c' with $0 < c' < c_{k-1}^{\pm}$ and $b \geq 10^5$. Then, $A_{2n} = B_{2l+1}$ has no solution. Moreover, if $A_{2n} = B_{2l}$ then $y_2 = 2$. In other cases, we have $y_2 = \pm 2$.*

PROOF. From (2.12) and (2.13), by induction on n, l, we easily get

$$A_{2n} \equiv y_1 \pmod b, \quad A_{2n+1} \equiv \frac{ty_1}{2} \pmod b,$$

$$B_{2l} \equiv y_2 \pmod b, \quad 2B_{2l+1} \equiv ry_2 \pmod b.$$

Since we know with certainty that t is even, in the continuation of our proof according to the case, we can multiply the congruences by 2 in order to make our transformations light. From [5, Lemma 8], we have

$$1 \leq |y_2| \leq \sqrt{(r-2)(b-1)}$$

and in the proof of [5, Lemma 10], we have the following bound on y_2, i.e. $|y_2| < b^{3/4}$.

If $A_{2n} = B_{2l}$, then we have $y_1 \equiv y_2 \pmod b$. As $y_1 = 2$ and $|y_2| < b^{3/4} < \frac{b}{2}$ for $b > 10^5$, then $y_2 = 2$.

If $A_{2n} = B_{2l+1}$, then $2 \equiv \frac{ry_2}{2} \pmod b$ so $4r \equiv r^2 y_2 \pmod b$. As $b+4 = r^2$, then we get $4r \equiv 4y_2 \pmod b$. Since, for $b > 10^5$, $|4y_2| < 4b^{3/4} < \frac{b}{2}$ and $|4r| < \frac{b}{2}$, thus $4r = 4y_2$ and then $y_2 = r$. In this case, from (2.3), we obtain $x_2^2 = \frac{1}{b}\left(y_2^2 - 4 + 4b\right) = 5$, which is not possible.

If $A_{2n+1} = B_{2l}$, then from Lemma 2.3 we have $\frac{rt}{2} \equiv y_2 \pmod b$. From (1.6), we conclude that $t = \pm 2T_k + 2bU_k \equiv \pm 2T_k \pmod b$. It is easy to check that $T_k \equiv 1 \pmod b$ or $T_k \equiv \frac{r}{2} \pmod b$, where T_k is given by $T_0 = 1, T_1 = \frac{r}{2}, T_{k+2} = rT_{k+1} - T_k$. We conclude that $t \equiv \pm 2 \pmod b$ or $t \equiv \pm r \pmod b$ and it follows that $y_2 \equiv \pm 2 \pmod b$ or $y_2 \equiv \pm r \pmod b$. Notice that the case $y_2 \equiv \pm r \pmod b$ leads to a contradiction thus we consider only $y_2 \equiv \pm 2 \pmod b$. It follows that $y_2 = \pm 2$.

If $A_{2n+1} = B_{2l+1}$, then by Lemma 2.3, we have $\frac{tr}{2} \equiv \frac{ry_2}{2} \pmod b$. This implies that $tr^2 \equiv r^2 y_2 \pmod b$. Since $r^2 = b + 4$ and $t \equiv \pm 2 \pmod b$ or $t \equiv \pm r \pmod b$. Then $y_2 \equiv \pm 2 \pmod{\frac{b}{\gcd(b,4)}}$ or $y_2 \equiv \pm r \pmod{\frac{b}{\gcd(b,4)}}$. As $b > 10^5$, then it is easy to show that the case $y_2 \equiv \pm r \pmod{\frac{b}{\gcd(b,4)}}$ gives a contradiction and we obtain $y_2 = \pm 2$. ◻

Therefore, the fundamental solutions of equation (2.3) are $(y_2, x_2) = (\pm 2, 2)$.

Finally, we will determine the integer solutions (x, y, z) of the following system

$$\begin{cases} y^2 - bx^2 = 4(1-b), \\ z^2 - cx^2 = 4(1-c). \end{cases}$$

From the above result, we have to solve the equation

$$x = P_l = Q_m, \tag{2.15}$$

for non-negative integers l and m, where the sequences $(P_l)_{l\geq 0}$ and $(Q_m)_{m\geq 0}$ are obtained using (2.4) and (2.6) and given by

$$P_0 = x_2, \quad P_1 = \frac{1}{2}(rx_2 + y_2), \quad P_{l+2} = rP_{l+1} - P_l, \tag{2.16}$$

$$Q_0 = x_0, \quad Q_1 = \frac{1}{2}(sx_0 + z_0), \quad Q_{m+2} = sQ_{m+1} - Q_m. \tag{2.17}$$

Using again (2.4) and (2.6), we get

$$P_l = \frac{1}{2\sqrt{b}}\left[(y_2 + x_2\sqrt{b})\alpha^l - (y_2 - x_2\sqrt{b})\alpha^{-l}\right], \tag{2.18}$$

$$Q_m = \frac{1}{2\sqrt{c}}\left[(z_0 + x_0\sqrt{c})\beta^m - (z_0 - x_0\sqrt{c})\beta^{-m}\right], \tag{2.19}$$

where $\alpha = \frac{r+\sqrt{b}}{2}$ and $\beta = \frac{s+\sqrt{c}}{2}$ are solutions of Pell equations $T^2 - bU^2 = 1$ and $W^2 - cV^2 = 1$, respectively. In view of the above results, we also conclude that there are two types of fundamental solutions as follows:
Type A: If $l \equiv m \equiv 0 \pmod 2$, then $z_0 = \pm 2$, $x_0 = y_2 = 2$ and $x_2 = 2$.
Type B: If $m \equiv 1 \pmod 2$, then $z_0 = \pm t$, $x_0 = r$, $y_2 = \pm 2$ and $x_2 = 2$.

The next result will help us to determine a relation of indices l and m if the equation $P_l = Q_m$ has a solution.

LEMMA 2.5. *If $P_l = Q_m$ has a solution (l, m) with $m \geq 1$, then $m < l$.*

PROOF. The proof of this lemma is done in two parts taking into account Type A and Type B. In the case of Type A, the proof is similar to that of the second author, He and the third author in [12, Lemma 9]. Finally, in the case of type B, we use the same strategy as [12, Lemma 9] by focusing on the cases $z_0 = t$ and $z_0 = -t$. ∎

3. Linear forms in three logarithms

Using technique from [3], first we transform equation (2.15) into an inequality for a linear forms in three logarithms of algebraic numbers. So, we will consider the following linear form in logarithms

$$\Lambda = l \log\alpha - m\log\beta + \log\gamma, \tag{3.1}$$

where $\gamma = \frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})}$.

LEMMA 3.1. *If $P_l = Q_m$ has a solution (l, m) with $m \geq 1$, then*

$$0 < \Lambda < 1.0064\beta^{-2m}.$$

PROOF. Put

$$E = \frac{y_2 + x_2\sqrt{b}}{\sqrt{b}}\alpha^l \quad \text{and} \quad F = \frac{z_0 + x_0\sqrt{c}}{\sqrt{c}}\beta^m.$$

It is clear that E, $F > 1$ if $m \geq 1$. Then equation $P_l = Q_m$ becomes

$$E + 4\left(\frac{b-1}{b}\right)E^{-1} = F + 4\left(\frac{c-1}{c}\right)F^{-1}. \tag{3.2}$$

Since $c > b > 10^5$, we have $\frac{c-1}{c} > \frac{b-1}{b}$. It follows that

$$E + 4\left(\frac{b-1}{b}\right)E^{-1} > F + 4\left(\frac{b-1}{b}\right)F^{-1} \tag{3.3}$$

and hence

$$(E-F)\left(EF - 4\left(\frac{b-1}{b}\right)\right) > 0.$$

So we get $E > F$. Moreover, by (3.3) we have

$$0 < E - F < 4\left(\frac{c-1}{c}\right)E^{-1} < 4E^{-1} < 4F^{-1}.$$

Therefore, we have $\Lambda > 0$ and

$$\Lambda = \log\frac{E}{F} = \log\left(1 + \frac{E-F}{F}\right) < \frac{E-F}{F} < 4F^{-2}.$$

Type A:

$$\Lambda < 4\frac{c}{(\pm 2 + 2\sqrt{c})^2}\beta^{-2m} = \frac{c}{(\pm 1 + \sqrt{c})^2}\beta^{-2m} < 1.0064\beta^{-2m},$$

for $c > b > 10^5$.

Type B:

• The case $z_0 = t$. We have

$$\Lambda < 4\frac{c}{(t + r\sqrt{c})^2}\beta^{-2m} < \frac{4}{r^2}\frac{c}{(1+\sqrt{c})^2}\beta^{-2m} < \beta^{-2m},$$

for $c > b > 10^5$ and $r > 100$.

• The case $z_0 = -t$. From (3.2), we get

$$\begin{aligned} F = E + 4\left(\frac{b-1}{b}\right)E^{-1} - 4\left(\frac{c-1}{c}\right)F^{-1} &> E - 4\left(\frac{c-1}{c}\right)F^{-1} \\ &> E - 4\left(\frac{c-1}{c}\right) > 0. \end{aligned}$$

In above, we use the fact that $F > 1$. Thus,

$$F^{-1} < \left(E - 4\left(\frac{c-1}{c}\right)\right)^{-1}.$$

Furthermore,

$$E - F = 4\left(\frac{c-1}{c}\right)F^{-1} - 4\left(\frac{b-1}{b}\right)E^{-1}$$

$$< 4\left(\frac{c-1}{c}\right)\left(E - 4\left(\frac{c-1}{c}\right)\right)^{-1} - 4\left(\frac{b-1}{b}\right)E^{-1}. \tag{3.4}$$

Moreover in type B and for $m \geq 3$, we have

$$F \geq \frac{r\sqrt{c}-t}{\sqrt{c}}\beta^3 = \frac{r^2c - t^2}{\sqrt{c}(r\sqrt{c}+t)} \cdot \left(\frac{s+\sqrt{c}}{2}\right)^3$$

$$> \frac{4(c-1)}{\sqrt{c}\cdot 2r\sqrt{c}} \cdot (\sqrt{c})^3 > 4(c-1), \tag{3.5}$$

which implies $E > F > 4(c-1)$ and then

$$\frac{c-1}{c}\left(E - 4\left(\frac{c-1}{c}\right)\right)^{-1} < E^{-1}. \tag{3.6}$$

In fact, the case $m = 1$ in the equation $P_l = Q_m$ is the subject of further study in the paper. Note also that inequality (3.5) holds if $\sqrt{c} > 2r$ which is true whenever $c > r^2 + 2r - 3$. So, combining (3.4) and (3.6), we obtain

$$E - F < 4E^{-1} - 4\left(\frac{b-1}{b}\right)E^{-1} = \frac{4}{b}E^{-1} < \frac{4}{b}F^{-1}.$$

Therefore,

$$\Lambda = \log\frac{E}{F} = \log\left(1 + \frac{E-F}{F}\right) < \frac{E-F}{F} < \frac{4}{b}F^{-2}$$

and

$$\frac{4}{b}F^{-2} = \frac{4}{b}\cdot\frac{c}{(r\sqrt{c}-t)^2}\beta^{-2m} < \frac{c^2r^2}{b(c-1)^2}\beta^{-2m}.$$

Using $c > b > 10^5$, we get

$$r^2 = b + 4 < 1.00004b \quad \text{and} \quad \frac{c^2}{(c-1)^2} < 1.00003.$$

Hence, $\Lambda = \log\dfrac{E}{F} < 1.0001\beta^{-2m}$. Considering all cases in types A, B, we have $\Lambda < 1.0064\beta^{-2m}$. This completes the proof of lemma 3.1. □

Put

$$\lambda = \begin{cases} 0 & \text{if the solution } (l, m) \text{ is of Type } A, \\ 2 & \text{if the solution } (l, m) \text{ is of Type } B, \text{ with } z_0 = t, \\ -2, & \text{if the solution } (l, m) \text{ is of Type } B, \text{ with } z_0 = -t. \end{cases}$$

LEMMA 3.2. *If the equation $P_l = Q_m$ has a solution (l, m) with $m \geq 1$, then for $r > 316$, we have*

$$\left|(l - \frac{1}{2}\lambda)\log\alpha - m\log\beta\right| < \frac{2}{\sqrt{b}}. \tag{3.7}$$

PROOF. According to the definition of Λ in (3.1), we have

$$\left|\left(l - \frac{1}{2}\lambda\right)\log\alpha - m\log\beta\right| = \left|\Lambda - \log\gamma - \frac{1}{2}\lambda\log\alpha\right| \leq |\Lambda| + \left|\log(\gamma\alpha^{\frac{1}{2}\lambda})\right|. \tag{3.8}$$

Notice that one can easily get

$$0 < \Lambda < \frac{1.0064}{c}. \tag{3.9}$$

In order to estimate inequality (3.8), we will consider three cases according to the values of λ. Let us start with the following first case.

Case I: If $\lambda = 0$, then the solution (l, m) is of Type A. Since, $c > r^2 + 2r - 3$ then we easily get

$$\gamma = \frac{\sqrt{c}(2 + 2\sqrt{b})}{\sqrt{b}(\pm 2 + 2\sqrt{c})} = \frac{1 + \sqrt{b}}{\sqrt{b}} \cdot \frac{\sqrt{c}}{\sqrt{c} \pm 1} > 1.$$

Thus, according to the basic inequality, for $x \in (0, \infty)$, $\log(1 + x) < x$, we have

$$0 < \log\gamma \leq \log\left(1 + \frac{1}{\sqrt{b}}\right) + \log\left(1 + \frac{1}{\sqrt{c} - 1}\right) < \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c} - 1} < \frac{2}{\sqrt{b}},$$

which implies

$$\left|\left(l - \frac{1}{2}\lambda\right)\log\alpha - m\log\beta\right| = |\Lambda - \log\gamma| < \max\left\{\frac{1.0064}{c}, \frac{2}{\sqrt{b}}\right\} = \frac{2}{\sqrt{b}}.$$

This completes the proof in case $\lambda = 0$.

Case II: If $\lambda = 2$, then the solution (l, m) is of Type B, with $x_0 = r$, $x_2 = 2$, $z_0 = t$ and $y_2 = \pm 2$. Here, we get

$$\gamma = \frac{\sqrt{c}(\pm 2 + 2\sqrt{b})}{\sqrt{b}(t + r\sqrt{c})}.$$

Since

$$\gamma\alpha^{\frac{1}{2}\lambda} - 1 = \gamma\alpha - 1 = \frac{\pm\sqrt{c}(r + \sqrt{b}) + (b\sqrt{c} - t\sqrt{b})}{\sqrt{b}(t + r\sqrt{c})} = \frac{\pm\sqrt{c}(r + \sqrt{b}) - 4\frac{\sqrt{b}}{t + \sqrt{bc}}}{\sqrt{b}(t + r\sqrt{c})}$$

and $\sqrt{c}(r + \sqrt{b}) - (t + r\sqrt{c}) = \sqrt{bc} - t < 0$, we see that

$$\left|\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1\right| < \frac{1 + 4\frac{\sqrt{b}}{t + \sqrt{bc}} \cdot \frac{1}{t + r\sqrt{c}}}{\sqrt{b}}.$$

From $c > r^2 + 2r - 3$ and $b > 10^5$, we get

$$\left|\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1\right| < \frac{1.00000003149}{\sqrt{b}} < \frac{1}{51}.$$

Notice that $|\log(1+x)| < 1.01|x|$, for $|x| < \frac{1}{51}$. Then we have

$$\left|\log(\gamma \cdot \alpha^{\frac{1}{2}\lambda})\right| = \left|\log(1 + (\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1))\right| < \frac{1.01}{\sqrt{b}}.$$

Using the above inequality and (3.8), we find

$$\left|(l - \frac{1}{2}\lambda)\log\alpha - m\log\beta\right| < \frac{1.0064}{c} + \frac{1.01}{\sqrt{b}} < \frac{1.02}{\sqrt{b}} < \frac{2}{\sqrt{b}}$$

which prove the lemma in the case $\lambda = 2$.

Case III: If $\lambda = -2$, then the solution (l, m) is of Type B, with $x_0 = r$, $x_2 = 2$, $z_0 = -t$ and $y_2 = \pm 2$. So, we have

$$\gamma = \frac{\sqrt{c}(\pm 2 + 2\sqrt{b})}{\sqrt{b}(-t + r\sqrt{c})}.$$

Thus, we get

$$\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1 = \gamma \cdot \alpha^{-1} - 1 = \frac{4\sqrt{c}(\pm 1 + \sqrt{b})}{\sqrt{b}(-t + r\sqrt{c})(r + \sqrt{b})} - 1.$$

This implies that

$$\gamma \cdot \alpha^{-1} - 1 = \frac{\pm\sqrt{c}(t + r\sqrt{c}) + \sqrt{b}(r + \sqrt{b}) + \sqrt{bc}(t - \sqrt{bc})}{\sqrt{b}(c-1)(r + \sqrt{b})}.$$

Since

$$t - \sqrt{bc} = \frac{4}{t + \sqrt{bc}} \quad \text{and} \quad \frac{\sqrt{bc}}{t + \sqrt{bc}} < \frac{1}{2},$$

then

$$\left|\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1\right| < \frac{2}{\sqrt{b}(c-1)(r + \sqrt{b})} + \frac{1}{c-1} + \frac{\sqrt{c}(t + r\sqrt{c})}{\sqrt{b}(c-1)(r + \sqrt{b})}.$$

Moreover,

$$\frac{\sqrt{c}(t + r\sqrt{c})}{(c-1)(r + \sqrt{b})} - 1 = \frac{r + \sqrt{b} + 4\frac{\sqrt{c}}{t + \sqrt{bc}}}{(c-1)(r + \sqrt{b})}, \quad \text{and} \quad \frac{\sqrt{c}}{t + \sqrt{bc}} < \frac{1}{2\sqrt{b}},$$

then we can see that

$$\frac{\sqrt{c}(t + r\sqrt{c})}{(c-1)(r + \sqrt{b})} - 1 < \frac{1.00001}{c-1} < 0.0000099.$$

So, we have

$$\left|\gamma \cdot \alpha^{\frac{1}{2}\lambda} - 1\right| < \frac{2}{\sqrt{b}(c-1)(r + \sqrt{b})} + \frac{1}{c-1} + \frac{1.0000099}{\sqrt{b}} < \frac{1.00317}{\sqrt{b}} < \frac{1}{51}.$$

Hence, we deduce that

$$\left|(l-\frac{1}{2}\lambda)\log\alpha - m\log\beta\right| < \frac{1.0064}{c} + \frac{1.0132017}{\sqrt{b}} < \frac{1.0232017}{\sqrt{b}} < \frac{2}{\sqrt{b}}.$$

This completes the proof of the lemma. □

Put

$$\Delta = l - \frac{1}{2}\lambda - km. \tag{3.10}$$

Lemma 3.3. *If $P_l = Q_m$ has a solution (l,m) with $m>1$, then $\Delta \neq 0$.*

Proof. Assume that $\Delta = l - \frac{1}{2}\lambda - km = 0$. For (1.4) and (1.5), by induction, we get $T_{mk+2k} = 2T_kT_{mk+k} - T_{mk}$ and $U_{mk+2k} = 2T_kU_{mk+k} - U_{mk}$. Hence, from (2.16), we get $P_{mk+2k} = 2T_kP_{mk+k} - P_{mk}$. Also, it is easy to show that $P_k = y_2U_k + x_2T_k$. Therefore, we have

$$P_0 = x_2, \quad P_k = y_2U_k + x_2T_k, \quad P_{mk+2k} = 2T_kP_{mk+k} - P_{mk}. \tag{3.11}$$

Our proof will be in three parts according to the fundamental solutions.

Part I: $\lambda = 0$. This is Type A with $x_0 = x_2 = 2$, $z_0 = \pm 2$, $y_2 = 2$. We have $l = km$. By (3.11) and (2.17), we get

$$P_0 = 2, \quad P_k = 2U_k + 2T_k, \quad P_{mk+2k} = 2T_kP_{mk+k} - P_{mk} \tag{3.12}$$

and

$$Q_0 = 2, \quad Q_1 = s \pm 1, \quad Q_{m+2} = sQ_{m+1} - Q_m. \tag{3.13}$$

Let us remember that $s = s_k^{(\pm)} = 2T_k \pm 2U_k$.

• Case of $s = s_k^{(-)}$. We have $P_k = 2U_k + 2T_k > 2T_k - 2U_k \pm 1 = Q_1$ and $P_{2k} = 2T_k(2U_k + 2T_k) - 2 > (2T_k - 2U_k)(2T_k - 2U_k \pm 1) - 2 = Q_2$. Before doing the induction, let us notice that $2T_k - 1 > s$ and that $(P_{km})_m$ is an increasing sequence. So, assume now that $P_{km} > Q_m$ and $P_{km+k} > Q_{m+1}$. Then we get

$$\begin{aligned} P_{mk+2k} &= 2T_kP_{mk+k} - P_{mk} \\ &= (2T_k - 1)P_{mk+k} + P_{mk+k} - P_{mk} > sQ_{m+1} - Q_m = Q_{m+2}. \end{aligned}$$

Thus, we have in this case $P_l = P_{km} > Q_m$, for $m \geq 1$.

• Case of $s = s_k^{(+)}$ and $Q_1 = s+1$. We have $Q_1 = 2U_k + 2T_k + 1 > 2T_k + 2U_k = P_k$ and $P_{2k} = 2T_k(2T_k+2U_k) - 2 < (2T_k+2U_k)(2T_k+2U_k+1) - 2 = Q_2$. We assume that $P_{mk} < Q_m$ and $P_{mk+k} < Q_{m+1}$, which imply

$$\begin{aligned} P_{mk+2k} &= 2T_kP_{mk+k} - P_{mk} \\ &< 2T_kQ_{m+1} + (2U_kQ_{m+1} - Q_m) = (2T_k + 2U_k)Q_{m+1} - Q_m = Q_{m+2}. \end{aligned}$$

It should be noted that the above sequences $(Q_m)_{m\geq 2}$, $(T_k)_{k\geq 0}$ and $(U_k)_{k\geq 0}$ are in positive terms. Hence, we conclude that $P_l = P_{km} < Q_m$, for $m \geq 1$.

• Case of $s = s_k^{(+)}$ and $Q_1 = s-1$. We get $P_k > Q_1$. Notice that $P_2 = 2T_k(2T_k+2U_k)-2 < (2T_k+2U_k)(2T_k+2U_k-2)-2 = Q_2$ and

$$Q_3 - P_{2k} = 16T_k^2U_k + 24T_kU_k^2 + 8U_k^3 + s^2 - 4U_k > 0.$$

Therefore, we assume that $P_{mk} < Q_m$ and $P_{mk+k} < Q_{m+1}$, then

$$\begin{aligned} P_{mk+2k} &= 2T_kP_{mk+k} - P_{mk} \\ &< 2T_kQ_{m+1} + (2U_kQ_{m+1} - Q_m) = (2T_k+2U_k)Q_{m+1} - Q_m = Q_{m+2}. \end{aligned}$$

We have $P_l = P_{mk} < Q_m$, for $m \geq 2$. Thus, we obtain $P_l = P_{km} \neq Q_m$ in Type A. This contradicts the fact that $l = km$.

Part II: $\lambda = 2$. We are in Type B with $x_0 = r$, $x_2 = 2$, $z_0 = t$, $y_2 = \pm 2$. If $\Delta = 0$, then $l = km+1$. By (3.11) and (2.17), we get

$$P_0 = 2, \quad P_k = 2T_k \pm 2U_k, \quad P_{mk+2k} = 2T_kP_{mk+k} - P_{mk} \tag{3.14}$$

and

$$Q_0 = r, \quad Q_1 = \frac{1}{2}(rs+t), \quad Q_{m+2} = sQ_{m+1} - Q_m. \tag{3.15}$$

If $m = 0$, then $l = 1$, $P_1 = \frac{rx_2+y_2}{2} = r \pm 1 \neq r = Q_0$.

If $m = 1$, then $l = k+1$. It is easy to show that $P_l = P_{k+1} = r(T_k \pm U_k) + bU_k \pm T_k$, $Q_1 = \frac{rs_k^{(\pm)} + t_k^{(\pm)}}{2} = r(T_k \pm U_k) + bU_k \pm T_k$. We conclude that when $s = s_k^{(\pm)}$, then we get

$$P_{k+1} = Q_1. \tag{3.16}$$

So, by induction $2T_k \neq s$ provides $P_{km+1} \neq Q_m$, for $m \geq 2$. Notice that

if $s = s_k^{(+)}$ and $y_2 = -2$, then $P_{2k+1} < Q_2$ and $2T_k < s$ imply $P_{km+1} < Q_m$;

if $s = s_k^{(-)}$ and $y_2 = 2$, then $P_{2k+1} < Q_2$ and $2T_k > s$ imply $P_{km+1} > Q_m$. Therefore, $P_{km+1} \neq Q_m$. This contradicts the fact that $l = km+1$.

Part III: $\lambda = -2$. We are in Type B with $x_0 = r$, $x_2 = 2$, $z_0 = -t$, $y_2 = \pm 2$. The proof is similar to that of Part II. ∎

The following result gives us the lower bound for m for a solution (l,m) of equation (2.15).

Lemma 3.4. *If $P_l = Q_m$ has a solution (l,m) with $m \geq 1$, then for $r > 316$, we have*

$$m > 0.983|\Delta|\sqrt{b} \cdot \log\alpha.$$

Proof. From Lemma 3.2, we have $\left|(l - \frac{1}{2}\lambda)\log\alpha - m\log\beta\right| < \frac{2}{\sqrt{b}}$ and then

$$\left|\frac{l-\frac{1}{2}\lambda}{m} - \frac{\log\beta}{\log\alpha}\right| < \frac{2}{m\sqrt{b}\cdot\log\alpha}.$$

Using that and (3.10), we further obtain

$$\frac{|\Delta|}{m}=\left|\frac{l-\frac{1}{2}\lambda}{m}-\frac{\log\beta}{\log\alpha}+\frac{\log\beta}{\log\alpha}-k\right|<\left|\frac{\log\beta}{\log\alpha}-k\right|+\frac{2}{m\sqrt{b}\cdot\log\alpha}. \tag{3.17}$$

As $\alpha^k=(\frac{r+\sqrt{b}}{2})^k=T_k+U_k\sqrt{b}$, $\beta=\frac{s+\sqrt{c}}{2}$, with $s=s_k^{(\pm)}=2T_k\pm 2U_k$ and

$$\left|\frac{\log\beta}{\log\alpha}-k\right|=\left|\frac{\log(\frac{\beta}{\alpha^k})}{\log\alpha}\right|=\left|\frac{\log(1+\frac{\beta-\alpha^k}{\alpha^k})}{\log\alpha}\right|, \tag{3.18}$$

then we have

$$\begin{aligned}\left|\frac{\beta-\alpha^k}{\alpha^k}\right| &= \left|\frac{\frac{s+\sqrt{c}}{2}-(\frac{r+\sqrt{b}}{2})^k}{(\frac{r+\sqrt{b}}{2})^k}\right|=\left|\frac{s-\frac{2}{s+\sqrt{c}}-(2T_k-\frac{1}{T_k+U_k\sqrt{b}})}{T_k+U_k\sqrt{b}}\right|\\ &= \left|\frac{\pm 2U_k-\frac{2}{s+\sqrt{c}}-\frac{1}{T_k+U_k\sqrt{b}}}{T_k+U_k\sqrt{b}}\right|<\frac{2U_k+0.0063169}{2U_k\sqrt{b}}<\frac{1.0063169}{\sqrt{b}}.\end{aligned}$$

Hence we get

$$\left|\log\left(1+\frac{\beta-\alpha^k}{\alpha^k}\right)\right|<1.01\left|\frac{\beta-\alpha^k}{\alpha^k}\right|<\frac{1.01638}{\sqrt{b}}. \tag{3.19}$$

From (3.17), (3.18), and (3.19), we obtain

$$\frac{|\Delta|}{m}\le\frac{2}{m\sqrt{b}\log\alpha}+\frac{1.01638}{\sqrt{b}\log\alpha}=\frac{1.01638+\frac{2}{m}}{\sqrt{b}\log\alpha}$$

and then

$$1.01638m+2\ge|\Delta|\sqrt{b}\log\alpha.$$

Therefore, we get

$$m\ge 0.983|\Delta|\sqrt{b}\cdot\log\alpha.$$

This is the end of the proof. ∎

4. Linear forms in two logarithms

As in [13, Lemma 8], we apply a result due to Laurent (see [15, Corollary 2]) on linear forms in two logarithms. For any non-zero algebraic α of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a_0\prod_{j=1}^{d}(X-\alpha^{(j)})$, we denote by

$$h(\alpha)=\frac{1}{d}\left(\log|a_0|+\sum_{j=1}^{d}\log\max\left(1,\left|\alpha^{(j)}\right|\right)\right)$$

its absolute logarithmic height.

As started in Section 1, we only need to consider the extensibility of the $D(4)$-triples $\{1, b, c_1^{(+)}\}$ and $\{1, b, c_k^{(\pm)}\}$, for $2 \leq k \leq 4$. Assume that $r > 316$ and $P_l = Q_m$ has a solution (l, m) with $l, m \geq 1$. We have

$$\Lambda = l \log \alpha - m \log \beta + \log \gamma.$$

In order to apply the result due to Laurent, we put $\Delta = l - \frac{1}{2}\lambda - km$ and rewrite Λ into the form

$$\Lambda = \log\left(\alpha^{\Delta+\frac{1}{2}\lambda}\gamma\right) - m \log\left(\frac{\beta}{\alpha^k}\right) = m \log\left(\frac{\alpha^k}{\beta}\right) - \log\left(\alpha^{-\Delta-\frac{1}{2}\lambda}\gamma^{-1}\right). \tag{4.1}$$

Lemma 4.1. *For a $D(4)$-triple $\{1, b, c_k^{(\pm)}\}$, $(1 \leq k \leq 4)$, if $P_l = Q_m$ has a solution (l, m), with $m \geq 1$ and $r > 316$, then we have*

$$m < 66972\left(\left|\Delta + \frac{1}{2}\lambda\right| + 2k + 3\right) \cdot \log \alpha + 186091. \tag{4.2}$$

Proof. Let $\alpha_1 = \frac{\alpha^k}{\beta}$ and $\alpha_2 = \alpha^{\Delta+\frac{1}{2}\lambda}\gamma$. Then, α_1 is a zero of the polynomial

$$X^4 - 2sT_kX^3 + (4T_k^2 + c + 2)X^2 - 2sT_kX + 1,$$

whose all roots are

$$\frac{2(T_k + U_k\sqrt{b})}{s + \sqrt{c}}, \quad \frac{2(T_k - U_k\sqrt{b})}{s - \sqrt{c}}, \quad \frac{2(T_k - U_k\sqrt{b})}{s + \sqrt{c}}, \quad \text{and} \quad \frac{2(T_k + U_k\sqrt{b})}{s - \sqrt{c}}.$$

Since α_1 is an algebraic unit and γ is not an algebraic unit then we conclude that α_1 and α_2 are multiplicatively independent. Here we have a linear form (4.1) in two algebraic numbers α_1 and α_2 over $\mathbb{Q}$. We note that at most two conjugates of α_1 are greater than 1, depending on whether $\alpha^k > \beta$ or $\alpha^k < \beta$. So, we get

$$h(\alpha_1) \leq \frac{k}{2}\log \alpha \quad \text{or} \quad h(\alpha_1) \leq \frac{1}{2}\log \beta.$$

Also, it is easy to see that $h(\alpha^{\Delta+\frac{1}{2}\lambda}) = \frac{1}{2}\left|\Delta + \frac{1}{2}\lambda\right| \cdot \log \alpha$ and

$$\begin{aligned} h(\gamma) &= h\left(\frac{\sqrt{c}(y_2 + x_2\sqrt{b})}{\sqrt{b}(z_0 + x_0\sqrt{c})}\right) \leq h\left(\frac{y_2 + x_2\sqrt{b}}{\sqrt{b}}\right) + h\left(\frac{z_0 + x_0\sqrt{b}}{\sqrt{c}}\right) \\ &\leq \frac{1}{2}\log(4b-4) + \frac{1}{2}\log(rc + t\sqrt{c}) < \frac{1}{2}\log(8bcr) \\ &< \frac{3}{2}\log \alpha + \log \beta + \log 4. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} h(\alpha_2) = h\left(\alpha^{\Delta+\frac{1}{2}\lambda}\gamma\right) &\leq h\left(\alpha^{\Delta+\frac{1}{2}\lambda}\right) + h(\gamma) \\ &\leq \frac{1}{2}\left(\left|\Delta + \frac{1}{2}\lambda\right| + 3\right)\log \alpha + \log \beta + \log 4. \end{aligned}$$

By (3.19), we get $\left|\log\alpha^k - \log\beta\right| < 0.003$. So we take

$$\begin{aligned} h_1 &= \frac{k}{2}\log\alpha + 0.003 > h(\alpha_1), \\ h_2 &= \frac{1}{2}\left(\left|\Delta + \frac{1}{2}\lambda\right| + 3 + 2k\right)\log\alpha + 1.389 > h(\alpha_2). \end{aligned}$$

Further, since by (4.1) $b_1 = m$, $b_2 = 1$ and $D = 4$, we have $\frac{|b_2|}{Dh_1} = \frac{1}{2k\log\alpha + 0.012} < 0.087$. Let us define

$$b' = \frac{m}{2(\left|\Delta + \frac{1}{2}\lambda\right| + 3 + 2k)\log\alpha + 5.5572} + 0.087. \tag{4.3}$$

If $\log b' + 0.38 \le \frac{30}{D}$, then $b' \le 1236$. Otherwise, we have

$$\log|\Lambda| \ge -17.9 \cdot 4^4(\log b' + 0.38)^2 h_1 h_2. \tag{4.4}$$

Moreover, from Lemma 3.1, it is easy to get $\log|\Lambda| < -1.9988m\log\beta$. Thus, we have

$$1.9988m\log\beta < 17.9 \cdot 4^4(\log b' + 0.38)^2 h_1 h_2.$$

Since $\log\beta > \log\alpha^k - 0.003 > 2h_1 - 0.009$, then

$$1.9988m < 17.9 \cdot 4^4 \cdot 1.002 \cdot 0.5(\log b' + 0.38)^2 h_2.$$

It follows that

$$b' - 0.087 = \frac{m}{4h_2} < 287.15(\log b' + 0.38)^2.$$

Using Maple, we obtain $b' < 33486.41$. So, from this and inequality (4.3) we get the result of Lemma 4.1. ◻

We use Lemma 3.4 and Lemma 4.1 to get the following proposition.

Proposition 4.2. *For a $D(4)$-triple $\{1, b, c_k^{(\pm)}\}$, $(1 \le k \le 4)$, if $r > 68131(2k+5) + 32889$, then the equation $P_l = Q_m$ has no solution (l, m) satisfying $m > 1$.*

Proof. Assume that $r > 316$. Since $\Delta \neq 0$, then $\Delta \ge 1$. By Lemma 3.4 and Lemma 4.1, we have

$$0.983|\Delta|\sqrt{b}\log\alpha < 66972\left(\left|\Delta + \frac{1}{2}\lambda\right| + 3 + 2k\right)\log\alpha + 186091.$$

This implies

$$\begin{aligned} r - 2 &< \sqrt{b} < \frac{66972\left(\left|\Delta + \frac{1}{2}\lambda\right| + 3 + 2k\right)}{0.983|\Delta|} + \frac{186091}{0.983|\Delta|\log\alpha} \\ &< 68131(5 + 2k) + 32887. \end{aligned}$$

Therefore, this completes the proof of Proposition 4.2. ◻

5. Proof of Theorem 1.5

In this section, we will use another theorem for the lower bounds of linear forms in logarithms which differs from that in above section and the Baker-Davenport reduction method to deal with the remaining cases. We recall the following result due to Matveev [16] (see also Lemma 10 in [13]).

Lemma 5.1. *Denote by $\alpha_1,\dots,\alpha_j$ algebraic numbers, not 0 or 1, by $\log\alpha_1,\dots,\ \log\alpha_j$ determinations of their logarithms, by D the degree over $\mathbb{Q}$ of the number field $\mathbb{K}=\mathbb{Q}(\alpha_1,\dots,\alpha_j)$, and by $b_1,\dots,b_j$ integers. Define $B=\max\{|b_1|,\dots,|b_j|\}$, and $A_i=\max\{Dh(\alpha_i),|\log\alpha_i|,0.16\}$ $(1\le i\le j)$, where $h(\alpha)$ denotes the absolute logarithmic Weil height of α. Assume that the number*

$$\Lambda=b_1\log\alpha_1+\dots+b_n\log\alpha_j$$

does not vanish; then

$$|\Lambda|\ge\exp\{-C(j,\chi)D^2A_1\cdots A_j\log(eD)\log(eB)\},$$

where $\chi=1$ if $\mathbb{K}\subset\mathbb{R}$ and $\chi=2$ otherwise and

$$C(j,\chi)=\min\left\{\frac{1}{\chi}\left(\frac{1}{2}ej\right)^{\chi}30^{j+3}j^{3.5},2^{6j+20}\right\}.$$

Now, we apply the above lemma with $j=3$ and $\chi=1$ for

$$\Lambda=l\log\alpha-m\log\beta+\log\gamma.$$

Here, we take

$$D=4,b_1=l,b_2=-m,b_3=1,\alpha_1=\alpha,\alpha_2=\beta,\alpha_3=\frac{\sqrt{c}(y_2+x_2\sqrt{b})}{\sqrt{b}(z_0+x_0\sqrt{c})}.$$

From the computations done in the previous section, we put

$$h(\alpha_1)=\frac{1}{2}\log\alpha,\quad h(\alpha_2)=\frac{1}{2}\log\beta.$$

We see also that α_3 is a zero of

$$\begin{aligned}&b^2(c-1)^2X^4-4b^2c(c-1)x_0x_2X^3\\&-2bc\left((b-1)(c-1)-2b(c-1)x_2^2-2c(b-1)x_0^2\right)X^2\\&-4bc^2(b-1)x_0x_2X+c^2(b-1)^2.\end{aligned}$$

This implies

$$\begin{aligned} h(\alpha_3) &\leq \frac{1}{4}\left[\log(b^2(c-1)^2) + 4\log\frac{\max\{|\sqrt{c}(y_2 \pm x_2\sqrt{b})|\}}{\min\{|\sqrt{b}(z_0 \pm x_0\sqrt{c})|\}}\right] \\ &= \frac{1}{4}\left[\log(b^2(c-1)^2) + 4\log\frac{\sqrt{c}(2+2\sqrt{b})}{\sqrt{b}(-t+r\sqrt{c})}\right] \\ &\leq \frac{1}{4}\log\left[\frac{r^4c^4(1+\sqrt{b})^4}{(c-1)^2}\right] \leq \log(c\cdot\sqrt{2b}\cdot\sqrt{2b}) = \log(2bc). \end{aligned}$$

Hence we take

$$A_1 = 2\log\alpha, \quad A_2 = 2\log\beta, \quad A_3 = 4\log(2bc).$$

Using Lemma 5.1, we have

$$|\log\Lambda| > -1.3901\cdot 10^{11}\cdot 16^2\cdot\log\alpha\cdot\log\beta\cdot\log(2bc)\cdot\log(4e)\cdot\log(el). \tag{5.1}$$

By (3.1) and Lemma 3.1, we get the following inequalities

$$l\log\alpha < 2m\log\beta \quad \text{and} \quad \log|\Lambda| < -1.9988m\log\beta. \tag{5.2}$$

Also it is easy to see that

$$\log\beta < \frac{1}{4}\log(2c^2). \tag{5.3}$$

Combining (5.1), (5.2), and (5.3), we get the following inequality

$$\frac{l}{\log(el)} < 2.12427475\cdot 10^{13}\cdot\log^2(2c^2). \tag{5.4}$$

As $c = c_k^{(\pm)} \leq c_k^{(+)}$, $k \leq 4$ and $r \leq 68131(2k+5)+32889$, then we have

$$c < 4r^8 < 2.03\cdot 10^{48}.$$

Solving inequality (5.4), we get a bound of l that we summarize in the following result.

Lemma 5.2. *If $P_l = Q_m$ has a solution (l,m) with $m \geq 1$, then we have $l < 5\cdot 10^{19}$.*

In order to deal with the remaining cases, we will use a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method (See [8, Lemma 5a]).

Lemma 5.3. *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M$ and let*

$$\eta = \parallel \mu q \parallel - M\cdot \parallel \kappa q \parallel,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < l\kappa - m + \mu < AB^{-l}$$

in integers l and m with

$$\frac{\log(Aq/\eta)}{\log B} \le l \le M.$$

Dividing $0 < \Lambda < 1.0064\beta^{-2m}$ by $\log\beta$ and using the fact that we have $\beta^{-2m} < \alpha^{-l}$ leads us to the inequality

$$0 < l\kappa - m + \mu < AB^{-l}, \tag{5.5}$$

where

$$\kappa := \frac{\log\alpha}{\log\beta}, \quad \mu := \frac{\log\gamma}{\log\beta}, \quad A := \frac{1.0064}{\log\beta}, \quad B := \alpha.$$

We apply Lemma 5.3 to the inequality (5.5) with $M = 5 \cdot 10^{19}$.

For the remaining proof, we use Mathematica to apply Lemma 5.3. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions. After at most 2 steps of reduction, in all cases, we are able to prove that $l \le 13$. However, in some cases, we are not able to furthermore sharpen this bound applying the above mentioned reduction. Thus, we have to see what is happening for small indices l, i.e. $l \le 13$. We check all cases and find no solution to $P_l = Q_m$, for $2 \le m < l \le 13$. So, we have the following result.

Proposition 5.4. *For a $D(4)$-triple $\{1, b, c_k^{(\pm)}\}$, $(1 \le k \le 4)$, if $r \le 68131(2k+5)+32889$, then equation $P_l = Q_m$ has no solution (l, m) satisfying $m > 1$.*

Proposition 5.4 allows us to deduce that if $P_l = Q_m$ has a positive integer solution (l, m), then $m \le 1$. In fact, from (3.16) we know that a solution comes from $m = 1$, $l = k + 1$ and $z_0 = t$. Using equations (1.4) and (1.5), it is easy to show by induction that

$$U_k = \frac{1}{2}(rU_{k+1} - T_{k+1}) \quad \text{and} \quad T_k = \frac{1}{2}(rT_{k+1} - bU_{k+1}).$$

Thus, we have

$$\begin{aligned} x = P_{k+1} = Q_1 &= (r \pm 1)T_k + (b \pm r)U_k \\ &= 2T_{k+1} \pm 2U_{k+1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d = x^2 - 4 &= (2T_{k+1} \pm 2U_{k+1})^2 - 4 \\ &= \pm 8T_{k+1}U_{k+1} + 4(r^2 - 3)U_{k+1}^2 = c_{k+1}^{(\pm)}. \end{aligned}$$

This completes the proof of Theorem 1.5.

Acknowledgements.

The authors are grateful to the referee for the useful comments that help to improve the quality of the paper. They also thank Professor Bo He for the helpful discussion. The first author is supported by Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey-Calavi. The second author is supported by the Croatian Science Fund, grant HRZZ-IP-2018-01-1313. The third author is partially supported by Purdue University Northwest.

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Problem proširenja $D(4)$-trojki $\{1, b, c\}$

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Sažetak. U ovom članku promatramo proširenje $D(4)$-trojki oblika $\{1, b, c\}$, gdje je $1 < b < c$. Dokazali smo da taj skup ne može biti proširen do neregularne $D(4)$-četvorke za neke oblike broja c. U dokazu koristimo standardne metode bazirane na rješenjima binarno rekurzivnih nizova, zajedno s nekim novim pristupima kako bi potvrdili slutnju o jedinstvenosti takvog proširenja.

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Received: 17.7.2020.
Revised: 14.10.2020.
Accepted: 17.11.2020.