RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 26 = 551 (2022): 45-53
DOI: https://doi.org/10.21857/y26kec1809

DIOPHANTINE QUINTUPLES CONTAINING TWO PAIRS OF CONJUGATES IN SOME QUADRATIC FIELDS

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Abstract. In this paper, we describe constructions of Diophantine quintuples of the special form in rings $\mathbb{Z}[\sqrt{D}]$ for certain positive integer D. The term "special form" refers to Diophantine quintuples of the form $\{e, a+b\sqrt{D}, a-b\sqrt{D}, c+d\sqrt{D}, c-d\sqrt{D}\}$, where a, b, c, d, e are integers. Also, we assume these quintuples contain two regular Diophantine quadruples.

1. Introduction

The two most significant historical examples of Diophantine quadruples are $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ and $\{1, 3, 8, 120\}$ found by Diophantus and Fermat, respectively (see [5], pp. 517-520). Each of them has the property that the product of every two distinct elements increased by 1 gives perfect square of a rational number (in the first example) and an integer (in the second example). Therefore, it makes sense to set the following definition:

Definition 1.1. *Let $\mathcal{R}$ be a commutative ring with the unity.* A Diophantine m-tuple in $\mathcal{R}$ *is a set of m elements in $\mathcal{R}\backslash\{0\}$ with the property that the product of any two of its distinct elements increased by the unity is a square in $\mathcal{R}$.*

The reasonable question is how large these sets can be, i.e., can we find an upper bound on m for a particular ring? So far, the complete answer is known for the case of Diophantine m-tuples in the ring of integers ($\mathbb{Z}$) where it is proved that an integer Diophantine quintuple does not exist (see [15]). For the case of Diophantine m-tuples in the field of rational integers ($\mathbb{Q}$), i.e., for so called rational Diophantine m-tuples, infinitely many rational Diophantine sextuples have been found (see [9]) but there is no example of rational Diophantine septuple. Also, no Diophantine quintuples were found in imaginary quadratic number rings ($\mathbb{Z}[\sqrt{D}]$, $D < 0$) although the only known result on the upper bound says that m is less then 43 (see [1]). It is easy to

2020 *Mathematics Subject Classification.* 11D09, 11R11.

Key words and phrases. Diophantine quintuples, regular Diophantine quadruples, quadratic fields.

see that in some real quadratic number rings ($\mathbb{Z}[\sqrt{D}]$, $D > 0$ and D is not a perfect square) there exist Diophantine quintuples since every Diophantine triple can be extended to a Diophantine quadruple (see [3]). So, for instance if $\{a_1, a_2, a_3, a_4\}$ is a Diophantine quadruple in $\mathbb{Z}$ then the triple $\{a_2, a_3, a_4\}$ can be extended by the fourth element a_5 and $\{a_1, a_2, a_3, a_4, a_5\}$ represents a Diophantine quintuple in $\mathbb{Z}[\sqrt{a_1a_5+1}]$.

As just mentioned, any Diophantine triple $\{a_1, a_2, a_3\}$ can be extended to a Diophantine quadruple. It can be done by adding one of the following two elements

$$d_\pm = a_1 + a_2 + a_3 + 2a_1a_2a_3 \pm 2rst, \tag{1.1}$$

where $a_1a_2+1 = r^2$, $a_1a_3+1 = s^2$, $a_2a_3+1 = t^2$, but just in case the appended element (d_- or d_+) is not zero or one of the first three elements (a_1, a_2, a_3). Obviously, $\{a_1, a_2, a_3, d_-, d_+\}$ is a Diophantine quintuple if $d_-d_+ + 1 = \square$ and $d_-, d_+ \notin \{a_1, a_2, a_3, 0\}$. In this context we state the following definition:

Definition 1.2. *A Diophantine quadruple* $\{a_1, a_2, a_3, a_4\}$ *such that* $a_4 = d_-$ *or* $a_4 = d_+$*, where* $d_\pm$ *are given by* (1.1)*, is called* regular.

A Diophantine quintuple containing two regular Diophantine quadruples is called biregular.

A classification of Diophantine quadruples and quintuples with examples in $\mathbb{Q}$ can be found in [13]. Also, biregular quintuples in $\mathbb{Q}$ were applied to construction of high-rank elliptic curves and rational Diophantine sextuples (see [6, 10]).

In this paper, we deal with the construction of explicit examples of Diophantine quintuples of the *special form* in $\mathbb{Z}[\sqrt{D}]$ for infinite families of positive integers D. Namely, in [14], Gibbs listed 160 examples of Diophantine quintuples in real quadratic number rings $\mathbb{Z}[\sqrt{D}]$ for all square free D with $1 < D < 50$ except for $D \in \{23, 35, 42, 43, 47\}$. (The example of Diophantine quintuple in $\mathbb{Z}[\sqrt{43}]$ was found in [11]). Among these, we observed the examples of biregular Diophantine quintuples of the form

$$\{e, a + b\sqrt{D}, a - b\sqrt{D}, c + d\sqrt{D}, c - d\sqrt{D}\} \tag{1.2}$$

where $e, a, b, c, d \in \mathbb{Z}$, i.e., Diophantine quintuples containing two pairs of conjugate elements. So here, by "Diophantine quintuples of the special form" we mean on Diophantine quintuples of the form (1.2) where elements $c \pm d\sqrt{D}$ are regular extensions (1.1) of the set $\{e, a \pm b\sqrt{D}\}$. In [11], some explicit polynomial formulas for such quintuples for certain values of D (e.g. for $D = n^2(n+1)^2 + 1$) were found. It has also been shown that the set (1.2) is a biregular Diophantine quintuple if the following conditions hold:

$$e(a + b\sqrt{D}) + 1 = (u + v\sqrt{D})^2, \tag{1.3}$$

$$(a + b\sqrt{D})(a - b\sqrt{D}) + 1 = x^2D, \tag{1.4}$$

$$c \pm d\sqrt{D} = e + 2a + 2e(a^2 - Db^2) \pm 2(u^2 - Dv^2)x\sqrt{D}, \tag{1.5}$$

$$(c + d\sqrt{D})(c - d\sqrt{D}) + 1 = y^2 \text{ or } = y^2 D, \tag{1.6}$$

for some $u, v, x, y \in \mathbb{Z}$.

Here, we present some new families of biregular Diophantine quintuples of the form (1.2) in $\mathbb{Z}[\sqrt{D}]$, for certain values of D. The main theorems are as follows:

Theorem 1.3. *If $D = \frac{a^2+1}{10}$, where $a > 3$ is an integer solution of the equation $5\chi^2 - 2a^2 = 27$, then there exists a biregular Diophantine quintuple of the form (1.2) in the ring $\mathbb{Z}[\sqrt{D}]$.*

Theorem 1.4. *Let n be a positive integer and $D = n^2(8n\pm1)^2+1$. There exists a biregular Diophantine quintuple of the form (1.2) in the ring $\mathbb{Z}[\sqrt{D}]$.*

2. Proof of Theorem 1.3

The proof is carried out in several steps.

1) Let us consider the equation

$$5\chi^2 - 2a^2 = 27, \tag{2.1}$$

a Diophantine equation of Pellian type. Since $19 + 6\sqrt{10}$ is a fundamental solution of a related Pell's equation $X^2 - 10A^2 = 1$, and $15 \pm 3\sqrt{10}$, $25 \pm 7\sqrt{10}$ are fundamental solutions of a related Pellian equation $X^2 - 10A^2 = 135$, all integer solutions of equation (2.1) with $a > 0$ are given by:

$$\begin{aligned} \chi_n^{\pm}\sqrt{5} + a_n^{\pm}\sqrt{2} &= (3\sqrt{5} \pm 3\sqrt{2})(19 + 6\sqrt{10})^n, \\ {\chi'}_n^{\pm}\sqrt{5} + {a'}_n^{\pm}\sqrt{2} &= (5\sqrt{5} \pm 7\sqrt{2})(19 + 6\sqrt{10})^n,\ n \in \mathbb{N}_0. \end{aligned}$$

Sequences $(a_n^{\pm})$, $({a'}_n^{\pm})$, $(\chi_n^{\pm})$, $({\chi'}_n^{\pm})$ satisfy the same binary recurrence

$$X_{n+2} = 38X_{n+1} - X_n,\ n \geq 0, \tag{2.2}$$

with initial conditions:

$$\begin{array}{ll} (a_0^+, \chi_0^+) = (3, 3), & (a_1^+, \chi_1^+) = (147, 93), \\ (a_0^-, \chi_0^-) = (-3, 3), & (a_1^-, \chi_1^-) = (33, 21), \\ ({a'}_0^+, {\chi'}_0^+) = (7, 5), & ({a'}_1^+, {\chi'}_1^+) = (283, 179), \\ ({a'}_0^-, {\chi'}_0^-) = (-7, 5), & ({a'}_1^-, {\chi'}_1^-) = (17, 11). \end{array} \tag{2.3}$$

2) We show that $D = \frac{a^2+1}{10}$ is well-defined, i.e., D is an integer which is not a perfect square. Let $a \in \{a_n^{\pm}, {a'}_n^{\pm} : n \in \mathbb{N}_0\}$. Then $a \equiv \pm 3 \pmod{10}$. Indeed, using (2.2) and (2.3) and arguing by induction, we easily prove that

$$\begin{aligned} a_n^+, {a'}_n^- &\equiv (-1)^n 3 \pmod{10},\ n \geq 0, \\ a_n^-, {a'}_n^+ &\equiv (-1)^{n+1} 3 \pmod{10},\ n \geq 0. \end{aligned}$$

So, for any integer solution of equation (2.1), $D = \frac{a^2+1}{10}$ is an integer.

If, for some integer z, $D = z^2$, then the system of equations

$$5\chi^2 - 2a^2 = 27, \quad a^2 - 10z^2 = -1$$

is solvable in integers and vice versa. It is easy to see that the only integer solutions (in a) of this system are $a = \pm 3$ and they correspond to $D = 1$.

3) In this step we construct a Diophantine triple $\{e, a + b\sqrt{D}, a - b\sqrt{D}\}$ in $\mathbb{Z}[\sqrt{D}]$. Let $b = 3$ and $a \in \{a_n^{\pm}, {a'}_n^{\pm} : n \in \mathbb{N}_0\}$, $a > 3$. Further, let us define

$$u = \begin{cases} \frac{1}{3}\left(2a - \frac{\sqrt{2a^2+27}}{\sqrt{5}}\right), & \text{if } a \in (a_n^-), ({a'}_n^+), \\ \frac{1}{3}\left(2a + \frac{\sqrt{2a^2+27}}{\sqrt{5}}\right), & \text{if } a \in (a_n^+), ({a'}_n^-), \end{cases} \tag{2.4}$$

and

$$e = \frac{4u}{3}. \tag{2.5}$$

Note that u is a positive integer since $2a^2 + 27 = 5\chi^2$, for some $\chi \in \mathbb{Z}$ and

$$2a - \frac{\sqrt{2a^2+27}}{\sqrt{5}} = \frac{2a^2 - 27}{\sqrt{5}(2a + \sqrt{2a^2+27})} > 0,$$

for $a \geq 4$.

Now let us verify that e is a positive integer. Using (2.2), (2.3), and the induction principle, we conclude that $u = \frac{1}{3}(2a \pm \chi) \equiv 0 \pmod 3$ in each of the following four cases:

$a = a_n^-$: Since $a_0^-, a_1^- \equiv 6 \pmod 9$, then $a_n^- \equiv 6 \pmod 9$ for $n > 1$. Also $\chi_0^-, \chi_1^- \equiv 3 \pmod 9$ and so is $\chi_n^- \equiv 3 \pmod 9$ for $n > 1$. Therefore,

$$2a_n^- - \chi_n^- \equiv 0 \pmod 9,$$

for all $n > 0$.

$a = a_n^+$: It is easy to see that $a_n^+, \chi_n^+ \equiv 3 \pmod 9$ and therefore

$$2a_n^+ + \chi_n^+ \equiv 0 \pmod 9,$$

for all $n \geq 0$.

$a = {a'}_n^-$: We have

$$\begin{aligned} {a'}_n^- &\equiv 2, 8, 5, 2, 8, \dots \pmod 9, \\ {\chi'}_n^- &\equiv 5, 2, 8, 5, 2, \dots \pmod 9. \end{aligned}$$

So, $({a'}_n^-, {\chi'}_n^-) \bmod 9 \in \{(2,5), (8,2), (5,8)\}$ and $2{a'}_n^- + {\chi'}_n^- \equiv 0 \pmod 9$, for all $n \geq 0$.

$a = {a'}_n^+$: Since

$$\begin{aligned} {a'}_n^+ &\equiv 7, 4, 1, 7, 4 \dots \pmod 9, \\ {\chi'}_n^+ &\equiv 5, 8, 2, 5, 8 \dots \pmod 9, \end{aligned}$$

we get $2{a'}_n^+ - {\chi'}_n^+ \equiv 0 \pmod 9$, for all $n \geq 0$.

Finally, we have to check that the product of any two elements in $\{e, a \pm b\sqrt{D}\}$ increased by 1 is a square in $\mathbb{Z}[\sqrt{D}]$. So,

$$\begin{aligned} e(a+b\sqrt{D})+1 &= \frac{4u}{3}(a+3\sqrt{D})+1 = \frac{4(2a\pm\chi)}{9}a+4u\sqrt{D}+1 \\ &= \frac{8}{9}a^2 \pm \frac{4}{9}a\chi + 1 + 4u\sqrt{D} \\ &= \frac{1}{9}(2a\pm\chi)^2 + \frac{4}{9}a^2 - \frac{1}{9}\chi^2 + 1 + 4u\sqrt{D} \\ &= \frac{1}{9}(2a\pm\chi)^2 + \frac{4}{9}a^2 - \frac{1}{45}(2a^2+27) + 1 + 4u\sqrt{D} \\ &= u^2 + 4D + 4u\sqrt{D} = (u+2\sqrt{D})^2 =: r^2. \end{aligned}$$

Analogously,

$$e(a-b\sqrt{D})+1 = (u-2\sqrt{D})^2 =: s^2.$$

And, it is easy to see that

$$(a+b\sqrt{D})(a-b\sqrt{D})+1 = a^2-9D+1 = D =: t^2.$$

4) The triple $\{e, a\pm b\sqrt{D}\}$, defined in the previous step, can be extended to a Diophantine quadruple with its regular extensions given by (1.1):

$$\begin{aligned} c \pm d\sqrt{D} &= e + (a+b\sqrt{D}) + (a-b\sqrt{D}) + 2e(a+b\sqrt{D})(a-b\sqrt{D}) \pm 2rst \\ &= e + 2a + 2e\underbrace{(a^2-b^2D)}_{=D-1} \pm 2(u^2-4D)\sqrt{D} \\ &= 2a + e(2D-1) \pm 2(u^2-4D)\sqrt{D}, \end{aligned}$$

if each of them does not equal zero or repeat one of the first three numbers which is clearly not the case here. Finally, if the product of these two appended elements increased by 1 is a square, i.e., if

$$(c+d\sqrt{D})(c-d\sqrt{D})+1 = \square,$$

then $\{e, a+b\sqrt{D}, a-b\sqrt{D}, c+d\sqrt{D}, c-d\sqrt{D}\}$ is a biregular Diophantine quintuple. In our case, it turns out to be true. We have

$$(c+d\sqrt{D})(c-d\sqrt{D})+1 = (2a+e(2D-1))^2 - 4(u^2-4D)^2D.$$

Implementing (2.4), (2.5) and $D = (a^2+1)/10$ into previous expression, we get

$$\frac{74a^2}{81} \pm \frac{16}{81}\sqrt{5}\sqrt{2a^2+27}a + \frac{5}{3},$$

where "$-$" is obtained for $u^+ = \frac{1}{3}\left(2a + \frac{\sqrt{2a^2+27}}{\sqrt{5}}\right) = \frac{1}{3}(2a+\chi)$ and vice versa. Further, taking into account that a is a solution of $5\chi^2 - 2a^2 = 27$, we have

$$\frac{5}{3} + \frac{74}{81}a^2 \pm \frac{80}{81}a\chi = \frac{1}{81}(25\chi^2 \pm 80a\chi + 64a^2) = \frac{(5\chi\pm 8a)^2}{81}.$$

So,

$$(c+d\sqrt{D})(c-d\sqrt{D})+1=\left(\frac{5\chi\pm 8a}{9}\right)^2.$$

It remains to conclude that $(5\chi \pm 8a)/9$ is an integer. All cases are listed below:

$a=a_n^-$: $(a_n^-,\chi_n^-) \mod 9=(6,3)$ implies $5\chi_n^-+8a_n^-\equiv 0 \pmod 9$.
$a=a_n^+$: $(a_n^+,\chi_n^+) \mod 9=(3,3)$ implies $5\chi_n^+-8a_n^+\equiv 0 \pmod 9$.
$a={a'}_n^-$: $({a'}_n^-,{\chi'}_n^-) \mod 9 \in \{(2,5),(8,2),(5,8)\}$ implies $5{\chi'}_n^- - 8{a'}_n^- \equiv 0 \pmod 9$.
$a={a'}_n^+$: $({a'}_n^+,{\chi'}_n^+) \mod 9 \in \{(7,5),(4,8),(1,2)\}$ implies $5{\chi'}_n^+ + 8{a'}_n^+ \equiv 0 \pmod 9$.

We obtained examples expressed in the following table:

Table 1. Some Diophantine quintuples from Theorem 1.3

D	e	(a,b)	(c,d)
5	4	(7, 3)	(50, 22)
29	20	(17, 3)	(1174, 218)
109	20	(33, 3)	(4406, 422)
2161	172	(147, 3)	(743506, 15994)
8009	172	(283, 3)	(2755490, 30790)
42641	764	(653, 3)	(65155990, 315530)

3. Proof of Theorem 1.4

We show how we effectively constructed Diophantine quintuples of the form (1.2) in $\mathbb{Z}[\sqrt{D}]$ for $D=n^2(8n\pm 1)^2+1$, $n\in\mathbb{N}$.

Let us assume that $e,a,b,u,v,x\in\mathbb{Z}\backslash\{0\}$ satisfy (1.3) and (1.4), i.e.,

$$ea+1=u^2+v^2D, \tag{3.1}$$

$$eb=2uv, \tag{3.2}$$

and

$$a^2-Db^2+1=x^2D. \tag{3.3}$$

Therefore, $\{e,a+b\sqrt{D},a-b\sqrt{D}\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{D}]$. Observe that in the previous case we had $x=1$. Now, assume that x is "small" again, i.e., $x=2$. So, (3.3) gives

$$D=\frac{a^2+1}{b^2+4}. \tag{3.4}$$

Also suppose that there is a linear connection between a and u, and b and v:

$$a = u + k,\ b = v + l,\ k, l \in \mathbb{Z}. \tag{3.5}$$

Putting (3.2), (3.4), (3.5) into (3.1) yields

$$\frac{2uv(u+k)}{v+l} + 1 = u^2 + v^2 \frac{(u+k)^2+1}{(v+l)^2+1}.$$

The previous expression can be recognized as a quadratic equation in the variable u:

$$(4l+l^3-4v+l^2v)u^2 - 2kv(4+l^2+lv)u - (l+v)(4+l^2+2lv-k^2v^2) = 0. \tag{3.6}$$

Since $u \in \mathbb{Z}$, the discriminant of the previous equation

$$p(v) = 4k^2v^2(4+l^2+lv)^2 + 4(4l+l^3-4v+l^2v)(l+v)(4+l^2+2lv-k^2v^2)$$

should be a perfect square. This is fulfilled for $l = 2k-2$ because the expression $2k-l-2$ divides the discriminant of the quartic polynomial $p(v)$ (i.e., the discriminant of $p(v)$ is zero for $l = 2k-2$). In that case ($l = 2k-2$), solutions of (3.6) are:

$$u^- = \frac{1}{2}(v-2),\ u^+ = \frac{2k^3(v+2) + k^2\left(v^2-12\right) - 4k(v-4) + 4(v-2)}{4k^3 + 2k^2(v-6) - 4k(v-4) - 8}.$$

We can reject the solution u^- since it yields $D = \frac{1}{4}$.

For $k = 1$ we get $u^+ = -\frac{1}{2}(2+v)$ and $D = \frac{1}{4}$. For $k = 2$ we obtain

$$u^+ = \frac{1}{2}(2 + 3v + v^2)$$

which is an integer for all $v \in \mathbb{Z}$ and corresponding $D = \frac{1}{4}(5 + 2v + v^2)$ is an integer for $v = 2m-1$, $m \in \mathbb{Z}$. Taking all that ($k = l = 2$, $v = 2m-1$, $u = 2m^2 + m$) into account we get that

$$\{e, a \pm b\sqrt{D}\} = \{2m(2m-1), 2m^2 + m + 2 \pm (2m+1)\sqrt{m^2+1}\}$$

is a Diophantine triple in $\mathbb{Z}[\sqrt{m^2+1}]$, for $m \in \mathbb{Z}$, $m \neq 0$.

Next we try to find conditions on m such that regular extensions given by (1.5), i.e., by

$$c \pm d\sqrt{D} = e + 2a + 2e(a^2 - Db^2) \pm 4(u^2 - v^2D)\sqrt{D}$$

extend the triple $\{e, a \pm b\sqrt{D}\}$ to a quintuple. This is fulfilled, if

$$c^2 - d^2D + 1 = 32m + 1 = y^2,$$

for $y \in \mathbb{Z}$. Assuming that $32m + 1 = y^2D = y^2(m^2+1)$, $y \in \mathbb{Z}$, we get only finitely many solutions of which only one corresponds to a Diophantine quintuple ($m = 32$). On the other hand, $32m + 1 = y^2$, $y \in \mathbb{Z}$, implies that $y = 16n \pm 1$, $n \in \mathbb{Z}$, i.e.,

$$m = ((16n \pm 1)^2 - 1)/32 = n(8n \pm 1).$$

Finally, for

$$\begin{aligned}
D &= n^2(8n\pm 1)^2+1,\\
e &= 2n(8n\pm 1)(2n(8n\pm 1)-1),\\
a &= n(8n\pm 1)(2n(8n\pm 1)+1)+2 = 128n^4 \pm 32n^3+10n^2\pm n+2,\\
b &= 2n(8n\pm 1)+1 = 16n^2\pm 2n+1,\\
c &= 4(8n^4(8n\pm 1)^4-4n^3(8n\pm 1)^3+8n^2(8n\pm 1)^2-3n(8n\pm 1)+1\\
&= 4(32768n^8\pm 16384n^7+1024n^6\mp 512n^5+424n^4\pm 124n^3-16n^2\mp 3n+1),\\
d &= 4(8n^3(8n\pm 1)^3-4n^2(8n\pm 1)^2+4n(8n\pm 1)-1),\\
&= 4(4096n^6\pm 1536n^5-64n^4\mp 56n^3+28n^2\pm 4n-1),
\end{aligned}$$

$\{e, a\pm b\sqrt{D}, c\pm d\sqrt{D}\}$ is a biregular Diophantine quintuple in $\mathbb{Z}[\sqrt{D}]$.

Examples obtained by evaluating the above expressions for the first few values of n are listed in the following table:

TABLE 2. Some Diophantine quintuples from Theorem 1.4

D	e	(a,b)	(c,d)
50	182	(107, 15)	(72832, 10300)
82	306	(173, 19)	(200776, 22172)
901	3540	(1832, 61)	(25516444, 850076)
1157	4556	(2348, 69)	(42170476, 1239772)

ACKNOWLEDGEMENTS.

The author acknowledges support from the QuantiXLie Center of Excellence, a project co-financed by the Croatian Government and European Union through the European Regional Development Fund- the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004)

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Diofantove četvorke koje sadrže dva para konjugata u nekim kvadratnim poljima

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Sažetak. U ovom radu opisujemo konstrukcije Diofantovih petorki posebnog oblika u prstenima $\mathbb{Z}[\sqrt{D}]$, za neke prirodne brojeve D. Pojam Diofantove petorke "posebnog oblika" odnosi se na petorke oblika $\{e, a+b\sqrt{D}, a-b\sqrt{D}, c+d\sqrt{D}, c-d\sqrt{D}\}$, gdje su a, b, c, d, e cijeli brojevi. Nadalje, pretpostavljamo da ove petorke sadrže dvije regularne Diofantove četvorke.

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Received: 14.5.2021.
Revised: 13.7.2021.
Accepted: 14.9.2021.