

ON A VARIANT AND EXTENSION OF GABLER INEQUALITY

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ABSTRACT. We propose a Jensen-Mercer type variant and a Niezgoda type extension of Gabler inequality along with applications.

1. INTRODUCTION AND PRELIMINARIES

It would not be an exaggeration if we say that “Jensen inequality is among the most celebrated inequalities of all time”. That is why most of the researcher are continuously working on this inequality for long time. In recent past we can find number of different variants, extensions, generalizations and refinements of this renowned inequality, for reference see [1–3, 8–11, 14, 15, 17–19, 22, 31–34] and the references given therein. We also adduce to [25] and [29] for detailed discussion on Jensen’s inequality and for some remarks on literature and history of the topic. Throughout the article we assume that J is an interval in \mathbb{R} and for weights w_1, \dots, w_n we define the notation

$$W_i = \sum_{j=1}^i w_j, \quad i \in \{1, \dots, n\} \quad \text{and clearly} \quad W_n = \sum_{j=1}^n w_j.$$

Now we start with Jensen’s inequality [29].

PROPOSITION 1.1. *Let \mathbf{x} be an n -tuple with $x_i \in J$ for $i \in \{1, \dots, n\}$ and let \mathbf{w} be a nonnegative n -tuple with $W_n > 0$, then for a convex function f on J following inequality holds*

$$(1.1) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i).$$

The following variant of the Jensen’s inequality was introduced by Mercer in [24], which is usually referred as “Jensen-Mercer inequality”.

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PROPOSITION 1.2. *Let all the assumptions of Proposition 1.1 be true. Then the following inequality holds*

$$(1.2) \quad f\left(a + b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) \leq f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i),$$

where

$$a = \min_{x_i \in J} \{x_i\} \quad \text{and} \quad b = \max_{x_i \in J} \{x_i\}.$$

Before going on to our next preliminary, let us recall a prerequisite concept of majorization from [23].

Let $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ denote two m -tuples and $x_{[1]} \geq \dots \geq x_{[m]}$, $y_{[1]} \geq \dots \geq y_{[m]}$, be their ordered components.

DEFINITION 1.3. *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,*

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} & , \quad k \in \{1, \dots, m-1\}, \\ \sum_{i=1}^m x_{[i]} = \sum_{i=1}^m y_{[i]} \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, \mathbf{x} is said to be majorized by \mathbf{y} or \mathbf{y} majorizes \mathbf{x} .

This notion and notation of majorization was first introduced by Hardy et al. in [13]. In the same book [13] we find a very powerful result, namely majorization theorem (see also [23]).

THEOREM 1.4. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then following inequality is true for all continuous convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$,*

$$\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i),$$

if and only if $\mathbf{x} \prec \mathbf{y}$.

We now state Niezgoda's inequality which is actually an extension of (1.2) by Niezgoda [27].

PROPOSITION 1.5. *Suppose that \mathbf{a} be an m -tuple such that $a_i \in J$ and let $\mathbf{X} = (\mathbf{x}_j) = (x_{ij})$, $n \times m$ be a matrix with $x_{ij} \in J$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.*

If \mathbf{a} majorizes each row of \mathbf{X} , that is,

$$\mathbf{x}_i = (x_{i1}, \dots, x_{im}) \prec (a_1, \dots, a_m) = \mathbf{a} \quad \text{for each } i \in \{1, \dots, n\},$$

then for a continuous convex function f on J following inequality holds:

$$(1.3) \quad f\left(\sum_{j=1}^m a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i x_{ij}\right) \leq \sum_{j=1}^m f(a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i f(x_{ij}),$$

with $w_i \geq 0$.

In [12] Gabler defined a special case of convex functions namely sequentially convex functions by employing the following double index function.

DEFINITION 1.6. For $\mathbf{x} = (x_1, \dots, x_n) \in J^n$ and a real-valued function $f : J \rightarrow \mathbb{R}$, define

$$(1.4) \quad f_{k,n} = f_{k,n}(\mathbf{x}) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \dots < i_k \leq n} f\left(\frac{1}{k}(x_{i_1} + \dots + x_{i_k})\right).$$

Gabler termed this double index function as an arithmetic mean of all possible convex functions generated by arithmetic means of any k values chosen from (x_1, \dots, x_n) . Gabler then defined sequentially convex functions as follows:

DEFINITION 1.7. Let $f : J \rightarrow \mathbb{R}$ and let $f_{k,n}$ be defined as in (1.4), then f is said to be sequentially convex if $(f_{k,n})$ is a convex sequence in k for all $n > 2$ and all $x_1, \dots, x_n \in J$.

While investigating sequentially convex functions Gabler also made the following important observation which we shall call as Gabler inequality.

PROPOSITION 1.8. For a sequentially convex function f of type (1.4) the following inequality holds where $k \in \{1, \dots, n-1\}$

$$(1.5) \quad f_{k,n} \geq f_{k+1,n}.$$

Through the proof in [12] it is interesting to notice that (1.5) is also true for midconvex functions see [26, 30].

It was 1994 when Pečarić upgraded the double index function (1.4) and came up with the following weighted version (see [28]).

DEFINITION 1.9. For $\mathbf{x} = (x_1, \dots, x_n) \in J^n$ and a real-valued function $f : J \rightarrow \mathbb{R}$, define

$$(1.6) \quad f_{k,n} = f_{k,n}(\mathbf{x}, w) = \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} (w_{i_1} + \dots + w_{i_k}) f\left(\frac{w_{i_1} x_{i_1} + \dots + w_{i_k} x_{i_k}}{w_{i_1} + \dots + w_{i_k}}\right)$$

where w_i 's are positive weights for $i \in \{1, \dots, n\}$.

In the same article the Gabler result was further strengthened by defining it for convex functions in the following way.

PROPOSITION 1.10. For a convex function f of type (1.6) the following inequality holds where $k \in \{1, \dots, n-1\}$

$$(1.7) \quad f_{k,n} \geq f_{k+1,n}.$$

Furthermore, it was proved that the inequality (1.7) is an interpolating inequality for Jensen's inequality, i.e.,

$$(1.8) \quad f\left(\frac{1}{W_n} \sum_{i=1}^n w_i x_i\right) = f_{n,n} \leq \cdots \leq f_{k+1,n} \leq f_{k,n} \leq \cdots \leq f_{1,n} = \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i).$$

For recent work on Gabler inequality we refer the reader to [6].

In the present article a variant of Gabler inequality in terms of Jensen-Mercer inequality and its extension via Niezgod'a's inequality will be stated along with some refinements similar to (1.8).

This article is organized in the following manner. The first section states preliminaries and introduction. In second section we give a variant of Gabler inequality through Jensen-Mercer inequality. Third section is devoted to an extension of Gabler inequality via Niezgod'a's inequality. While the fourth section is devoted to applications of our obtained results in terms of generalized means.

Throughout this article we follow techniques provided in [28].

2. JENSEN-MERCER TYPE VARIANT OF GABLER INEQUALITY

THEOREM 2.1. *Under the assumptions of Proposition 1.2, if we define*

$$(2.1) \quad f_{k,n} = f_{k,n}(\mathbf{x}, w, a, b) = \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) f\left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_k} x_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right),$$

then the inequality (1.7) holds.

PROOF. By using the definition of convex functions and rearrangements, we have

$$\begin{aligned} & (w_{i_1} + \cdots + w_{i_{k+1}}) f\left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_{k+1}} x_{i_{k+1}}}{w_{i_1} + \cdots + w_{i_{k+1}}}\right) \\ &= (w_{i_1} + \cdots + w_{i_{k+1}}) f\left[\frac{1}{\sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l})}\right. \\ & \quad \left. \times \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) \left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_{k+1}} x_{i_{k+1}} - w_{i_l} x_{i_l}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}}\right)\right] \end{aligned}$$

$$\begin{aligned}
&\leq (w_{i_1} + \cdots + w_{i_{k+1}}) \left[\frac{1}{\sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l})} \right. \\
&\quad \times \left. \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) f \left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_{k+1}}x_{i_{k+1}} - w_{i_l}x_{i_l}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right) \right] \\
&= \frac{1}{k} \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) f \left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_{k+1}}x_{i_{k+1}} - w_{i_l}x_{i_l}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right).
\end{aligned}$$

In order to use above result we consider

$$\begin{aligned}
f_{k+1,n} &= \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} (w_{i_1} + \cdots + w_{i_{k+1}}) f \left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_{k+1}}x_{i_{k+1}}}{w_{i_1} + \cdots + w_{i_{k+1}}} \right) \\
&\leq \frac{1}{k \binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) \\
&\quad \times f \left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_{k+1}}x_{i_{k+1}} - w_{i_l}x_{i_l}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right) \\
&= \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) f \left(a + b - \frac{w_{i_1}x_{i_1} + \cdots + w_{i_k}x_{i_k}}{w_{i_1} + \cdots + w_{i_k}} \right) = f_{k,n},
\end{aligned}$$

which concludes our proof. \square

COROLLARY 2.2. *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned}
f \left(a + b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \right) &= f_{n,n} \\
&\leq \cdots \leq f_{k+1,n} \leq f_{k,n} \leq \cdots \leq f_{1,n} \leq f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i).
\end{aligned}$$

PROOF. For $k = n$, double index function (2.1) yields to

$$\begin{aligned}
f_{n,n} &= \frac{1}{\binom{n-1}{n-1} W_n} (w_1 + \cdots + w_n) f \left(a + b - \frac{w_1x_1 + \cdots + w_nx_n}{w_1 + \cdots + w_n} \right) \\
&= f \left(a + b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i \right).
\end{aligned}$$

Similarly for $k = 1$, double index function (2.1) and Lemma 1.3 of [24] gives us

$$\begin{aligned}
f_{1,n} &= \frac{1}{\binom{n-1}{1-1} W_n} \sum_{i_1=1}^n w_{i_1} f\left(a + b - \frac{w_{i_1} x_{i_1}}{w_{i_1}}\right) \\
&= \frac{1}{W_n} \sum_{i=1}^n w_i f(a + b - x_i). \\
&\leq \frac{1}{W_n} \sum_{i=1}^n w_i [f(a) + f(b) - f(x_i)] \\
&= f(a) + f(b) - \frac{1}{W_n} \sum_{i=1}^n w_i f(x_i).
\end{aligned}$$

Above calculations in accordance to inequality (1.7) concludes our proof. \square

3. NIEZGODA TYPE EXTENSION OF GABLER INEQUALITY

THEOREM 3.1. *Under the assumptions of Proposition 1.5, if we define*

$$f_{k,n} = f_{k,n}(\mathbf{x}, \mathbf{a}, w)$$

$$\begin{aligned}
(3.1) \quad &= \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} (w_{i_1} + \dots + w_{i_k}) f\left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \dots + w_{i_k} x_{i_k j}}{w_{i_1} + \dots + w_{i_k}}\right)
\end{aligned}$$

then the inequality (1.7) holds.

PROOF. By using the definition of convex functions and rearrangements we have

$$\begin{aligned}
&(w_{i_1} + \dots + w_{i_{k+1}}) f\left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \dots + w_{i_{k+1}} x_{i_{k+1} j}}{w_{i_1} + \dots + w_{i_{k+1}}}\right) \\
&= (w_{i_1} + \dots + w_{i_{k+1}}) f\left[\frac{1}{\sum_{l=1}^{k+1} (w_{i_1} + \dots + w_{i_{k+1}} - w_{i_l})}\right] \\
&\quad \times \sum_{l=1}^{k+1} (w_{i_1} + \dots + w_{i_{k+1}} - w_{i_l}) \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \dots + w_{i_{k+1}} x_{i_{k+1} j} - w_{i_l} x_{i_l j}}{w_{i_1} + \dots + w_{i_{k+1}} - w_{i_l}}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq (w_{i_1} + \cdots + w_{i_{k+1}}) \left[\frac{1}{\sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l})} \right. \\
&\quad \times \left. \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \cdots + w_{i_{k+1}} x_{i_{k+1} j} - w_{i_l} x_{i_l j}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right) \right] \\
&= \frac{1}{k} \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \cdots + w_{i_{k+1}} x_{i_{k+1} j} - w_{i_l} x_{i_l j}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right).
\end{aligned}$$

In order to use the above result we consider

$$\begin{aligned}
f_{k+1,n} &= \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} (w_{i_1} + \cdots + w_{i_{k+1}}) \\
&\quad \times f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \cdots + w_{i_{k+1}} x_{i_{k+1} j}}{w_{i_1} + \cdots + w_{i_{k+1}}} \right) \\
&\leq \frac{1}{k \binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \sum_{l=1}^{k+1} (w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}) \\
&\quad \times f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \cdots + w_{i_{k+1}} x_{i_{k+1} j} - w_{i_l} x_{i_l j}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}} \right) \\
&= \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (w_{i_1} + \cdots + w_{i_k}) f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i_1} x_{i_1 j} + \cdots + w_{i_k} x_{i_k j}}{w_{i_1} + \cdots + w_{i_k}} \right) \\
&= f_{k,n},
\end{aligned}$$

which concludes our proof. \square

COROLLARY 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\begin{aligned}
f \left(\sum_{j=1}^m a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i x_{ij} \right) &= f_{n,n} \leq \cdots \leq f_{k+1,n} \\
&\leq f_{k,n} \leq \cdots \leq f_{1,n} \leq \sum_{j=1}^m f(a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i f(x_{ij}).
\end{aligned}$$

PROOF. For $k = n$, double index function (3.1) yields to

$$\begin{aligned} f_{n,n} &= \frac{1}{\binom{n-1}{n-1}W_n} (w_1 + \cdots + w_n) f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_1 x_{1j} + \cdots + w_n x_{nj}}{w_1 + \cdots + w_n} \right) \\ &= f \left(\sum_{j=1}^m a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i x_{ij} \right). \end{aligned}$$

Now by using similar technique for majorization theorem as in Theorem 2.1 of [27] and double index function (3.1) for $k = 1$, we have

$$\begin{aligned} f_{1,n} &= \frac{1}{\binom{n-1}{1-1}W_n} \sum_{i=1}^n w_i f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} \frac{w_{i1} x_{i1j}}{w_{i1}} \right) \\ &= \frac{1}{W_n} \sum_{i=1}^n w_i f \left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1} x_{ij} \right) \\ &\leq \sum_{j=1}^m f(a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i f(x_{ij}). \end{aligned}$$

Above calculations in accordance to inequality (1.7) concludes our proof. \square

REMARK 3.3. If we set $k = m = 2$, $a_1 = a$, $a_2 = b$ and $x_{i1} = x_i$ for $i \in \{1, \dots, n\}$, then Theorem 2.1 and related results will become special cases of Theorem 3.1.

4. APPLICATIONS

4.1. *For Jensen-Mercer type variant of Gabler inequality.* For $[a, b] \subset J$, $0 < a < b$ and positive weights w_i for $i \in \{1, \dots, \alpha\}$ where $\alpha \in \{1, \dots, n\}$, we define the following (modified) arithmetic, geometric and harmonic means along with power mean of order $r \in \mathbb{R}$ for all $x_i \in [a, b]$:

$$\begin{aligned} A(x_1, \dots, x_\alpha; w_1, \dots, w_\alpha) &= a + b - \frac{1}{W_\alpha} \sum_{i=1}^\alpha w_i x_i, \\ G(x_1, \dots, x_\alpha; w_1, \dots, w_\alpha) &= \frac{ab}{\left(\prod_{i=1}^\alpha x_i^{w_i} \right)^{\frac{1}{W_\alpha}}}, \\ H(x_1, \dots, x_\alpha; w_1, \dots, w_\alpha) &= \left(a^{-1} + b^{-1} - \frac{1}{W_\alpha} \sum_{i=1}^\alpha w_i \frac{1}{x_i} \right)^{-1}, \\ M^{[r]}(x_1, \dots, x_\alpha; w_1, \dots, w_\alpha) &= \begin{cases} \left(a^r + b^r - \frac{1}{W_\alpha} \sum_{i=1}^\alpha w_i x_i^r \right)^{\frac{1}{r}}, & r \neq 0, \\ G(x_1, \dots, x_\alpha; w_1, \dots, w_\alpha), & r = 0. \end{cases} \end{aligned}$$

Also for $i \in \{1, \dots, n\}$ and $p \in \{1, \dots, k, k+1, \dots, n\}$ we define

$$\begin{aligned} A_{p,n} &= A(x_{i_1}, \dots, x_{i_p}; w_{i_1}, \dots, w_{i_p}), \\ G_{p,n} &= G(x_{i_1}, \dots, x_{i_p}; w_{i_1}, \dots, w_{i_p}), \\ H_{p,n} &= H(x_{i_1}, \dots, x_{i_p}; w_{i_1}, \dots, w_{i_p}), \\ M_{p,n}^{[r]} &= M^{[r]}(x_{i_1}, \dots, x_{i_p}; w_{i_1}, \dots, w_{i_p}). \end{aligned}$$

For $x_{i_p} \mapsto (1 - x_{i_p})$, $a \mapsto (1 - a)$ and $b \mapsto (1 - b)$ we put

$$\begin{aligned} A'_{p,n} &= A(1 - x_{i_1}, \dots, 1 - x_{i_p}; w_{i_1}, \dots, w_{i_p}), \\ G'_{p,n} &= G(1 - x_{i_1}, \dots, 1 - x_{i_p}; w_{i_1}, \dots, w_{i_p}), \\ H'_{p,n} &= H(1 - x_{i_1}, \dots, 1 - x_{i_p}; w_{i_1}, \dots, w_{i_p}). \end{aligned}$$

Clearly for $n = p$,

$$\begin{aligned} A_{n,n} &= a + b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i = A_n, \\ G_{n,n} &= \frac{ab}{\left(\prod_{i=1}^n x_i^{w_i}\right)^{\frac{1}{W_n}}} = G_n, \\ H_{n,n} &= \left(a^{-1} + b^{-1} - \frac{1}{W_n} \sum_{i=1}^n w_i \frac{1}{x_i}\right)^{-1} = H_n, \\ M_{n,n}^{[r]} &= \begin{cases} \left(a^r + b^r - \frac{1}{W_n} \sum_{i=1}^n w_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ G_{n,n}, & r = 0, \end{cases} = M^{[n]}, \\ A'_{n,n} &= (1 - a) + (1 - b) - \frac{1}{W_n} \sum_{i=1}^n w_i (1 - x_i) = A'_n, \\ G'_{n,n} &= \frac{(1 - a)(1 - b)}{\left(\prod_{i=1}^n (1 - x_i)^{w_i}\right)^{\frac{1}{W_n}}} = G'_n, \\ H'_{n,n} &= \left((1 - a)^{-1} + (1 - b)^{-1} - \frac{1}{W_n} \sum_{i=1}^n w_i \frac{1}{(1 - x_i)}\right)^{-1} = H'_n. \end{aligned}$$

We now introduce mixed symmetric means as follows:

$$M_{k,n}^{[s,r]} = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{l=1}^k w_{i_l}\right) \left(M_{k,n}^{[r]}\right)^s\right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} M_{k,n}^{[r]}\right)^{\frac{1}{\binom{n-1}{k-1} W_n}}, & s = 0. \end{cases}$$

Under the assumptions of Theorem 2.1 here we state a couple of applications starting with the following refinement series of the arithmetic-geometric and Ky Fan inequalities (see [7] and reference therein).

THEOREM 4.1.

(i)

$$\begin{aligned} A_n &\leq \cdots \leq \left(\prod_{1 \leq i_1 < \cdots < i_{k+1} \leq n} (A_{k+1,n})^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}} \\ &\leq \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} (A_{k,n})^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}} \\ &\leq \cdots \leq \left(\prod_{i=1}^n (A_{1,n})^{w_i} \right)^{\frac{1}{W_n}} \leq G_n. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{A'_n}{A_n} &\leq \cdots \leq \left(\prod_{1 \leq i_1 < \cdots < i_{k+1} \leq n} \left(\frac{A'_{k+1,n}}{A_{k+1,n}} \right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}} \\ &\leq \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \left(\frac{A'_{k,n}}{A_{k,n}} \right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}} \\ &\leq \cdots \leq \left(\prod_{i=1}^n \left(\frac{A'_{1,n}}{A_{1,n}} \right)^{w_i} \right)^{\frac{1}{W_n}} \leq \frac{G'_n}{G_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

PROOF.

(i) By applying convex function $f(x) = -\ln(x)$, to Corollary 2.2 we obtain required result.

(ii) For $x \in (0, \frac{1}{2}]$, applying convex function $f(x) = \ln(\frac{1-x}{x})$ to the Corollary 2.2 we get

$$\begin{aligned} \ln \left(\frac{1-a-b + \frac{1}{W_n} \sum_{i=1}^n w_i x_i}{a+b - \frac{1}{W_n} \sum_{i=1}^n w_i x_i} \right) &\leq \cdots \leq \\ \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} &W_{i_{k+1}} \ln \left(\frac{1-a-b + \frac{1}{W_{i_{k+1}}} \sum_{l=1}^{k+1} w_{i_l} x_{i_l}}{a+b - \frac{1}{W_{i_{k+1}}} \sum_{l=1}^{k+1} w_{i_l} x_{i_l}} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\binom{n-1}{k-1}W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} W_{i_k} \ln \left(\frac{1-a-b + \frac{1}{W_{i_k}} \sum_{l=1}^k w_{i_l} x_{i_l}}{a+b - \frac{1}{W_{i_k}} \sum_{l=1}^k w_{i_l} x_{i_l}} \right) \\
&\leq \dots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \ln \left(\frac{1-a-b+x_i}{a+b-x_i} \right) \\
&\leq \ln \left(\frac{1-a}{a} \right) + \ln \left(\frac{1-b}{b} \right) - \frac{1}{W_n} \sum_{i=1}^n w_i \ln \left(\frac{1-x_i}{x_i} \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\ln \left(\frac{A'_n}{A_n} \right) &\leq \dots \leq \ln \left(\prod_{1 \leq i_{k+1} < \dots < i_1 \leq n} \left(\frac{A'_{k+1}}{A_{k+1}} \right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k}W_n}} \\
&\leq \ln \left(\prod_{1 \leq i_k < \dots < i_1 \leq n} \left(\frac{A'_k}{A_k} \right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1}W_n}} \\
&\leq \dots \leq \ln \left(\prod_{i=1}^n \left(\frac{A'_{1,n}}{A_{1,n}} \right)^{w_i} \right)^{\frac{1}{W_n}} \leq \ln \left(\frac{G'_n}{G_n} \right).
\end{aligned}$$

Finally, by taking exponential, we deduced our result. \square

The refinements series of the variant of arithmetic-geometric inequality is given as:

THEOREM 4.2.

$$\begin{aligned}
\frac{G_n}{G_n + G'_n} &\leq \dots \leq \frac{1}{\binom{n-1}{k}W_n} \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{G_{k+1,n}}{G'_{k+1,n} + G_{k+1,n}} \right) \\
&\leq \frac{1}{\binom{n-1}{k-1}W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} W_{i_k} \left(\frac{G_{k,n}}{G'_{k,n} + G_{k,n}} \right) \\
&\leq \dots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{G_{1,n}}{G'_{1,n} + G_{1,n}} \right) \leq A_n,
\end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \dots + w_{i_{k+1}}$.

PROOF. By applying the convex function $f(x) = \frac{1}{1+\exp x}$ to Corollary 2.2 and replacing a by $\ln \frac{1-a}{a}$, b by $\ln \frac{1-b}{b}$ and x_{i_l} by $\ln \frac{1-x_{i_l}}{x_{i_l}}$ we obtain the result. \square

Now, we present refinement series of the arithmetic and harmonic mean as follow:

THEOREM 4.3.

(i)

$$\begin{aligned} \frac{1}{A_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{A_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{1}{A_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{A_{1,n}} \right) \leq \frac{1}{H_n}. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{1}{A'_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{A'_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{1}{A'_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{A'_{1,n}} \right) \leq \frac{1}{H'_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

PROOF.

(i) By applying convex function $f(x) = \frac{1}{x}$ to Corollary 2.2 we obtain required result.

(ii) By applying convex function $f(x) = \frac{1}{1-x}$, $x \in (0, \frac{1}{2}]$ to Corollary 2.2 we obtain required result. \square

In the following theorem we establish a refinement series of the difference of the arithmetic and harmonic mean.

THEOREM 4.4.

$$\begin{aligned} \frac{1}{A_n} - \frac{1}{A'_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{A_{k+1,n}} - \frac{1}{A'_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{1}{A_{k,n}} - \frac{1}{A'_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{A_{1,n}} - \frac{1}{A'_{1,n}} \right) \leq \frac{1}{H_n} - \frac{1}{H'_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \dots + w_{i_{k+1}}$.

PROOF. By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Corollary 2.2 we obtain required result. \square

We now prove some results in terms of mixed symmetric means.

THEOREM 4.5. *Let $r, s \in \mathbb{R}$ such that $s \leq r$. Then we have*

$$(4.1) \quad M^{[s]} = M_{n,n}^{[r,s]} \leq \dots \leq M_{k+1,n}^{[r,s]} \leq M_{k,n}^{[r,s]} \leq \dots \leq M_{1,n}^{[r,s]} \leq M^{[r]}.$$

$$(4.2) \quad M^{[s]} \leq M_{1,n}^{[s,r]} \leq \dots \leq M_{k,n}^{[s,r]} \leq M_{k+1,n}^{[s,r]} \leq \dots \leq M_{n,n}^{[s,r]} = M^{[r]}.$$

PROOF. Let $r, s \neq 0$. By applying the function $f(x) = x^{\frac{s}{r}}$, in the Corollary 2.2 and replacing a, b and x_{i_l} by a^r, b^r and $(x_{i_l})^r$ respectively, and then raising the power $\frac{1}{s}$, we get (4.2). Similarly, using the function $f(x) = x^{\frac{r}{s}}$ in the Corollary 2.2 and replacing a, b and x_{i_l} by a^s, b^s and $(x_{i_l})^s$ respectively then raising the power $\frac{1}{r}$, we get (4.1). For $s = 0$ or $r = 0$, we obtain the required result by taking limit. \square

Let $\phi, \psi : J \rightarrow \mathbb{R}$ be continuous strictly monotonic functions. We define the quasi-arithmetic means with respect to Theorem 2.1, as follow:

$$(4.3) \quad \hat{M}_{k,n}^{[\phi,\psi]} = \phi^{-1} \left[\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{l=1}^k w_{i_l} \right) \times (\phi \circ \psi^{-1}) \left(\psi(a) + \psi(b) - \frac{\sum_{l=1}^k w_{i_l} \psi(x_{i_l})}{\sum_{l=1}^k w_{i_l}} \right) \right],$$

where $\phi \circ \psi^{-1}$ is convex function.

COROLLARY 4.6. *If we define a continuous and strictly monotonic function $\varphi : J \rightarrow \mathbb{R}$ as*

$$\hat{M}^{[\varphi]} = \varphi^{-1} \left[\varphi(a) + \varphi(b) - \frac{1}{W_n} \sum_{i=1}^n w_i \varphi(x_i) \right],$$

then the following monotonicity of generalized quasi-arithmetic means holds

$$(4.4) \quad \hat{M}^{[\phi]} \geq \hat{M}_{1,n}^{[\phi,\psi]} \geq \dots \geq \hat{M}_{k,n}^{[\phi,\psi]} \geq \hat{M}_{k+1,n}^{[\phi,\psi]} \geq \dots \geq \hat{M}_{n,n}^{[\phi,\psi]} = \hat{M}^{[\psi]}.$$

PROOF. Setting $f \mapsto \phi \circ \psi^{-1}$ and replacing a, b and x_{i_l} by $\psi(a), \psi(b)$ and $\psi(x_{i_l})$ respectively in Corollary 2.2 and then applying ϕ^{-1} , we get (4.4). \square

4.2. *For Niezgoda type extension of Gabler inequality.* For $[a, b] \subset J$, $0 < a < b$ and positive weights w_i for $i \in \{1, \dots, \alpha\}$, where $\alpha \in \{1, \dots, n\}$, we define the following (generalized) arithmetic, geometric and harmonic means along with power mean of the order $r \in \mathbb{R}$ for all $x_{ij} \in [a, b]$, $j \in \{1, \dots, m\}$:

$$\tilde{A}(x_{1j}, \dots, x_{\alpha j}; w_1, \dots, w_\alpha) = \sum_{j=1}^m a_j - \frac{1}{W_\alpha} \sum_{j=1}^{m-1} \sum_{i=1}^{\alpha} w_i x_{ij},$$

$$\tilde{G}(x_{1j}, \dots, x_{\alpha j}; w_1, \dots, w_\alpha) = \frac{\prod_{j=1}^m a_j}{\left(\prod_{j=1}^{m-1} \prod_{i=1}^{\alpha} x_{ij}^{w_i} \right)^{\frac{1}{W_\alpha}}},$$

$$\tilde{H}(x_{1j}, \dots, x_{\alpha j}; w_1, \dots, w_\alpha) = \left(\sum_{j=1}^m a_j^{-1} - \frac{1}{W_\alpha} \sum_{j=1}^{m-1} \sum_{i=1}^{\alpha} w_i \frac{1}{x_{ij}} \right)^{-1},$$

$$\tilde{M}^{[r]}(x_{1j}, \dots, x_{\alpha j}; w_1, \dots, w_\alpha) = \begin{cases} \left(\sum_{j=1}^m (a_j)^r - \frac{1}{W_\alpha} \sum_{j=1}^{m-1} \sum_{i=1}^{\alpha} w_i (x_{ij})^r \right)^{\frac{1}{r}}, & r \neq 0, \\ G(x_{1j}, \dots, x_{\alpha j}; w_1, \dots, w_\alpha), & r = 0. \end{cases}$$

Also for $i \in \{1, \dots, n\}$ and $p \in \{1, \dots, k+1, k, \dots, n\}$, we define

$$\begin{aligned} \tilde{A}_{p,n} &= \tilde{A}(x_{i_1j}, \dots, x_{i_pj}; w_{i_1}, \dots, w_{i_p}), \\ \tilde{G}_{p,n} &= \tilde{G}(x_{i_1j}, \dots, x_{i_pj}; w_{i_1}, \dots, w_{i_p}), \\ \tilde{H}_{p,n} &= \tilde{H}(x_{i_1j}, \dots, x_{i_pj}; w_{i_1}, \dots, w_{i_p}), \\ \tilde{M}_{p,n}^{[r]} &= \tilde{M}_{[r]}(x_{i_1j}, \dots, x_{i_pj}; w_{i_1}, \dots, w_{i_p}). \end{aligned}$$

For $x_{i_pj} \mapsto (1 - x_{i_pj})$ and $a_j \mapsto (1 - a_j)$ we define following means

$$\begin{aligned} \tilde{A}'_{p,n} &= \tilde{A}(1 - x_{i_1j}, \dots, 1 - x_{i_pj}; w_{i_1}, \dots, w_{i_p}), \\ \tilde{G}'_{p,n} &= \tilde{G}(1 - x_{i_1j}, \dots, 1 - x_{i_pj}; w_{i_1}, \dots, w_{i_p}), \\ \tilde{H}'_{p,n} &= \tilde{H}(1 - x_{i_1j}, \dots, 1 - x_{i_pj}; w_{i_1}, \dots, w_{i_p}). \end{aligned}$$

Clearly for $n = p$,

$$\begin{aligned} \tilde{A}_{n,n} &= \sum_{j=1}^m a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i x_{ij} = \tilde{A}_n, \\ \tilde{G}_{n,n} &= \frac{\prod_{j=1}^m a_j}{\left(\prod_{j=1}^{m-1} \prod_{i=1}^n x_{ij}^{w_i}\right)^{\frac{1}{W_n}}} = \tilde{G}_n, \\ \tilde{H}_{n,n} &= \left(\sum_{j=1}^m a_j^{-1} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i \frac{1}{x_{ij}}\right)^{-1} = \tilde{H}_n, \\ \tilde{M}_{n,n}^{[r]} &= \begin{cases} \left(\sum_{j=1}^m (a_j)^r - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i (x_{ij})^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \tilde{G}_{n,n}, & r = 0, \end{cases} \\ &= \tilde{M}^{[n]}, \\ \tilde{A}'_{n,n} &= \sum_{j=1}^m (1 - a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i (1 - x_{ij}) = \tilde{A}'_n, \\ \tilde{G}'_{n,n} &= \frac{\prod_{j=1}^m (1 - a_j)}{\left(\prod_{j=1}^{m-1} \prod_{i=1}^n (1 - x_{ij})^{w_i}\right)^{\frac{1}{W_n}}} = \tilde{G}'_n, \\ \tilde{H}'_{n,n} &= \left(\sum_{j=1}^m (1 - a_j)^{-1} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i \frac{1}{(1 - x_{ij})}\right)^{-1} = \tilde{H}'_n. \end{aligned}$$

We now introduce mixed symmetric means as follows:

$$\tilde{M}_{k,n}^{[s,r]} = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{l=1}^k w_{i_l}\right) \left(\tilde{M}_{k,n}^{[r]}\right)^s\right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \tilde{M}_{k,n}^{[r]}\right)^{\frac{1}{\binom{n-1}{k-1} W_n}}, & s = 0. \end{cases}$$

Under the assumptions of Theorem 3.1, here we state a couple of applications starting with the following refinement series of the arithmetic-geometric and Ky Fan inequalities.

THEOREM 4.7.

(i)

$$\begin{aligned} \tilde{A}_n &\leq \dots \leq \left(\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n} (\tilde{A}_{k+1,n})^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} w_n}} \\ &\leq \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} (\tilde{A}_{k,n})^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} w_n}} \\ &\leq \dots \leq \left(\prod_{i=1}^n (\tilde{A}_{1,n})^{w_i} \right)^{\frac{1}{w_n}} \leq \tilde{G}_n. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{\tilde{A}'_n}{\tilde{A}_n} &\leq \dots \leq \left(\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(\frac{\tilde{A}'_{k+1,n}}{\tilde{A}_{k+1,n}} \right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} w_n}} \\ &\leq \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(\frac{\tilde{A}'_{k,n}}{\tilde{A}_{k,n}} \right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} w_n}} \\ &\leq \dots \leq \left(\prod_{i=1}^n \left(\frac{\tilde{A}'_{1,n}}{\tilde{A}_{1,n}} \right)^{w_i} \right)^{\frac{1}{w_n}} \leq \frac{\tilde{G}'_n}{\tilde{G}_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \dots + w_{i_{k+1}}$.

PROOF.

(i) By applying convex function $f(x) = -\ln(x)$ to Corollary 3.2, we obtain required result.

(ii) For $x \in (0, \frac{1}{2}]$ and $a_j < 1$ for all $j \in \{1, \dots, m\}$, applying convex function $f(x) = \ln\left(\frac{1-x}{x}\right)$ and adopting the technique of Theorem 4.1, we get the proof. \square

The refinements series of the variant of arithmetic-geometric inequality is given as:

THEOREM 4.8.

$$\begin{aligned} \frac{\tilde{G}_n}{\tilde{G}_n + \tilde{G}'_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{\tilde{G}_{k+1,n}}{\tilde{G}_{k+1,n} + \tilde{G}'_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{\tilde{G}_{k,n}}{\tilde{G}_{k,n} + \tilde{G}'_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{\tilde{G}_{1,n}}{\tilde{G}_{1,n} + \tilde{G}'_{1,n}} \right) \leq \tilde{A}_n, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

PROOF. By applying the convex function $f(x) = \frac{1}{1+\exp x}$ with $a_j < 1$ for all $j \in \{1, \dots, m\}$, to Corollary 3.2 and replacing a_j by $\ln \frac{1-a_j}{a_j}$ and x_{i_j} by $\ln \frac{1-x_{i_j}}{x_{i_j}}$, we obtain the result. \square

Now, we present refinement series of the arithmetic and harmonic mean as follow:

THEOREM 4.9.

(i)

$$\begin{aligned} \frac{1}{\tilde{A}_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{\tilde{A}_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{1}{\tilde{A}_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{\tilde{A}_{1,n}} \right) \leq \frac{1}{\tilde{H}_n}. \end{aligned}$$

(ii)

$$\begin{aligned} \frac{1}{\tilde{A}'_n} &\leq \cdots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \cdots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{\tilde{A}'_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} W_{i_k} \left(\frac{1}{\tilde{A}'_{k,n}} \right) \\ &\leq \cdots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{\tilde{A}'_{1,n}} \right) \leq \frac{1}{\tilde{H}'_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

PROOF.

(i) By applying convex function $f(x) = \frac{1}{x}$ to Corollary 3.2 we obtain required result.

(ii) By applying convex function $f(x) = \frac{1}{1-x}$ with $x \in (0, \frac{1}{2}]$ and $a_j < 1$ for all $j \in \{1, \dots, m\}$, to Corollary 3.2 we obtain required result. \square

In the following theorem we establish a refinement series of the difference of the arithmetic and harmonic mean.

THEOREM 4.10.

$$\begin{aligned} \frac{1}{\tilde{A}_n} - \frac{1}{\tilde{A}'_n} &\leq \dots \leq \frac{1}{\binom{n-1}{k} W_n} \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} W_{i_{k+1}} \left(\frac{1}{\tilde{A}_{k+1,n}} - \frac{1}{\tilde{A}'_{k+1,n}} \right) \\ &\leq \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} W_{i_k} \left(\frac{1}{\tilde{A}_{k,n}} - \frac{1}{\tilde{A}'_{k,n}} \right) \\ &\leq \dots \leq \frac{1}{W_n} \sum_{i=1}^n w_i \left(\frac{1}{\tilde{A}_{1,n}} - \frac{1}{\tilde{A}'_{1,n}} \right) \leq \frac{1}{\tilde{H}_n} - \frac{1}{\tilde{H}'_n}, \end{aligned}$$

where $W_{i_{k+1}} = w_{i_1} + \dots + w_{i_{k+1}}$.

PROOF. By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}$ with $x \in (0, \frac{1}{2}]$ and $a_j < 1$ for all $j \in \{1, \dots, m\}$, to Corollary 3.2 we obtain required result. \square

We now prove some results in terms of mixed symmetric means:

THEOREM 4.11. *Let $r, s \in \mathbb{R}$ such that $s \leq r$. Then we have*

$$(4.5) \quad \tilde{M}^{[s]} = \tilde{M}_{n,n}^{[r,s]} \leq \dots \leq \tilde{M}_{k+1,n}^{[r,s]} \leq \tilde{M}_{k,n}^{[r,s]} \leq \dots \leq \tilde{M}_{1,n}^{[r,s]} \leq \tilde{M}^{[r]}.$$

$$(4.6) \quad \tilde{M}^{[s]} \leq \tilde{M}_{1,n}^{[s,r]} \leq \dots \leq \tilde{M}_{k,n}^{[s,r]} \leq \tilde{M}_{k+1,n}^{[s,r]} \leq \dots \leq \tilde{M}_{n,n}^{[s,r]} = \tilde{M}^{[r]}.$$

PROOF. Let $r, s \neq 0$. By applying the function $f(x) = x^{\frac{s}{r}}$, in the Corollary 3.2 and replacing a_j and $x_{i_{ij}}$ by $(a_j)^r$ and $(x_{i_{ij}})^r$ respectively, and then raising the power $\frac{1}{s}$, we get (4.6). Similarly, using the function $f(x) = x^{\frac{r}{s}}$ in the Corollary 3.2 and replacing a_j and $x_{i_{ij}}$ by $(a_j)^s$ and $(x_{i_{ij}})^s$ respectively then raising the power $\frac{1}{r}$, we get (4.5). For $s = 0$ or $r = 0$, we obtain the required result by taking limit. \square

Let $\phi, \psi : J \rightarrow \mathbb{R}$ be continuous strictly monotonic functions. We define the quasi-arithmetic means with respect to Theorem 3.1, as follow:

$$\begin{aligned} \hat{M}_{k,n}^{[\phi,\psi]} &= \phi^{-1} \left[\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{l=1}^k w_{i_l} \right) \right. \\ &\quad \left. \times (\phi \circ \psi^{-1}) \left(\psi \left(\sum_{j=1}^m a_j \right) - \frac{\sum_{j=1}^{m-1} \sum_{l=1}^k w_{i_l} \psi(x_{i_{lj}})}{\sum_{l=1}^k w_{i_l}} \right) \right], \end{aligned}$$

where $\phi \circ \psi^{-1}$ is convex function.

COROLLARY 4.12. *If we define a continuous and strictly monotonic function $\varphi : J \rightarrow \mathbb{R}$ as*

$$\hat{M}^{[\varphi]} = \varphi^{-1} \left[\varphi \left(\sum_{j=1}^m a_j \right) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^n w_i \varphi(x_{ij}) \right],$$

then following monotonicity of generalized quasi-arithmetic means holds

$$(4.7) \quad \hat{M}^{[\phi]} \geq \hat{M}_{1,n}^{[\phi,\psi]} \geq \dots \geq \hat{M}_{k,n}^{[\phi,\psi]} \geq \hat{M}_{k+1,n}^{[\phi,\psi]} \geq \dots \geq \hat{M}_{n,n}^{[\phi,\psi]} = \hat{M}^{[\psi]}.$$

PROOF. Setting $f \mapsto \phi \circ \psi^{-1}$ and replacing a_j and x_{ij} by $\psi(a_j)$ and $\psi(x_{ij})$ respectively in Corollary 3.2 and then applying ϕ^{-1} we get (4.7). \square

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O varijanti i proširenju Gablerove nejednakosti

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