SOME INEQUALITIES ON TIME SCALES SIMILAR TO REVERSE HARDY'S INEQUALITY

Bouharket Benaissa

ABSTRACT. In this paper, we give new inverted Hardy inequalities on a general time scale by introducing two parameters and a new inequality with negative parameters. The main results are demonstrated using two new lemmas and an interesting proposition. We derive some new corollaries of continuous and discrete choice on time scale.

1. INTRODUCTION

Hardy inequality in the integral form plays an important role in the modern analysis of partial differential equations and is an indispensable tool in spectral theory of partial differential operators.

In 2012, Sulaiman presented a reverse Hardy's inequality [6, Theorem 3.1]: Let h be positive function defined on $[a, b]$, $H(x) = \int_a^x h(\tau)d\tau$, then

1. for $p \geq 1$,

$$p \int_a^b \left(\frac{H(x)}{x}\right)^p dx \leq (b-a)^p \int_a^b \left(\frac{h(x)}{x}\right)^p dx - \int_a^b (1 - \frac{a}{x}) h^p(x) dx,$$

2. for $0 < p < 1$,

$$p \int_a^b \left(\frac{H(x)}{x}\right)^p dx \geq (1 - \frac{a}{b})^p \int_a^b \left(\frac{h(x)}{x}\right)^p dx - \frac{1}{b^p} \int_a^b (x-a)^p h^p(x).$$

In 2013, B. Sroysang extends the inverse Hardy inequality [5, Theorem 2.1 and Theorem 2.2]: Let h be positive function defined on $[a, b]$, $H(x) = \int_a^x f(\tau)d\tau$ and $q > 0$, then

1. for $p \geq 1$,

$$p \int_a^b \frac{H(x)^p}{x^q} dx \leq (b-a)^p \int_a^b \frac{h(x)^p}{x^q} dx - \int_a^b \frac{(x-a)^p}{x^q} h^p(x) dx,$$

2. for $0 < p < 1$,

$$p \int_a^b \frac{H(x)^p}{x^q} dx \geq \frac{(b-a)^p}{b^q} \int_a^b h^p(x) dx - \frac{1}{b^q} \int_a^b (x-a)^p h^p(x).$$

In 2018, B. Benaissa gives a further generalization [2, Theorem 2.2]. Let h, ϕ be positive functions defined on $[a, b]$ and $H(x) = \int_a^x h(\tau) d\tau$. If ϕ is non-decreasing then

(i) for $p \geq 1$,

$$(1.1) \qquad p \int_a^b \frac{H(x)^p}{\phi(x)} dx \leq (b-a)^p \int_a^b \frac{h(x)^p}{\phi(x)} dx - \int_a^b \frac{(x-a)^p}{\phi(x)} h^p(x) dx,$$

(ii) for $0 < p < 1$,

$$(1.2) \qquad p \int_a^b \frac{H(x)^p}{\phi(x)} dx \geq \frac{(b-a)^p}{\phi(b)} \int_a^b h^p(x) dx - \frac{1}{\phi(b)} \int_a^b (x-a)^p h^p(x) dx.$$

The aim of this work is to give a generalization for (1.1), (1.2) and to conclude new results using calculus on time scales.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with standard topology.

DEFINITION 2.1. *[3] Let $\mathbb{T}$ be a time scales. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by:*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

We put $\inf \emptyset = \sup \mathbb{T}$, $\quad \sup \emptyset = \inf \mathbb{T}$.

2.1. Delta derivative.

DEFINITION 2.2. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. We define $f^\Delta(t)$ to be the number, if it exists, defined as follows: for every $\epsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s| \text{ for all } s \in U, \ s \neq \sigma(t).$$

We call $f^\Delta(t)$ the delta derivative of f at t.

THEOREM 2.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following.*

1. *If f is Δ-differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

3. *If t is right-dense, then f is Δ-differentiable iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is Δ-differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

For $a, b \in \mathbb{T}$ and a delta differentiable function h, the Cauchy integral of h^Δ is defined by

$$\int_a^b h^\Delta(\tau)\Delta\tau = h(b) - h(a).$$

THEOREM 2.4. *[1, Theorem 1.1.2] Let $h, \phi \in C_{rd}(\mathbb{T}, \mathbb{R})$ be rd-continuous functions, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then, the following are true:*

1. $\displaystyle\int_a^b \left(\alpha h^\Delta(\tau) + \beta\phi^\Delta(\tau)\right)\Delta\tau = \alpha\int_a^b h^\Delta(\tau)\Delta\tau + \beta\int_a^b \phi^\Delta(\tau)\Delta\tau,$

2. $\displaystyle\int_a^b h^\Delta(\tau)\Delta\tau = -\int_b^a h^\Delta(\tau)\Delta\tau,$

3. $\displaystyle\int_a^c h^\Delta(\tau)\Delta\tau = \int_a^b h^\Delta(\tau)\Delta\tau + \int_b^c h^\Delta(\tau)\Delta\tau,$

4. $\displaystyle\left|\int_a^b h^\Delta(\tau)\Delta\tau\right| \leq \int_a^b \left|h^\Delta(\tau)\right|\Delta\tau.$

2.2. *Chain rule.*

THEOREM 2.5. *[3, Theorem 1.90] Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta = g^\Delta \int_0^1 f'\left(s\, g^\sigma + (1-s)g\right)ds,$$

holds.

LEMMA 2.6. *(Hölder's inequality) [1, Theorem 1.1.10; (1.1.8)] Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and h, $\phi \in C_{rd}(\mathbb{T}, \mathbb{R})$ be two positive functions. If $\frac{1}{p} + \frac{1}{p'} = 1$ with $p \geq 1$, then*

$$(2.1) \qquad \int_a^b h(\tau)\phi(\tau)\Delta\tau \leq \left(\int_a^b h^p(\tau)\Delta\tau \right)^{\frac{1}{p}} \left(\int_a^b \phi^{p'}(\tau)\Delta\tau \right)^{\frac{1}{p'}},$$

this inequality is reversed if $0 < p < 1$.

PROPOSITION 2.7. *[1, Fubini's Theorem] Let f be $\Delta\Delta$-integrable on $\mathbf{R} = [a, b) \times [c, d)$ and suppose that the single integral*

$$I(x) = \int_c^d f(x, y)\Delta_2 y,$$

exists for each $x \in [a, b)$ and the single integral

$$K(y) = \int_a^b f(x, y)\Delta_1 x,$$

exists for each $y \in [c, d)$. Then the iterated integrals

$$\int_a^b I(x)\Delta_1 x = \int_a^b \Delta_1 x \int_c^d f(x, y)\Delta_2 y$$

and

$$\int_c^d K(y)\Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x, y)\Delta_1 x,$$

exist and the equalities

(2.2)
$$\int\int_{\mathbf{R}} f(x, y)\Delta_1 x \Delta_2 y = \int_a^b \Delta_1 x \int_c^d f(x, y)\Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x, y)\Delta_1 x$$

hold.

In this paper, we study some inverted Hardy dynamic integral inequalities on time scales and we give new ones with a negative parameter, also extend some continuous inequalities and their discrete analogues. We assume that all the integrals of the right and left side of the inequalities are convergent.

3. MAIN RESULTS

We state the following lemmas which are useful in the proof of the main theorem. Firstly, we extended the lemma [4, 3.6, p.15] for $p \neq 0$.

LEMMA 3.1. *Let ψ be a nonnegative monotone function on $[a, b]_{\mathbb{T}}$. If ψ in non-decreasing on $[a, b]_{\mathbb{T}}$, then*

$$(3.1) \qquad \qquad for\ 1 \leq p < \infty:\ (\psi^p)^\Delta \geq p\,\psi^\Delta \psi^{p-1},$$

$$(3.2) \qquad\qquad for \ 0 < p < 1: \ (\psi^p)^\Delta \le p \, \psi^\Delta \psi^{p-1}.$$

If ψ is non-increasing function on $[a, b]_\mathbb{T}$, then

$$(3.3) \qquad\qquad for \ p < 0: \ (\psi^p)^\Delta \ge p \, \psi^\Delta \psi^{p-1},$$

PROOF. Applying the chain rule for $1 \le p < \infty$, we get

$$
\begin{aligned}
(\psi^p)^\Delta \ &= \ p\,\psi^\Delta \int_0^1 (s\,\psi^\sigma + (1-s)\psi)^{p-1}\, ds \\
&\ge \ p\,\psi^\Delta \int_0^1 (s\,\psi + (1-s)\psi)^{p-1}\, ds \\
&= \ p\,\psi^\Delta \psi^{p-1}.
\end{aligned}
$$

For $0 < p < 1$, since $p - 1 < 0$ we have

$$
\begin{aligned}
(\psi^p)^\Delta \ &\le \ p\,\psi^\Delta \int_0^1 (s\,\psi + (1-s)\psi)^{p-1}\, ds \\
&= \ p\,\psi^\Delta \psi^{p-1}.
\end{aligned}
$$

Since $p < 0$ and $\psi^\Delta \le 0$, then

$$
\begin{aligned}
(\psi^p)^\Delta \ &\ge \ p\,\psi^\Delta \int_0^1 (s\,\psi + (1-s)\psi)^{p-1}\, ds \\
&= \ p\,\psi^\Delta \psi^{p-1}.
\end{aligned}
$$

$\square$

LEMMA 3.2. *Let $0 < p \le q < \infty$ and h, ν are nonnegative and rd-continuous functions on $[a, b]_\mathbb{T}$ and we suppose that $0 < \int_a^b h^q(\tau)\nu(\tau)\Delta\tau < \infty$, then*

$$(3.4) \qquad \int_a^b h^p(\tau)\nu(\tau)\Delta\tau \le \left(\int_a^b \nu(\tau)\Delta\tau \right)^{\frac{q-p}{q}} \left(\int_a^b h^q(\tau)\nu(\tau)\Delta\tau \right)^{\frac{p}{q}}.$$

The inequality (3.4) hold for $-\infty < q \le p < 0$ and inverted for $0 < q \le p < \infty$.

PROOF. By Hölder integral inequality (2.1) for using the parameter $\dfrac{q}{p} \ge 1$, we have

$$
\begin{aligned}
\int_a^b h^p(\tau)\nu(\tau)\Delta\tau \ &= \int_a^b \left(\nu^{\frac{q-p}{q}}(\tau) \right) \left(h^p(\tau)\nu^{\frac{p}{q}}(\tau) \right) \Delta\tau \\
&\le \left(\int_a^b \nu(\tau)\Delta\tau \right)^{\frac{q-p}{q}} \left(\int_a^b h^q(\tau)\nu(\tau)\Delta\tau \right)^{\frac{p}{q}}.
\end{aligned}
$$

$\square$

PROPOSITION 3.3. *Let $0 < B < A$ are two positive numbers, then*

$$(3.5) \qquad for \quad p \geq 1: \qquad (A - B)^p \leq A^p - B^p,$$

$$(3.6) \qquad for \quad 0 < p < 1: \quad (A - B)^p > A^p - B^p.$$

PROOF. We can write

$$for \ p \geq 1; \qquad 1 \leq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p,$$

$$for \ 0 < p < 1; \quad 1 \geq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p,$$

so for $p = 1$ we get equality.

Now we take $t = \dfrac{A}{A - B}$, then we get $t > 1$ and $B = A(1 - \dfrac{1}{t})$, thus

$$\left(\frac{A}{A - B}\right)^p - \left(\frac{B}{A - B}\right)^p = t^p - (t - 1)^p = g(t),$$

for all $t > 1$, we have

$$g'(t) = p\,t^p \left(1 - \left(\frac{t - 1}{t}\right)^{p-1}\right).$$

Since $0 < \frac{t-1}{t} < 1$, then

- for $p > 1$, g is increasing function, hence $t > 1 \Rightarrow g(t) \geq g(1) = 1$, consequently $1 \leq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p$.
- for $0 < p < 1$, g is decreasing function, hence $t > 1 \Rightarrow g(t) \leq g(1) = 1$, which is same as $1 \geq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p$.

$\square$

We can rewrite the above inequalities (3.5) and (3.6) in the following form

$$(3.7) \qquad for \quad p \geq 1: \qquad A - B \leq (A^p - B^p)^{\frac{1}{p}},$$

$$(3.8) \qquad for \quad 0 < p < 1: \qquad A - B \geq (A^p - B^p)^{\frac{1}{p}}.$$

3.1. *Reverse Hardy's inequality.* Now we give the dynamic reverse Hardy's inequality version in time scales.

THEOREM 3.4. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and h, ϕ be non-negative continuous functions on $[a, b]_{\mathbb{T}}$. Let*

$$H(\tau) = \int_a^\tau h(s)\Delta s.$$

If ϕ is increasing function and $(\sigma(s) - a)^\Delta = 1$, then
- *for $1 \leq p \leq q < \infty$:*

$$(3.9) \quad p \int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)} \Delta\tau \leq \left(\int_a^b \frac{1}{\phi(\sigma(s))} \Delta s \right)^{\frac{q-p}{q}}$$

$$\times \left(\int_a^b \frac{h^q(s) \left[(b-a)^q - (\sigma(s) - a)^q \right]}{\phi(\sigma(s))} \Delta s \right)^{\frac{p}{q}}$$

- *for $0 < q \leq p < 1$:*

$$(3.10)$$

$$p \int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)} \Delta\tau \geq \frac{(b-a)^{\frac{q-p}{q}}}{\phi(b)} \left(\int_a^b h^q(s) \left[(b-a)^q - (\sigma(s)-a)^q \right] \Delta s \right)^{\frac{p}{q}}.$$

PROOF. Let $p \geq 1$. Using Hölder inequality for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we get

$$H(\sigma(\tau)) = \int_a^{\sigma(\tau)} h(s)\Delta s \leq \left(\int_a^{\sigma(\tau)} h^p(s)\Delta s \right)^{\frac{1}{p}} \left(\int_a^{\sigma(\tau)} \Delta s \right)^{\frac{1}{p'}}$$

$$= \left(\int_a^{\sigma(\tau)} h^p(s)\Delta s \right)^{\frac{1}{p}} (\sigma(\tau) - a)^{\frac{p-1}{p}},$$

hence by using Fubini Theorem, we have

$$\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))} \Delta\tau = \int_a^b \frac{1}{\phi(\sigma(\tau))} \left(\int_a^{\sigma(\tau)} h(s)\Delta s \right)^p \Delta\tau$$

$$\leq \int_a^b \frac{1}{\phi(\sigma(\tau))} (\sigma(\tau) - a)^{p-1} \left(\int_a^{\sigma(\tau)} h^p(s)\Delta s \right) \Delta\tau$$

$$= \int_a^b \Delta\tau \int_a^{\sigma(\tau)} \frac{1}{\phi(\sigma(\tau))} (\sigma(\tau) - a)^{p-1} h^p(s)\Delta s$$

$$= \int_a^b h^p(s)\Delta s \left(\int_{\sigma(s)}^b \frac{1}{\phi(\sigma(\tau))} (\sigma(\tau) - a)^{p-1} \Delta\tau \right).$$

The hypothesis that ϕ is increasing yield

$$\forall \tau \in [\sigma(s), b] : \frac{1}{\phi(\sigma(\tau))} \leq \frac{1}{\phi(\tau)} \leq \frac{1}{\phi(\sigma(s))},$$

and this gives us that

$$(3.11) \qquad \int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))} \Delta\tau \leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))} \Delta s \left(\int_{\sigma(s)}^b (\sigma(\tau) - a)^{p-1} \Delta\tau \right).$$

Let $\psi(\tau) = \sigma(\tau) - a$, by applying (3.1) with the supposition that $\psi^\Delta = 1$, we get

$$(\sigma(\tau) - a)^{p-1} \leq \frac{1}{p}((\sigma(\tau) - a)^p)^\Delta.$$

Integrating on $[\sigma(s), b]_\mathbb{T}$, then

$$\int_{\sigma(s)}^b (\sigma(\tau) - a)^{p-1} \Delta\tau \quad \leq \quad \frac{1}{p} \int_{\sigma(s)}^b (\psi^p)^\Delta(\tau) \Delta t$$

$$= \quad \frac{1}{p} \left[(b-a)^p - (\sigma(s) - a)^p \right],$$

using the inequality (3.7) for $A = (b-a)^p$, $B = (\sigma(s) - a)^p$ and $\frac{q}{p} \geq 1$, we get

$$(b-a)^p - (\sigma(s) - a)^p \leq ((b-a)^q - (\sigma(s) - a)^q)^{\frac{p}{q}},$$

therefore

$$(3.12) \qquad \int_{\sigma(s)}^b (\sigma(\tau) - a)^{p-1} \Delta\tau \leq \frac{1}{p} \left[(b-a)^q - (\sigma(s) - a)^q \right]^{\frac{p}{q}}.$$

Putting (3.12) in (3.11) and applying (3.4), we get

$$p \int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))} \Delta\tau \quad \leq \quad \int_a^b \frac{h^p(s)}{\phi(\sigma(s))} \left[(b-a)^q - (\sigma(s) - a)^q \right]^{\frac{p}{q}} \Delta s$$

$$= \quad \int_a^b \frac{\left(h(s) \left[(b-a)^q - (\sigma(s) - a)^q \right]^{\frac{1}{q}} \right)^p}{\phi(\sigma(s))} \Delta s$$

$$\leq \quad \left(\int_a^b \frac{1}{\phi(\sigma(s))} \Delta s \right)^{\frac{q-p}{q}}$$

$$\times \quad \left(\int_a^b \frac{h^q(s) \left[(b-a)^q - (\sigma(s) - a)^q \right]}{\phi(\sigma(s))} \Delta s \right)^{\frac{p}{q}},$$

and this completes the proof.

(ii) for $0 < q \leq p < 1$, by using the reverse Hölder inequality and the assumption ϕ is increasing function on $[s, b]$ we get

$$\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))} \Delta\tau \geq \frac{1}{\phi(b)} \int_a^b h^p(s)\Delta s \left(\int_{\sigma(s)}^b (\sigma(\tau) - a)^{p-1} \Delta\tau \right).$$

Let $\psi(\tau) = \sigma(\tau) - a$. Applying (3.2) with the supposition that $\psi^\Delta = 1$, we get

$$(\sigma(\tau) - a)^{p-1} \geq \frac{1}{p}((\sigma(\tau) - a)^p)^\Delta$$

Integrating on $[\sigma(s), b]_{\mathbb{T}}$, we obtain

$$\int_{\sigma(s)}^b (\sigma(\tau) - a)^{p-1}\Delta\tau \quad \geq \quad \frac{1}{p}\left[(b - a)^p - (\sigma(s) - a)^p\right],$$

using the inequality (3.7) and we take $\nu = 1$, so the proof is similar to the proof of inequality (3.9). □

THEOREM 3.5. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and h, ϕ be non-negative continuous functions on $[a, b]_{\mathbb{T}}$. Let*

$$\widetilde{H}(\tau) = \int_\tau^b h(s)\Delta s.$$

If ϕ is decreasing function and $(b - \sigma(s))^\Delta = -1$, then
• for $-\infty < q \leq p < 0$:
(3.13)

$$-p\int_a^b \frac{(\widetilde{H}^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau \quad \leq \quad \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s \right)^{\frac{q-p}{q}}$$
$$\times \left(\int_a^b \frac{h^q(s)\left[(b - \sigma(s))^q - (b - a)^q\right]}{\phi(\sigma(s))}\Delta s \right)^{\frac{p}{q}}.$$

PROOF. Let $p < 0$. By using Hölder inequality, we have

$$\widetilde{H}(\sigma(\tau)) = \int_{\sigma(\tau)}^b h(s)\Delta s \quad \geq \quad \left(\int_{\sigma(\tau)}^b h^p(s)\Delta s \right)^{\frac{1}{p}} \left(\int_{\sigma(\tau)}^b \Delta s \right)^{\frac{1}{p'}}$$

$$= \quad \left(\int_{\sigma(\tau)}^b h^p(s)\Delta s \right)^{\frac{1}{p}} (b - \sigma(\tau))^{\frac{p-1}{p}}.$$

Now we apply Fubini theorem,

$$\int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \;=\; \int_a^b \frac{1}{\phi(\sigma(\tau))}\left(\int_{\sigma(\tau)}^b h(s)\Delta s\right)^p \Delta\tau$$

$$\leq\; \int_a^b \frac{1}{\phi(\sigma(\tau))}(b-\sigma(\tau))^{p-1}\left(\int_{\sigma(\tau)}^b h^p(s)\Delta s\right)\Delta\tau$$

$$=\; \int_a^b h^p(s)\Delta s\left(\int_a^{\sigma(s)} \frac{1}{\phi(\sigma(\tau))}(b-\sigma(\tau))^{p-1}\Delta\tau\right),$$

since ϕ is decreasing, this yields

$$\forall\tau\in[a,\sigma(s)]:\;\frac{1}{\phi(\tau)}\leq\frac{1}{\phi(\sigma(\tau))}\leq\frac{1}{\phi(\sigma(s))}.$$

Therefore

$$(3.14)\qquad \int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))}\Delta s\left(\int_a^{\sigma(s)}(b-\sigma(\tau))^{p-1}\Delta\tau\right).$$

Let $\psi(\tau)=b-\sigma(\tau)$. By applying (3.3) with the supposition that $\psi^\Delta = -1$, we get

$$(b-\sigma(\tau))^{p-1}\leq -\frac{1}{p}(b-(\sigma(\tau))^p)^\Delta.$$

Integrating on $[a,\sigma(s)]_{\mathbb{T}}$, then

$$\int_a^{\sigma(s)}(b-\sigma(\tau))^{p-1}\Delta\tau \;\leq\; -\frac{1}{p}\int_a^{\sigma(s)}(\psi^p)^\Delta(\tau)\Delta\tau$$

$$=\; -\frac{1}{p}\left[(b-\sigma(s))^p-(b-a)^p\right].$$

Using the inequality (3.7) for $A=(b-\sigma(s))^p$, $B=(b-a)^p$ and $\frac{q}{p}\geq 1$, we obtain

$$(b-\sigma(s))^p-(b-a)^p\leq\left[(b-\sigma(s))^q-(b-a)^q\right]^{\frac{p}{q}},$$

hence

$$(3.15)\qquad \int_a^{\sigma(s)}(b-\sigma(\tau))^{p-1}\Delta\tau\leq -\frac{1}{p}\left[(b-\sigma(s))^q-(b-a)^q\right]^{\frac{p}{q}}.$$

Putting (3.15) in (3.14) and applying (3.4), we get

$$-p \int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))} \Delta\tau \ \leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))} \left[(b-\sigma(s))^q - (b-a)^q\right]^{\frac{p}{q}} \Delta s$$

$$= \int_a^b \frac{\left(h(s) \left[b-\sigma(s))^q - (b-a)^q\right]^{\frac{1}{q}}\right)^p}{\phi(\sigma(s))} \Delta s$$

$$\leq \left(\int_a^b \frac{1}{\phi(\sigma(s))} \Delta s\right)^{\frac{q-p}{q}}$$

$$\times \left(\int_a^b \frac{h^q(s) \left[b-\sigma(s))^q - (b-a)^q\right]}{\phi(\sigma(s))} \Delta s\right)^{\frac{p}{q}},$$

and this completes the proof.

□

4. Applications

By taking $q = p$ in Theorem 3.4 and Theorem 3.5, we obtain the following Corollaries.

COROLLARY 4.1. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and suppose that $(\sigma(s) - a)^\Delta = 1$. Let h, ϕ be non-negative continuous functions on $[a, b]_\mathbb{T}$ and $H(\tau) = \int_a^\tau h(s)\Delta s$. If ϕ is increasing function, then*
• *for $1 \leq p < \infty$:*

$$(4.1) \qquad p \int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)} \Delta\tau \leq \int_a^b \frac{h^p(s) \left[(b-a)^p - (\sigma(s)-a)^p\right]}{\phi(\sigma(s))} \Delta s,$$

• *for $0 < p < 1$:*

$$(4.2) \qquad p \int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)} \Delta\tau \geq \frac{1}{\phi(b)} \int_a^b h^p(s) \left[(b-a)^p - (\sigma(s)-a)^p\right] \Delta s.$$

COROLLARY 4.2. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and suppose that $(b-\sigma(s))^\Delta = -1$. Let h, ϕ be non-negative continuous functions on $[a, b]_\mathbb{T}$ and $\widetilde{H}(\tau) = \int_\tau^b h(s)\Delta s$. If ϕ is decreasing function, then*
• *for $p < 0$:*

$$(4.3) \qquad -p \int_a^b \frac{(\widetilde{H}^\sigma(\tau))^p}{\phi^\sigma(\tau)} \Delta\tau \ \leq \int_a^b \frac{h^p(s) \left[(b-\sigma(s))^p - (b-a)^p\right]}{\phi(\sigma(s))} \Delta s.$$

If we put $\mathbb{T} = \mathbb{R}$ in Theorem 3.4 and Theorem 3.5, we get the following Corollaries.

COROLLARY 4.3. *Let $a, b \in \mathbb{R}$ with $a < b$ and h, ϕ be non-negative contin-uous functions on $[a, b]$, let $H(\tau) = \int_a^\tau h(s)d\,s$. If ϕ is increasing function, then*

• *for $1 \le p \le q < \infty$:*

(4.4)
$$p \int_a^b \frac{(H(\tau))^p}{\phi(\tau)} d\,\tau \le \left(\int_a^b \frac{1}{\phi(s)} d\,\tau \right)^{\frac{q-p}{q}} \left(\int_a^b \frac{h^q(\tau)\left[(b-a)^q - (\tau-a)^q\right]}{\phi(\tau)} d\,\tau \right)^{\frac{p}{q}},$$

• *for $0 < q \le p < 1$:*

(4.5)
$$p \int_a^b \frac{(H(\tau))^p}{\phi(\tau)} d\,\tau \ge \frac{(b-a)^{\frac{q-p}{q}}}{\phi(b)} \left(\int_a^b h^q(\tau)\left[(b-a)^q - (\tau-a)^q\right] d\,\tau \right)^{\frac{p}{q}}.$$

The inequalities (4.4) and (4.5) are the new generalizations of the reverse Hardy inequalities.

REMARK 4.4. By taking $q = p$, then Corollary 4.3 also coincides with Theorem 2.2 in [2].

COROLLARY 4.5. *Let $a, b \in \mathbb{R}$ with $a < b$ and h, ϕ be non-negative contin-uous functions on $[a, b]$. Let $\widetilde{H}(\tau) = \int_\tau^b h(s)\Delta s$. If ϕ is decreasing function, then*

• *for $-\infty < q \le p < 0$:*

(4.6)
$$-p \int_a^b \frac{(\widetilde{H}(\tau))^p}{\phi(\tau)} d\,\tau \le \left(\int_a^b \frac{1}{\phi(s)} d\,s \right)^{\frac{q-p}{q}} \left(\int_a^b \frac{h^q(s)\left[(b-s)^q - (b-a)^q\right]}{\phi(s)} d\,s \right)^{\frac{p}{q}}.$$

REMARK 4.6. By putting $q = p$ in Corollary 4.5, we get for $p < 0$:

(4.7)
$$\int_a^b \frac{(\widetilde{H}(\tau))^p}{\phi(\tau)} d\,\tau \le -\frac{1}{p} \int_a^b \frac{h^p(s)\left[(b-s)^p - (b-a)^p\right]}{\phi(s)} d\,s.$$

The inequalities (4.6) and (4.7) are the new inverse Hardy inequalities for negative parameters.

If we put $\mathbb{T} = \mathbb{Z}$ in Theorem 3.4 and Theorem 3.5, we get the following corollary.

COROLLARY 4.7. *Let $\{u_j\}, \{U_j\}, \{V_j\}$ and $\{w_j\}$ for $j = 0, 1, 2, ..., n, \ n \in \mathbb{N}^\star$ be positive sequences of real numbers where $U_j = \sum_{i=0}^{j-1} u_i$ and $V_j = \sum_{i=j}^{n-1} u_i$. If $\{w_j\}$ is increasing, then*

• *for $1 \leq p \leq q < \infty$;*

$$(4.8) \qquad p \sum_{j=0}^{n-1} \frac{U_j^p}{w_j} \leq \left(\sum_{j=0}^{n-1} \frac{1}{w_j} \right)^{\frac{q-p}{q}} \left(\sum_{j=0}^{n-1} \frac{u_j^q (n^q - j^q)}{w_j} \right)^{\frac{p}{q}},$$

• *for $0 < q \leq p < 1$:*

$$(4.9) \qquad p \sum_{j=0}^{n-1} \frac{U_j^p}{w_j} \geq \frac{n^{\frac{q-p}{q}}}{w_{n-1}} \left(\sum_{j=0}^{n-1} u_j^q (n^q - j^q) \right)^{\frac{p}{q}}.$$

If $\{w_j\}$ is decreasing, then for $-\infty < q \leq p < 0$;

$$(4.10) \qquad -p \sum_{j=0}^{n-1} \frac{V_j^p}{w_j} \leq \left(\sum_{j=0}^{n-1} \frac{1}{w_j} \right)^{\frac{q-p}{q}} \left(\sum_{j=0}^{n-1} \frac{u_j^q ((n-j)^q - n^q)}{w_j} \right)^{\frac{p}{q}}.$$

The inequalities (4.8), (4.9) and (4.10) are the new ones of the reverse Hardy inequalities in the discrete form.

REFERENCES

[1] R. P. Agarwal, D. O'Regan and S. H. Saker, Hardy Type Inequalities on Time Scales, Springer, Cham, 2016.

[2] B. Benaissa, *More on reverses of Minkowski's inequalities and Hardy's integral inequalities,* Asian-Eur. J. Math. **13(3)** (2020), 2050064.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales. An Introduction with Applications, Birkhäuser, Boston, 2001.

[4] M. Bohner and S. H. Saker, *Gehring inequalities on time scales,* J. Computational Analysis and Applications **28** (2020), 11–23.

[5] B. Sroysang, *A generalization of some integral inequalities similar to Hardy's inequality,* Math. Æterna **3** (2013), 593–596.

[6] W. T. Sulaiman, *Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities,* Int. J. Mod. Math. Sci. **1** (2012), 14–24.

Neke nejednakosti na vremenskim skalama slične obrnutoj Hardyjevoj nejednakosti

Bouharket Benaissa

SAŽETAK. U ovom članku dajemo nove obrnute Hardyjeve nejednakosti na općoj vremenskoj skali uvođenjem dva parametra i novu nejednakost s negativnim parametrima. Glavni rezultati demonstrirani su pomoću dvije nove leme i zanimljive propozicije. Izvodimo neke nove korolare kontinuiranog i diskretnog izbora na vremenskoj skali.

Bouharket Benaissa
Faculty of Material Sciences, University of Tiaret-Algeria
Laboratory of Informatics and Mathematics
University of Tiaret, Algeria
E-mail: bouharket.benaissa@univ-tiaret.dz