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SOME NEW GRONWALL-BIHARI TYPE INEQUALITIES ASSOCIATED WITH GENERALIZED FRACTIONAL OPERATORS AND APPLICATIONS

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Abstract. In this paper, we derive some generalizations of certain Gronwall-Bihari type inequality for generalized fractional operators unifying Riemann-Liouville and Hadamard fractional operators for functions in one variable, which provide explicit bounds on unknown functions.To show the feasibility of the obtained inequalities, two illustrative examples are also introduced.

1. Introduction and preliminaries

It is well known that the Gronwall-Bellman inequality [2, 11] and their generalizations can provide explicit bounds for solutions to differential and integral equations as well as difference equations. Many authors have researched various inequalities and investigated the boundedness, global existence, uniqueness, stability, and continuous dependence on the initial value and parameters of solutions to differential equations, integral equations (see [3–6, 14]). However, we notice that the existing results in the literature are inadequate for researching the qualitative and quantitative properties of solutions to some fractional integral equations (see [13–15, 17, 22, 23]).

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order (see [21]). However, in this branch of mathematics we are not looking at the usual integer order but at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals. The first appearance of the concept of a fractional derivative is found in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz in 1695. As far as the existence of such a theory is concerned the foundations of the subject were laid by Liouville in a paper from 1832. The autodidact Oliver Heaviside introduce

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the practical use of fractional differential operators in electrical transmission line analysis circa 1890. Many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations, for example, we refer the reader to [14, 15, 22, 23] and the references therein.

In [16], the authors proved the following results.

THEOREM 1.1. *Let $k, \lambda \in \mathbb{R}^+$. Also, let h and u be nonnegative and locally integrable functions defined on $[0, X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, non-decreasing, and continuous function on $[0, X)$ which is bounded on $[0, X)$, that is, $\phi(x) \leq M$ for all $x \in [0, X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \leq h(x) + k\phi(x) \int_0^x (x-\rho)^{\frac{\lambda}{k}-1} u(\rho)\, d\rho,\ x \in [0, X).$$

Then

$$u(x) \leq h(x) + \sum_{n=1}^{\infty} \frac{\{k\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_0^x (x-\rho)^{n\frac{\lambda}{k}-1} h(\rho) d\rho\ ,\ x \in [0, X)\ .$$

COROLLARY 1.2. *Let $k, \lambda \in \mathbb{R}^+$ Also, let h and u be nonnegative and locally integrable functions defined on $[1, X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a nonnegative, nondecreasing, and continuous function on $[0, X)$ which is bounded on $[1, X)$, that is, $\phi(x) \leq M$ for all $x \in [1, X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \leq h(x) + k\phi(x) \int_0^x \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} u(\rho)\, \frac{d\rho}{\rho}\ , \qquad (x \in [1, X)).$$

Then

$$u(x) \leq h(x) + \sum_{n=1}^{\infty} \frac{\{k\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_1^x \left(\ln \frac{x}{\rho}\right)^{n\frac{\lambda}{k}-1} h(\rho)\, \frac{d\rho}{\rho}, \quad (x \in [1, X)).$$

In [1], the authors proved the following result:

THEOREM 1.3. *Let $\alpha > 0$, $x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$,$T > 0$, $b(t) \leq M$, where M is a constant. If*

$$x(t) \leq a(t) + b(t) \int_{t_0}^t (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} x(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{1.1}$$

then

$$x(t) \leq a(t) + \int_{t_0}^t \sum_{n=1}^{\infty} \frac{(b(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1} a(\tau) \frac{d\tau}{\tau^{1-\rho}}, t \in [t_0, T). \tag{1.2}$$

In this paper, we establish some new Gronwall-Bihari-type inequalities associated with the generalized fractional integral operator given by (1.12) (see Definition 1.6), which generalize some results given in [1]. We also present some nonlinear integral inequalities with singular kernels of Bihari type, we apply the results established to research boundedness, uniqueness for the solution to some certain initial value problems within generalized fractional derivatives given by (1.13) (see Definition 1.7).

Now, some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows :

Definition 1.4. *The Riemann-Liouville fractional integral of order α on the interval $[0,x]$ is defined by*

$$(I^{\alpha} f) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau \qquad (x>0)\,, \tag{1.3}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} \exp(-s)\, ds,$$

which is well defined for $\alpha > 0$.

Definition 1.5. *i) The modified Riemann-Liouville derivative of order α is defined by*

$$(D_x^{\alpha} f)(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x (x-\zeta)^{-\alpha} (f(\zeta) - f(0))\, d\zeta, & 0 < \alpha < 1, \\ \left(f^{(n)}(x)\right)^{(\alpha - n)}, & n \le \alpha < n+1, \quad n \ge 1. \end{cases} \tag{1.4}$$

ii) The Hadamard fractional integral ${}_H D_{1,x}^{\mu} f$ of order $\mu > 0$ is defined by

$${}_H D_{1,x}^{\mu} f = \frac{1}{\Gamma(\mu)} \int_1^x \left(\ln \frac{x}{\tau}\right)^{\mu-1} f(\tau) \frac{d\tau}{\tau} \qquad (x>1)\,. \tag{1.5}$$

iii) The Hadamard fractional derivative ${}_H D_{1,x}^{\mu} f$ of order $\mu > 0$ is defined by

$${}_H D_{1,x}^{\mu} f = \frac{1}{\Gamma(n-\mu)} \left(x \frac{d}{dx}\right)^n \int_1^x \left(\ln \frac{x}{\tau}\right)^{n-\mu-1} f(\tau) \frac{d\tau}{\tau} \qquad (x>1) \tag{1.6}$$

$$[n = [\mu] + 1, \ (x > 0)\,.$$

Here and in the following, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{R}_+, \mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, nonnegative real numbers, positive integers, and non-positive integer, respectively.

Dĭaz and Pariguan [8] introduced k-gamma function Γ_k defined by

$$\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1} dt \qquad \left[[\Re(z)] > 0, k \in \mathbb{R}^+\right), \tag{1.7}$$

which satisfies the following relationships:

$$\Gamma_k(z+k) = z\Gamma_k(z), \ \Gamma_k(k) = 1, \tag{1.8}$$

and

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right). \tag{1.9}$$

Also, k-beta function $B_k(\alpha,\beta)$ is defined by

$$B_k(\alpha,\beta) = \begin{cases} \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\,dt & (\min\{\Re(\alpha),\Re(\beta)\}>0) \\ \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} & \left(\alpha,\beta\in\mathbb{C}\backslash k\mathbb{Z}_0^-\right), \end{cases} \tag{1.10}$$

where $k\mathbb{Z}_0^-$ denotes the set of k-multiples of the elements in $\mathbb{Z}_0^-$.

Among many generalizations of the Mittag-Leffler function, one of them is recalled (see [18, 20]):

$$E_{\lambda,\beta} = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\beta)} \qquad (\lambda,\beta\in\mathbb{C};\ \Re(\lambda)>0), \tag{1.11}$$

Definition 1.6. *The generalized fractional integral operator of order $\alpha\in[n-1,n),\rho>0,t_0\geq 0$ and $t\in[t_0,\infty)$ is defined by*

$$(I_{t_0+}^{\alpha,\rho}g)t = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}(t^\rho-\tau^\rho)^{\alpha-1}g(\tau)\frac{d\tau}{\tau^{1-\rho}}. \tag{1.12}$$

Definition 1.7. *The generalized fractional derivative operator is defined by*

$$(D_{t_0+}^{\alpha,\rho}g)t = \frac{\gamma^n}{\Gamma(n-\alpha)}\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{n-\alpha-1}g(\tau)\frac{d\tau}{\tau^{1-\rho}}, \alpha\in[n-1,n), \tag{1.13}$$

where $\gamma=(t^{1-\rho}\frac{d}{dt})$.

The relation between the above latter two fractional operators is as follows:

$$(D_{t_0+}^{\alpha,\rho}g)t = \gamma^n(I_{t_0+}^{n-\alpha,\rho}g)(t), \alpha\in[n-1,n). \tag{1.14}$$

Note that the generalized operators (1.12)-(1.13) are reduced to Riemann–Liouville fractional operators as $\rho\to 1$ and Hadamard fractional operators as $\rho\to 0^+$.

The generalized Caputo fractional derivatives were discussed in [10].

Lemma 1.8. *([10]) (i) Let $\alpha\in(0,1],\beta\geq 0, 0\leq t_0,\rho>0$. Then we have*

$$(I_{t_0+}^{\alpha,\rho})((\frac{\tau^\rho-t_0^\rho}{\rho})^\beta)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(\frac{t^\rho-t_0^\rho}{\rho})^{\alpha+\beta}. \tag{1.15}$$

In particular

$$(I_{t_0+}^{\alpha,\rho}1)(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}\frac{d\tau}{\tau^{1-\rho}} = \frac{1}{\Gamma(\alpha+1)}(\frac{t^\rho-t_0^\rho}{\rho})^{\alpha}. \tag{1.16}$$

(ii) If $\beta > 0$ and $0 < \alpha \leq 1$ then

$$D^{\alpha,\rho}_{t_{0+}}((\frac{\tau^\rho - t_0^\rho}{\rho})^\beta)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\frac{t^\rho - t_0^\rho}{\rho})^{\beta-\alpha}. \tag{1.17}$$

In particular

$$(D^{\alpha,\rho}_{t_{0+}}1)(t) = \frac{(\frac{t^\rho - t_0^\rho}{\rho})^{-\alpha}}{\Gamma(1-\alpha)}, \tag{1.18}$$

and for $k = 0, 1.., [\alpha] + 1$, we have

$$(D^{\alpha,\rho}_{t_{0+}}(\frac{\tau^\rho - t_0^\rho}{\rho})^{\alpha-k})(t) = 0. \tag{1.19}$$

LEMMA 1.9. *([9]) Suppose that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}} a + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}. \tag{1.20}$$

for any $\varepsilon > 0$.

2. Main results

In this section, we establish a new version of Gronwall type integral inequality, which generalizes some previous ones.

THEOREM 2.1. *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$ $, T > 0, b(t) \leq M$, where M is a constant. If*

$$x^p(t) \leq a(t) + b(t)\int_{t_0}^t (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} x^q(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.1}$$

where $p \neq 0$,$p \geq q > 0$, are constants, then

$$x(t) \leq [\ \widetilde{a}(t) + \int_{t_0}^t \sum_{n=1}^\infty \frac{(\widetilde{b}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}\ ,\ t \in [t_0, T), \tag{2.2}$$

where

$$\widetilde{a}(t) = a(t) + \frac{p-q}{p\alpha}\varepsilon^{\frac{q}{p}} b(t)(\frac{t^\rho - t_0^\rho}{\rho})^\alpha, \widetilde{b}(t) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} b(t). \tag{2.3}$$

PROOF. Denote the right-hand side of (2.1) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t \in [t_0, T)). \tag{2.4}$$

So it follows that

$$z(t) \leq a(t) + b(t)\int_{t_0}^t (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} z^{\frac{q}{p}}(\tau)\frac{d\tau}{\tau^{1-\rho}},\ \ (t \in [t_0, T)). \tag{2.5}$$

Using Lemma 1.9, we obtain that

(2.6)

$$z(t) \leq a(t)+b(t)\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}\left(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}z(\tau)+\frac{p-q}{p}\varepsilon^{\frac{q}{p}}\right)\frac{d\tau}{\tau^{1-\rho}},\ \ (t\in[t_0,T)).$$

Using Lemma 1.8, one gets

$$z(t) \leq \widetilde{a}(t)+\widetilde{b}(t)\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}z(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.7}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.3).

Applying Theorem 1.3 to (2.7) and using (2.4), we can get the desired inequality in (2.2). ∎

Remark 2.2. If $p = q = 1$, then Theorem 2.1 reduces to Theorem 1.3.

Theorem 2.3. *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0,T)$, $T > 0$, $b(t) \leq M$, where M is a constant. Further, let $S \in C\left(\mathbb{R}_+^2, \mathbb{R}_+\right)$ be a continuous function such that*

$$0 \leq S(t,x)-S(t,y) \leq L(x-y), \quad x \geq y \geq 0, \tag{2.8}$$

for $t \in [t_0,T)$,where $L > 0$. If

$$x^p(t) \leq a(t)+b(t)\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}S(\tau,x^q(\tau))\frac{d\tau}{\tau^{1-\rho}}. \tag{2.9}$$

where $p \neq 0$,$p \geq q > 0$, are constants, then

(2.10)

$$x(t) \leq [\ \widetilde{a}(t)+\int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(\widetilde{b}(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho-\tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}, t\in[t_0,T),$$

where

(2.11)

$$\widetilde{a}(t) = a(t)+b(t)\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}S(\tau,\frac{p-q}{p}\varepsilon^{\frac{q}{p}})\frac{d\tau}{\tau^{1-\rho}}, \widetilde{b}(t) = L\frac{q}{p}\varepsilon^{\frac{q-p}{p}}b(t).$$

Proof. Denote the right-hand side of (2.9) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t\in[t_0,T)). \tag{2.12}$$

So it follows that

$$z(t) \leq a(t)+b(t)\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}S(\tau,z^{\frac{q}{p}}(\tau))\frac{d\tau}{\tau^{1-\rho}}, \quad (t\in[t_0,T)). \tag{2.13}$$

By Lemma 1.9, we obtain for any $\varepsilon > 0$,

$$z^{\frac{q}{p}}(t) \leq \frac{q}{p}\varepsilon^{\frac{q-p}{p}}z(t)+\frac{p-q}{p}\varepsilon^{\frac{q}{p}}. \tag{2.14}$$

Using (2.8) and (2.14), one has for any $\varepsilon > 0$ that

$$S(t, z^{\frac{q}{p}}(t)) \leq S(t, \tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) \leq S(t, \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) + L\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}} z(t). \tag{2.15}$$

From (2.13) and (2.15), for $t \in [t_0, T)$, we have

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}[S(\tau, \frac{p-q}{p}\varepsilon^{\frac{q}{p}}) + L\frac{q}{p}\varepsilon^{\frac{q-p}{p}} z(\tau)]\frac{d\tau}{\tau^{1-\rho}}. \tag{2.16}$$

The inequality (2.16) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + \widetilde{b}(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} z(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.17}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.11).

Applying Theorem 1.3 to (2.17) and using (2.12), we can get the desired inequality in (2.10). □

REMARK 2.4. If $p = q = 1$ and $S(t, x) = x$, then Theorem 2.3 reduces to Theorem 1.3.

THEOREM 2.5. *Let $\alpha > 0, x(t), a(t)$ be nonnegative functions and $b(t)$ be nonnegative and nondecreasing function for $t \in [t_0, T)$, $T > 0$, $b(t) \leq M$, where M is a constant. Further, let $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0, +\infty[$ with continuous nonincreasing first derivative g' on $]0, +\infty[$. If*

$$x^p(t) \leq a(t) + b(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g(x^q(\tau))\frac{d\tau}{\tau^{1-\rho}}, \tag{2.18}$$

where $p \neq 0, p \geq q > 0$, are constants, then

$$x(t) \leq [\ \widetilde{a}(t) + \int_{t_0}^{t}\sum_{n=1}^{\infty}\frac{(\widetilde{b}(\tau)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1}\widetilde{a}(\tau)\frac{d\tau}{\tau^{1-\rho}}]^{\frac{1}{p}}, t \in [t_0, T), \tag{2.19}$$

where

$$\widetilde{a}(t) = a(t) + \frac{1}{\alpha}(\frac{t^\rho - t_0^\rho}{\rho})^\alpha b(t) g(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}),\ \widetilde{b}(t) = \frac{q}{p}\varepsilon^{\frac{q-p}{p}} g'(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}) b(t). \tag{2.20}$$

PROOF. Denote the right-hand side of (2.18) by $z(t)$. Then we have

$$x(t) \leq z^{\frac{1}{p}}(t), \qquad (t \in [t_0, T)). \tag{2.21}$$

So it follows that

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g(z^{\frac{q}{p}}(\tau))\frac{d\tau}{\tau^{1-\rho}}, \quad (t \in [t_0, T)). \tag{2.22}$$

By Lemma 1.9, we obtain for any $\varepsilon > 0$ that

$$g(z^{\frac{q}{p}}(t)) \leq g(\frac{q}{p}\varepsilon^{\frac{q-p}{p}}z(t) + \frac{p-q}{p}\varepsilon^{\frac{q}{p}}). \tag{2.23}$$

Applying the mean value theorem for the function g, then for every $x \geq y > 0$ there exists $c \in]y, x[$ such that

$$g(x) - g(y) = g^{'}(c)(x-y) \leq g^{'}(y)(x-y).$$

Then

$$\begin{aligned} g(z^{\frac{q}{p}}(t)) &\leq g(\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}}z(t) + \tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) \\ &\leq g(\tfrac{p-q}{p}\varepsilon^{\frac{q}{p}}) + g^{'}(\tfrac{p-q}{p}\varepsilon^{\frac{q}{p}})\tfrac{q}{p}\varepsilon^{\frac{q-p}{p}}z(t). \end{aligned} \tag{2.24}$$

From (2.22) and (2.24), for $t \in [t_0, T)$, we have

$$z(t) \leq a(t) + b(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}[g(\frac{p-q}{p}\varepsilon^{\frac{q}{p}}) + \frac{q}{p}\varepsilon^{\frac{q-p}{p}}g^{'}(\frac{p-q}{p}\varepsilon^{\frac{q}{p}})z(\tau)]\frac{d\tau}{\tau^{1-\rho}}. \tag{2.25}$$

The inequality (2.25) can be reformulated as

$$z(t) \leq \widetilde{a}(t) + \widetilde{b}(t)\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}z(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.26}$$

where $\widetilde{a}$ and $\widetilde{b}$ are defined as in (2.20).

Applying Theorem 1.3 to (2.26) and using (2.21), we can get the desired inequality in (2.19). ∎

3. Applications

In this section, we will use the Gronwall inequality mentioned in the previous section in order to investigate the boundedness and uniqueness of a certain fractional differential equation with generalized derivatives, on the order and the initial conditions. Consider the following initial value problem within generalized fractional derivatives:

$$D_{t_0+}^{\alpha,\rho}x(t) = f(t, x(t)), \tag{3.1}$$

and

$$I_{t_0+}^{1-\alpha,\rho}x(t)\mid_{t=t_0} = c, \tag{3.2}$$

where $0 < \alpha \leq 1$, $0 \leq t < T \leq \infty$ and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to all its arguments. The Volterra integral equations corresponding to the problem (3.1)-(3.2) is as follows :

$$x(t) = c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1}f(\tau, x(\tau))\frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \tag{3.3}$$

In [1], the authors prove the equivalence of the Cauchy problem (3.1)-(3.2) and the Volterra equation (3.3).

EXAMPLE 3.1. Assume that $f(t,x(t))$ satisfies

$$|f(t,x(t))| \leq b(t)g\left(|x(t)|\right), \tag{3.4}$$

where g, b are defined as in Theorem 2.5, such that $g(0)=0$, then we have the following explicit estimate for $x(t)$

$$|x(t)| \leq \widetilde{a}(t) + \int_{t_0}^{t} \sum_{n=1}^{\infty} \frac{(g^{'}(0)b(\tau))^n}{\Gamma(n\alpha)} (\frac{t^\rho - \tau^\rho}{\rho})^{n\alpha-1} \widetilde{a}(\tau) \frac{d\tau}{\tau^{1-\rho}}, t \in [t_0, T), \tag{3.5}$$

where

$$\widetilde{a}(t) = \frac{|c|}{\Gamma(\alpha)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha+1)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha} b(t) g(0). \tag{3.6}$$

PROOF. The solution of the initial value problem (3.1)-(3.2) is given by

$$x(t) = \frac{c}{\Gamma(\alpha)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} f(\tau, x(\tau)) \frac{d\tau}{\tau^{1-\rho}}, 0 \leq t \leq \infty, \tag{3.7}$$

then

$$|x(t)| \leq \frac{|c|}{\Gamma(\alpha)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} b(\tau) g\left(|x(\tau)|\right) \frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty,$$

taking into-account that b is nondecreasing function, we obtain that

$$|x(t)| \leq \frac{|c|}{\Gamma(\alpha)} (\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1} + \frac{b(t)}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} g\left(|x(\tau)|\right) \frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty,$$

applying Theorem 2.5 to the last inequality, we obtain the desired inequality in (3.5). ∎

EXAMPLE 3.2. Assume that

$$|f(t,x) - f(t,\overline{x})| \leq b(t) g\left(|x - \overline{x}|\right),$$

where g, b are defined as in Theorem 2.5 such that $g(0) = 0$ and $b(t)$ is nondecreasing function in $t \geq 0$. Then the Cauchy problem (3.1)-(3.2) has at most one solution.

PROOF. Suppose $x(t)$, $\overline{x}(t)$ are two solutions of the Cauchy problem (3.1)-(3.2), then we have

$$\begin{aligned} x(t) &= c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} f(\tau, x(\tau)) \frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \\ \overline{x}(t) &= c\frac{(\frac{t^\rho - t_0^\rho}{\rho})^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha-1} f(\tau, \overline{x}(\tau)) \frac{d\tau}{\tau^{1-\rho}}, 0 \leq t_0 \leq t \leq \infty. \end{aligned}$$

It is clear that

$$x(t)-\overline{x}(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}(f(\tau,x(\tau))-f(\tau,\overline{x}(\tau)))\frac{d\tau}{\tau^{1-\rho}},$$

which implies that

$$|x(t)-\overline{x}(t)|\leq\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}b(\tau)g\left(|x(\tau)-\overline{x}(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}.$$

Taking into account that b is nondecreasing function, one gets

$$|x(t)-\overline{x}(t)|\leq\frac{b(t)}{\Gamma(\alpha)}\int_{t_0}^{t}(\frac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}g\left(|x(\tau)-\overline{x}(\tau)|\right)\frac{d\tau}{\tau^{1-\rho}}. \tag{3.8}$$

Through a suitable application of Theorem 2.5 to (3.8) (with $p=q=1$), we obtain that $|x(t)-\overline{x}(t)|\leq 0$, which implies $x(t)=\overline{x}(t)$. □

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Neke nove nejednakosti Gronwall-Biharijevog tipa povezane s generaliziranim frakcijskim operatorima i primjenama

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Sažetak. U ovom članku izvodimo neke generalizacije određenih nejednakosti Gronwall-Biharijevog tipa za generalizirane frakcijske operatore koji objedinjuju Riemann-Liouvilleove i Hadamardove frakcijske operatore za funkcije jedne varijable, koje daju eksplicitne granice na nepoznate funkcije. Za prikaz dobivenih nejednakosti, dana su i dva ilustrativna primjera.

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