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INVERSE SYSTEMS OF COMPACT HAUSDORFF SPACES AND (m,n)-DIMENSION

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Abstract. In 2012, V. Fedorchuk, using m-pairs and n-partitions, introduced the notion of the (m,n)-dimension of a space. It generalizes covering dimension; Fedorchuk showed that (m,n)-dimension is preserved in inverse limits of compact Hausdorff spaces. We separately have characterized those approximate inverse systems of compact metric spaces whose limits have a specified (m,n)-dimension. Our characterization is in terms of internal properties of the system. Here we are going to give a parallel internal characterization of those inverse systems of compact Hausdorff spaces whose limits have a specified (m,n)-dimension. Fedorchuk's limit theorem will be a corollary to ours.

1. Introduction

In [3], V. Fedorchuk introduced a new generalization of covering dimension which he called (m,n)-dimension, written (m,n)-dim, and such that for each normal T_1-space X, $(2,1)\text{-}\dim X = \dim X$. Fedorchuk's (m,n)-dim is defined using m-pairs and n-partitions; in Section 2 we will provide what is needed to define such pairs and partitions, and, with that in hand, we shall give the definition of the (m,n)-dimension of a space. We shall also cite in that section a few fundamental facts from this theory that will be used in the sequel.

Since the introduction of (m,n)-dimension, the theory has been developed in parallel to that of the classical notions of dimension which one can find in [2]. For example, a strong inductive version was presented in [5], a transfinite type in [11], and for (m,n)-dimension, both a factorization theorem and one about the existence of universal spaces were given in [10] and [13], respectively. The main result, Theorem 5.2 of [7], gives an internal characterization of those approximate inverse systems of compact metric spaces (see [8]) whose limits have a specified (m,n)-dimension. In [12], Martynchuk proved that for

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every strongly hereditarily normal space X, (m,n)-dim $X = \left\lfloor \frac{\dim X}{n} \right\rfloor$; therefore Fedorchuk's notion of dimension deviates from that of covering dimension in infinitely many cases. One may also consult [4] and [6] for additional contributions of Fedorchuk who proved:

THEOREM 1.1 (Theorem 2.21 from [3]). *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of compact Hausdorff spaces* X_a *with* (m,n)-dim$X_a \leq k$ *for all* $a \in A$*, and let* $X = \lim \mathbf{X}$*. Then* (m,n)-dim$X \leq k$.

We improve this in our main result, Theorem 4.1, which gives a characterization of the (m,n)-dimension of a space X, where X is the limit of an inverse system of compact Hausdorff spaces, strictly in terms of internal properties of the given system. Our result incorporates Theorem 1.1 in Corollary 5.3. In Section 3 we shall collect several facts dealing with finite covers of limits of inverse systems of compact Hausdorff spaces. These will be used in Section 4, which contains both the statement and the proof of our main result. Section 5 gathers some corollaries to Theorem 4.1.

2. INTRODUCTION TO (m,n)-dim

Throughout this paper, map will mean continuous function. We are going to use certain notation and concepts that were established in [3]. Thus, when we say that $(S_1, \ldots, S_m)$ is a finite family of subsets of a set X, we mean that $m \geq 1$ and for each $1 \leq i \leq m$, $S_i \subset X$. Repetitions are permitted, i.e., one may have $1 \leq i < j \leq m$ and $S_i = S_j$.

DEFINITION 2.1. *Let* $\Phi = (S_1, \ldots, S_m)$ *be a finite family of subsets of a set* X*. We shall say that the order of* Φ*, denoted* ord(Φ)*, is* 0 *if* $S_i = \emptyset$ *for all* $1 \leq i \leq m$*; otherwise* ord(Φ) *will mean the largest* $n \in \mathbb{N}$ *such that* Φ *contains a subset* Ψ *with* card$(\Psi) = n$ *and* $\bigcap \Psi \neq \emptyset$.

By this definition ord$(\Phi) = 1$ if and only if Φ is pairwise disjoint and there exists $1 \leq i \leq m$ such that $S_i \neq \emptyset$.

LEMMA 2.2. *Let* $f : X \to Y$ *be a surjective function,* $v = (V_1, \ldots, V_m)$ *a family of subsets of* X*, and* $t = (T_1, \ldots, T_m)$ *a family of subsets of* Y*. Suppose that for each* $1 \leq i \leq m$*,* $f^{-1}(T_i) \subset V_i$*. Then* ord$(t) \leq$ ord(v).

PROOF. In case ord$(v) = 0$, then for each $1 \leq i \leq m$, $T_i = \emptyset$ since $f^{-1}(T_i) \subset V_i = \emptyset$ and f is surjective. In case $T_i \neq \emptyset$, then the surjectivity of f implies that $f(f^{-1}(T_i)) = T_i$. Hence for $1 \leq i < j \leq m$, $T_i \neq T_j$ if and only if $f^{-1}(T_i) \neq f^{-1}(T_j)$. Now suppose that $n \in \mathbb{N}$ and there is a subset K of t with card$(K) = n$ and $\bigcap K \neq \emptyset$. By what we just showed, card$(f^{-1}(K)) = n$, and the surjectivity of f yields that $\bigcap f^{-1}(K) \neq \emptyset$. $\square$

DEFINITION 2.3 (Definition 2.1 from [3]). *Let* $u = (U_1, \ldots, U_m)$ *be a finite open cover of* X *and* $\Phi = (F_1, \ldots, F_m)$ *be a family of closed subsets of* X *such* *that*

$$F_j \subseteq U_j, \quad j = 1, \ldots, m;$$
$$\mathrm{ord}(\Phi) \leq 1.$$

Then (u, Φ) *is said to be an* m***-pair in*** X.

Definition 2.4 (Definition 2.5 from [3]). *Let* (u, Φ) *be an* m*-pair in* X *where* $u = (U_1, \ldots, U_m)$ *and* $\Phi = (F_1, \ldots, F_m)$. *A closed set* $P \subseteq X$ *is said to be an* n***-partition*** *of* (u, Φ) *if there exists a family of open sets* $v = (V_1, \ldots, V_m)$ *of* X *such that*

1. $F_j \subseteq V_j \subseteq U_j$, *for* $j = 1, \ldots, m$,
2. $\mathrm{ord}(v) \leq n$, *and*
3. $X \setminus P = \bigcup v$.

Definition 2.5 (Definition 2.7 from [3]). *For each* $i = 1, \ldots, r$, *let* (u_i, Φ_i) *be an* m*-pair in* X. *The sequence* (u_i, Φ_i), $i = 1, \ldots, r$, *is called* n***-inessential in*** X *if for each* i, *there exists an* n*-partition* P_i *of* (u_i, Φ_i) *such that* $P_1 \cap \cdots \cap P_r = \emptyset$.

Definition 2.6 (Definition 2.8 from [3]). *Let* $m, n \in \mathbb{N}$ *with* $n \leq m$. *To every space* X *one assigns the* (m, n)-***dimension*** (m, n)-dim$X$, *which is an element of* $\{-1\} \cup \{0\} \cup \mathbb{N} \cup \{\infty\}$ *in the following way.*

(1) (m, n)-dim$X = -1$ *if and only if* $X = \emptyset$.

In case $X \neq \emptyset$, *then:*

(2.1) (m, n)-dim$X = \infty$, *if for each* $k \in \{0\} \cup \mathbb{N}$, *there is a sequence* (u_i, Φ_i), $i = 1, \ldots, k + 1$, *of* m*-pairs in* X, *that is not* n*-inessential in* X*;*

(2.2) (m, n)-dim$X = r$, *where* $r \in \{0\} \cup \mathbb{N}$, *if* (m, n)-dim$X \neq \infty$ *and* r *is the minimum of those* $k \in \{0\} \cup \mathbb{N}$ *such that every sequence* (u_i, Φ_i), $i = 1, \ldots, k + 1$, *of* m*-pairs in* X, *is* n*-inessential in* X.

Theorem 2.7 (Theorem 2.9 from [3]). *Let* X *be a normal* T_1*-space. Then* $(2, 1)$-dim$X = \dim X$.

Definition 2.8. *Let* Y *be a space,* $f : X \to Y$ *a map, and* (u, Φ) *an* m*-pair in* Y *where* $u = (U_1, \ldots, U_m)$ *and* $\Phi = (F_1, \ldots, F_m)$. *Put* $f^{-1}(u) = (f^{-1}(U_1), \ldots, f^{-1}(U_m))$ *and* $f^{-1}(\Phi) = (f^{-1}(F_1), \ldots, f^{-1}(F_m))$. *Then by* $f^{-1}(u, \Phi)$ *we mean* $(f^{-1}(u), f^{-1}(\Phi))$.

Lemma 2.9. *The pair* $f^{-1}(u, \Phi)$ *of* Definition 2.8 *is an* m*-pair in* X. *Therefore if* B *is a subspace of* Y, *and we define* $u \cap B = (U_1 \cap B, \ldots, U_m \cap B)$ *and* $\Phi \cap B = (F_1 \cap B, \ldots, F_m \cap B)$, *then* $(u \cap B, \Phi \cap B)$, *which we shall also denote* $(u, \Phi) \cap B$, *is an* m*-pair in* B. □

Proposition 2.10 (Proposition 2.19 from [3]). *Let* Y *be a space,* $f : X \to Y$ *be a map, and let a sequence* (u_i, Φ_i), $i = 1, \ldots, r$, *of* m*-pairs in* Y *be* n*-inessential in* Y. *Then* $(f^{-1}(u_i, \Phi))$, $i = 1, \ldots, r$, *is an* n*-inessential sequence of* m*-pairs in* X.

Proposition 2.11 (Proposition 2.20 from [3]). *Let* (u_i, Φ_i) *and* (w_i, Ψ_i), $i = 1, \dots, r$, *be sequences of* m*-pairs in* X *where* $u_i = (U_1^i, \dots, U_m^i)$, $\Phi_i = (F_1^i, \dots, F_m^i)$, $w_i = (W_1^i, \dots, W_m^i)$, *and* $\Psi_i = (G_1^i, \dots, G_m^i)$. *Assume that*

$$F_j^i \subseteq G_j^i \subseteq W_j^i \subseteq U_j^i, \quad i = 1, \dots, r, \quad j = 1, \dots, m.$$

If the sequence (w_i, Ψ_i), $i = 1, \dots, r$, *is* n*-inessential in* X, *then the sequence* (u_i, Φ_i), $i = 1, \dots, r$, *is* n*-inessential in* X.

3. Inverse System Facts

We shall gather several facts concerning inverse systems of compact Hausdorff spaces. One may consult [9] or [1] for a study of such objects. In this section, $\mathbf{X} = \{X_a, p_{aa'}, A\}$ will denote an inverse system of compact Hausdorff spaces and X will designate $\lim \mathbf{X}$. We require that $A = (A, \leq)$ is a nonempty partially ordered directed set.

Definition 3.1. *In case* $a \in A$ *and* $u = (U_1, \dots, U_m)$ *is a sequence of subsets of* X, *then by* $p_a(u)$, *we mean,* $(p_a(U_1), \dots, p_a(U_m))$.

Lemma 3.2. *If* $\mathcal{U}$ *is an open cover of* X *and* $a \in A$, *then there exists* $a' \geq a$ *such that for all* $b \geq a'$, *there is an open cover* $\mathcal{T}$ *of* X_b *such that* $p_b^{-1}(\mathcal{T})$ *refines* $\mathcal{U}$. □

Corollary 3.3. *For each finite open cover* $u = (U_1, \dots, U_m)$ *of* X *and* $a \in A$, *there exists* $a' \geq a$ *such that for all* $b \geq a'$, *there is a finite open cover* $t = (T_1, \dots, T_m)$ *of* X_b *with* $p_b^{-1}(T_j) \subseteq U_j$ *for all* $j = 1, \dots, m$.

Proof. Choose $a' \geq a$ as in Lemma 3.2 where $\mathcal{U}$ is replaced by u, and let $b \geq a'$. Select an open cover $\mathcal{T}$ of X_b such that $p_b^{-1}(\mathcal{T})$ refines u. For each $j = 1, \dots, m$, set $T_j = \bigcup\{T \in \mathcal{T} \mid p_b^{-1}(T) \subseteq U_j\}$. Then $t = (T_1, \dots, T_m)$ is a finite open cover of X_b, and $p_b^{-1}(T_j) \subseteq U_j$ for all $j = 1, \dots, m$. □

Lemma 3.4. *For each closed subset* F *of* X *and neighborhood* U *of* F *in* X, *there exists* $a' \in A$ *such that for all* $b \geq a'$, $p_b^{-1}(p_b(F)) \subseteq U$. □

Lemma 3.5. *For each collection* $u = (U_1, \dots, U_m)$ *of open subsets of* X, *collection* $\Phi = (F_1, \dots, F_m)$ *of closed subsets of* X *with* $F_j \subseteq U_j$ *for all* $j = 1, \dots, m$, *and* $a \in A$, *there exists* $a' \geq a$ *such that,*

1. *for all* $b \geq a'$ *and* $j = 1, \dots, m$, $p_b^{-1}(p_b(F_j)) \subseteq U_j$, *and*
2. *in case* $\operatorname{ord}(\Phi) \leq 1$, *we may require in addition to* (1) *that for all* $b \geq a'$, $\operatorname{ord}(p_b(\Phi)) \leq 1$.

Proof. Choose a collection $t = (T_1, \dots, T_m)$ of open subsets of X such that for each $j = 1, \dots, m$, $F_j \subseteq T_j \subseteq U_j$. From Lemma 3.4 and the fact that A is directed, we find $a' \geq a$ such that for all $b \geq a'$, $p_b^{-1}(p_b(F_j)) \subseteq T_j \subseteq U_j$, which gives us (1).

To prove (2), choose the collection $t = (T_1, \ldots, T_m)$ of open subsets of X so that in addition to the above:

(†) if $1 \le i < j \le m$, then $T_i = T_j$ if $F_i = F_j$ and $T_i \cap T_j = \emptyset$ if $F_i \ne F_j$.

Let $b \ge a'$. If it were the case that for some $1 \le i < j \le m$, $p_b(F_i) \ne p_b(F_j)$, then $F_i \ne F_j$, so by (†), $T_i \cap T_j = \emptyset$. If $p_b(F_i) \cap p_b(F_j) \ne \emptyset$, one can find $q_i \in F_i$, $q_j \in F_j$, and $x \in p_b(F_i) \cap p_b(F_j)$ such that $p_b(q_i) = x = p_b(q_j)$. But then, $\emptyset \ne \{q_i, q_j\} \subseteq p_b^{-1}(x) = p_b^{-1}(p_b(q_i)) = p_b^{-1}(p_b(q_j)) \subseteq p_b^{-1}(p_b(F_i)) \cap p_b^{-1}(p_b(F_j)) \subseteq T_i \cap T_j = \emptyset$, a contradiction. □

Lemma 3.6. *For each m-pair (u, Φ) in X, where $u = (U_1, \ldots, U_m)$ and $\Phi = (F_1, \ldots, F_m)$, there exist $a \in A$ and a finite open cover $v = (V_1, \ldots, V_m)$ of X_a such that $(v, p_a(\Phi))$ is an m-pair in X_a, that is:*

1. $p_a(F_j) \subseteq V_j$, *for each* $j = 1, \ldots, m$, *and*
2. $\operatorname{ord}(p_a(\Phi)) \le 1$,

and moreover we have,

3. $\operatorname{ord}(p_b(\Phi)) \le 1$, *for all* $b \ge a$, *and*
4. $p_a^{-1}(V_j) \subseteq U_j$, *for each* $j = 1, \ldots, m$.

Proof. Using Corollary 3.3 and Lemma 3.5, choose $a \in A$ and a finite open cover $t = (T_1, \ldots, T_m)$ of X_a such that $p_a^{-1}(T_j) \subseteq U_j$, $j = 1, \ldots, m$; for all $b \ge a$, $\operatorname{ord}(p_b(\Phi)) \le 1$; and for all $b \ge a$ and $j = 1, \ldots, m$, $p_b^{-1}(p_b(F_j)) \subseteq U_j$. Fix j. Since p_a is a closed map and $p_a^{-1}(p_a(F_j)) \subseteq U_j$, there exists an open neighborhood S_j of $p_a(F_j)$ in X_a such that $p_a^{-1}(S_j) \subseteq U_j$. Set $V_j = T_j \cup S_j$. Then, of course, $p_a^{-1}(V_j) = p_a^{-1}(T_j) \cup p_a^{-1}(S_j) \subseteq U_j$ and $p_a(F_j) \subseteq S_j \subseteq V_j$ as requested for (4) and (1). □

Lemma 3.7. *For every sequence (u_i, Φ_i), $i = 1, \ldots, k+1$, of m-pairs in X, where $u_i = (U_1^i, \ldots, U_m^i)$ and $\Phi_i = (F_1^i, \ldots, F_m^i)$, and every $a \in A$, there exists $b_0 \ge a$ such that for all $b \ge b_0$, there is a corresponding sequence $(y_i, p_b(\Phi_i))$, $i = 1, \ldots, k+1$, of m-pairs in X_b, where $y_i = (Y_1^i, \ldots, Y_m^i)$, so that for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,*

$$(*)\ F_j^i \subseteq p_b^{-1}(p_b(F_j^i)) \subseteq p_b^{-1}(Y_j^i) \subseteq U_j^i.$$

Proof. For each i, apply Lemma 3.6 to the m-pair (u_i, Φ_i) in X to select $a_i \in A$ and a finite open cover $v_i = (V_1^i, \ldots, V_m^i)$ of X_{a_i}, such that $(v_i, p_{a_i}(\Phi_i))$ is an m-pair in X_{a_i}; for all $b \ge a_i$, $\operatorname{ord}(p_b(\Phi_i)) \le 1$; and for each $j = 1, \ldots, m$,

($†_1$) $p_{a_i}^{-1}(V_j^i) \subseteq U_j^i$, and

($†_2$) $p_{a_i}(F_j^i) \subseteq V_j^i$.

Pick $b_0 \in A$ so that $b_0 \ge a$ and $b_0 \ge a_i$ for all $i = 1, \ldots, k+1$. Fix $b \ge b_0$. For each i and j, set $Y_j^i = p_{a_i b}^{-1}(V_j^i)$. Put $y_i = \left(Y_1^i, \ldots, Y_m^i\right)$. We claim that,

($†_3$) for each i, $(y_i, p_b(\Phi_i))$ is an m-pair in X_b.

Since v_i is an open cover of X_{a_i}, it follows that y_i is an open cover of X_b. So, since $\mathrm{ord}(p_b(\Phi_i)) \leq 1$, it is sufficient to show that,

($\dagger_4$) for each i and j, $p_b(F^i_j) \subseteq Y^i_j$.

Let $x \in p_b(F^i_j)$, and choose $z \in F^i_j$ such that $p_b(z) = x$. Employing ($\dagger_2$), we have that $p_{a_i}(z) \in V^i_j$, and of course $p_{a_i b}(x) = p_{a_i b} p_b(z)$. Thus, $x = p_b(z) \in p^{-1}_{a_i b}(V^i_j) = Y^i_j$ as needed for ($\dagger_4$).

To prove ($*$), fix i and j. The left inclusion is obvious, and the middle inclusion follows from ($\dagger_4$). To show the right inclusion, let $x \in p_b^{-1}(Y^i_j) = p_b^{-1}(p^{-1}_{a_i b}(V^i_j))$. Then $p_{a_i}(x) = p_{a_i b} p_b(x) \in p_{a_i b}(Y^i_j) = V^i_j$. Finally, apply ($\dagger_1$) to see that $x \in p^{-1}_{a_i}(V^i_j) \subseteq U^i_j$. □

4. Characterization

Here is our main result.

Theorem 4.1. *Let* $\mathbf{X} = \{X_a, p_{aa'}, A\}$ *be an inverse system of compact Hausdorff spaces,* $X = \lim \mathbf{X}$*,* $\{m, n\} \subseteq \mathbb{N}$*, and* $k \geq 0$*. The following are equivalent.*

1. (m, n)-$\dim X \leq k$.
2. *for each* $a \in A$ *and sequence* (w_i, Φ_i)*,* $i = 1, \ldots, k+1$*, of* m*-pairs in* X_a*, there exists* $b_0 \geq a$ *such that for all* $b \geq b_0$*, the sequence* $(p^{-1}_{ab}(w_i), p^{-1}_{ab}(\Phi_i)) \cap p_b(X)$*,* $i = 1, \ldots, k+1$*, of* m*-pairs in* $p_b(X)$ *is* n*-inessential in* $p_b(X)$*.*

Proof. ($\Leftarrow$) We assume that (2) is true and proceed to prove (1). Let (u_i, Ψ_i), $i = 1, \ldots, k+1$, be a sequence of m-pairs of X. We wish to show that this sequence is n-inessential in X. For each $i = 1, \ldots, k+1$, let $u_i = (U^i_1, \ldots, U^i_m)$ and $\Psi_i = (F^i_1, \ldots, F^i_m)$.

Using Lemma 3.6, for each $i = 1, \ldots, k+1$, we can find an index $a_i \in A$ and a finite open cover $v_i = (V^i_1, \ldots, V^i_m)$ of X_{a_i} such that,

($\dagger_1$) $p^{-1}_{a_i}(V^i_j) \subseteq U^i_j$,

($\dagger_2$) $p_{a_i}(F^i_j) \subseteq V^i_j$, and

($\dagger_3$) $\mathrm{ord}\, p_{a_i}(\Psi_i) \leq 1$.

It follows from ($\dagger_2$) and ($\dagger_3$) that for each $i = 1, ..., k+1$, $(v_i, p_{a_i}(\Psi_i))$ is an m-pair of X_{a_i}. Choose $a \in A$ so that $a \geq a_i$ for all $i = 1, \ldots, k+1$. For each $i = 1, \ldots, k+1$, set

$$v^0_i = (p^{-1}_{a_i a}(V^i_1), \ldots, p^{-1}_{a_i a}(V^i_m)), \text{ and}$$
$$\Psi^0_i = (p^{-1}_{a_i a}(p_{a_i}(F^i_1)), \ldots, p^{-1}_{a_i a}(p_{a_i}(F^i_m))).$$

It readily follows from Lemma 2.9 that, for each $i = 1, \ldots, k+1$, (v^0_i, Ψ^0_i) is an m-pair of X_a. Now apply (2) to choose $b \geq a$ such that the sequence $\left(p^{-1}_{ab}\left(v^0_i\right), p^{-1}_{ab}\left(\Psi^0_i\right)\right) \cap p_b(X)$, $i = 1, \ldots, k+1$, of m-pairs in $p_b(X)$ is n-inessential in $p_b(X)$. Treating $p_b : X \to p_b(X)$, it follows from Proposition

2.10 that the sequence $(p_b^{-1}(p_{ab}^{-1}(v_i^0) \cap p_b(X)), p_b^{-1}(p_{ab}^{-1}(\Psi_i^0) \cap p_b(X)))$, $i = 1, \ldots, k+1$, of m-pairs in X, which equals $\left(p_a^{-1}\left(v_i^0\right), p_a^{-1}\left(\Psi_i^0\right)\right)$, $i = 1, \ldots, k+1$, is n-inessential in X.

Using $(\dagger_1)$ and $(\dagger_2)$ we have,

$$F_j^i \subseteq p_{a_i}^{-1}(p_{a_i}(F_j^i)) \subseteq p_{a_i}^{-1}(V_j^i) \subseteq U_j^i.$$

We note that $p_{a_i} = p_{a_i a} \circ p_a$, so $p_{a_i}^{-1}(p_{a_i}(F_j^i)) = p_a^{-1}(p_{a_i a}^{-1}(p_{a_i}(F_j^i)))$ and $p_{a_i}^{-1}(V_j^i) = p_a^{-1}(p_{a_i a}^{-1}(V_j^i))$. Hence for each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,

$$F_j^i \subseteq p_a^{-1}(p_{a_i a}^{-1}(p_{a_i}(F_j^i))) \subseteq p_a^{-1}(p_{a_i a}^{-1}(V_j^i)) \subseteq U_j^i.$$

This shows that the sequence $\left(p_a^{-1}\left(v_i^0\right), p_a^{-1}\left(\Psi_i^0\right)\right)$, $i = 1, \ldots, k+1$, which is known to be n-inessential in X, "squeezes between" the given sequence (u_i, Ψ_i), $i = 1, \ldots, k+1$, of m-pairs of X. So by Proposition 2.11, the sequence of m-pairs (u_i, Ψ_i), $i = 1, \ldots, k+1$, is n-inessential in X, and thus (m, n)-dim$X \leq k$.

($\Rightarrow$) We will now assume (1) and let $a \in A$, and (w_i, Φ_i), $i = 1, \ldots, k+1$, be a sequence of m-pairs in X_a, where we denote $w_i = (W_1^i, \ldots, W_m^i)$ and $\Phi_i = (F_1^i, \ldots, F_m^i)$. We are required to prove (2). Recall that for each $i = 1, \ldots, k+1$, w_i is an open cover of X_a, $\operatorname{ord} \Phi_i \leq 1$, and for each $j = 1, \ldots, m$ we have $F_j^i \subseteq W_j^i$.

Since (m, n)-dim$X \leq k$, the sequence $(p_a^{-1}(w_i), p_a^{-1}(\Phi_i))$, $i = 1, \ldots, k+1$, of m-pairs in X is n-inessential in X. So, for each $i = 1, \ldots, k+1$, there exists an n-partition P_i of the pair $\left(p_a^{-1}(w_i), p_a^{-1}(\Phi_i)\right)$ such that

$$P_1 \cap \cdots \cap P_{k+1} = \emptyset.$$

By the definition of n-partition, for each $i = 1, \ldots, k+1$, we have a collection of open sets in X, $v_i = (V_1^i, \ldots, V_m^i)$, such that

$(\dagger_{4.1})$ $p_a^{-1}(F_j^i) \subseteq V_j^i \subseteq p_a^{-1}(W_j^i)$, $j = 1, \ldots, m$,

$(\dagger_{4.2})$ $\operatorname{ord} v_i \leq n$, and

$(\dagger_5)$ $X \setminus P_i = \bigcup v_i$.

By $(\dagger_{4.1})$ and $(\dagger_5)$ we have for each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,

$$p_a^{-1}\left(F_j^i\right) \subseteq V_j^i \subseteq \bigcup v_i = X \setminus P_i.$$

Thus, for each $i = 1, \ldots, k+1$,

$$\bigcup p_a^{-1}(\Phi_i) \subseteq X \setminus P_i,$$

and so,

$$P_i \subseteq X \setminus \bigcup p_a^{-1}(\Phi_i).$$

For each $i = 1, \ldots, k+1$, we choose an open set Q_i in X with the following properties:

$(\dagger_6)$ $P_i \subseteq Q_i$,

$(\dagger_7)$ $Q_1 \cap \cdots \cap Q_{k+1} = \emptyset$, and

($\dagger_8$) $Q_i \subseteq X \setminus \bigcup p_a^{-1}(\Phi_i)$.

Consider the closed set $X \setminus Q_i$. Then by ($\dagger_5$) and ($\dagger_6$), for each $i = 1, \ldots, k+1$, the collection

$$\left(V_1^i \cap (X \setminus Q_i), \ldots, V_m^i \cap (X \setminus Q_i)\right)$$

of open sets in $X \setminus Q_i$, covers $X \setminus Q_i$. By ($\dagger_{4.1}$) we have that $p_a^{-1}(F_j^i) \subseteq V_j^i$, and by definition, $p_a^{-1}(F_j^i) \subseteq \bigcup p_a^{-1}(\Phi_i)$. These and ($\dagger_8$) imply that $p_a^{-1}(F_j^i) \subseteq \bigcup p_a^{-1}(\Phi_i) \subseteq X \setminus Q_i$. And so, $p_a^{-1}(F_j^i) \subseteq V_j^i \cap (X \setminus Q_i)$. This shows that there exist closed sets $G_1^i, \ldots, G_m^i$ in X such that for $j = 1, \ldots, m$ and $i = 1, \ldots, k+1$, we have,

($\dagger_9$) $p_a^{-1}(F_j^i) \subseteq G_j^i \subseteq V_j^i \cap (X \setminus Q_i)$, and

($\dagger_{10}$) the collection $(G_1^i, \ldots, G_m^i)$ covers $X \setminus Q_i$.

Apply Lemma 3.5(1) to choose $b_1 \geq a$ so that for all $b \geq b_1$,

($\dagger_{11}$) $p_b^{-1}(p_b(G_j^i)) \subseteq V_j^i$ for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$.

We now apply Lemma 3.7 with $a = b_1$ and with the sequence,

$$(p_a^{-1}(w_i), p_a^{-1}(\Phi_i)), i = 1, \ldots, k+1,$$

of m-pairs in X. Note that here we substitute $p_a^{-1}(W_j^i)$ for the U_j^i and $p_a^{-1}(F_j^i)$ for the F_j^i of the lemma. This gives us a certain $b_0 \geq b_1 \geq a$, and this will be the b_0 requested in (2). Let $b \geq b_0$; Lemma 3.7 gives us a "corresponding sequence"

$$\left(y_i, p_b\left(p_a^{-1}(\Phi_i)\right)\right), i = 1, \ldots, k+1,$$

of m-pairs in X_b, say $y_i = (Y_1^i, \ldots, Y_m^i)$. So, for all $i = 1, \ldots, k+1$, we have,

$$p_b\left(p_a^{-1}(\Phi_i)\right) = \left(p_b\left(p_a^{-1}\left(F_1^i\right)\right), \ldots, p_b\left(p_a^{-1}\left(F_m^i\right)\right)\right),$$

and, using ($*$) of Lemma 3.7,

($\dagger_{12}$) $p_a^{-1}(F_j^i) \subseteq p_b^{-1}(p_b(p_a^{-1}(F_j^i))) \subseteq p_b^{-1}(Y_j^i) = p_b^{-1}(Y_j^i \cap p_b(X)) \subseteq p_a^{-1}(W_j^i)$.

We now use ($\dagger_{4.1}$), ($\dagger_{11}$), and the fact that p_b is a closed map to choose, for each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$, an open neighborhood T_j^i in $p_b(X)$ of $p_b(G_j^i)$ such that,

($\dagger_{13}$) $p_b^{-1}(p_b(G_j^i)) \subseteq p_b^{-1}(T_j^i) \subseteq V_j^i \subseteq p_a^{-1}(W_j^i)$.

Let $t_i = (T_1^i, \ldots, T_m^i)$; thus t_i is a collection of subsets of $p_b(X)$. Then since by ($\dagger_{4.2}$), $\text{ord}(v_i) \leq n$, and $p_b : X \to p_b(X)$ is surjective, Lemma 2.2 shows that $\text{ord}(t_i) \leq n$.

For each $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$, let $\widehat{Y}_j^i = (Y_j^i \cap p_b(X)) \cup T_j^i \subseteq p_b(X)$. For each $i = 1, \ldots, k+1$, put $\hat{y}_i = (\widehat{Y}_1^i, \ldots, \widehat{Y}_m^i)$. Since $(y_i, p_b(p_a^{-1}(\Phi_i)) \cap p_b(X)$, $i = 1, \ldots, k+1$, is a sequence of m-pairs in $p_b(X)$, then, *a fortiori*, $(\hat{y}_i, p_b(p_a^{-1}(\Phi_i)))$, $i = 1, \ldots, k+1$, is a sequence of m-pairs in $p_b(X)$, and using ($\dagger_{12}$) and ($\dagger_{13}$) we have that for all $i = 1, \ldots, k+1$ and $j = 1, \ldots, m$,

($\dagger_{14}$) $p_a^{-1}(F_j^i) \subseteq p_b^{-1}(p_b(p_a^{-1}(F_j^i))) \subseteq p_b^{-1}(\widehat{Y}_j^i) \subseteq p_a^{-1}(W_j^i)$.

We claim that the sequence $\left(\hat{y}_i, p_b\left(p_a^{-1}(\Phi_i)\right)\right) \cap p_b(X) = \left(\hat{y}_i, p_b\left(p_a^{-1}(\Phi_i)\right)\right)$, $i = 1, \dots, k+1$, is n-inessential in $p_b(X)$. To see this, for each $i = 1, \dots, k+1$, let $R_i = p_b(X) \setminus \bigcup_{j=1}^m T_j^i$. Using ($\dagger_9$) and the fact that $p_b(G_j^i) \subseteq T_j^i \subseteq p_b(X)$, we see that $p_b(p_a^{-1}(F_j^i)) \subseteq T_j^i \subseteq \widehat{Y}_j^i$ for $j = 1, \dots, m$. Since t_i is a family of open sets in $p_b(X)$ such that $p_b(p_a^{-1}(F_j^i)) \subseteq T_j^i \subseteq \widehat{Y}_j^i$, and $\operatorname{ord} t_i \leq n$, then by Definition 2.4, R_i is an n-partition of $(\hat{y}_i, p_b(p_a^{-1}(\Phi_i)))$ in $p_b(X)$. We will now show that $R_1 \cap \cdots \cap R_{k+1} = \emptyset$.

We first note that by ($\dagger_{10}$) and ($\dagger_{13}$) we have for each $i = 1, \dots, k+1$,

$$X \setminus Q_i \subseteq \bigcup_{j=1}^m G_j^i \subseteq \bigcup_{j=1}^m p_b^{-1}(p_b(G_j^i)) \subseteq \bigcup_{j=1}^m p_b^{-1}(T_j^i).$$

It follows that

$$p_b^{-1}(R_i) = p_b^{-1}\left(p_b(X) \setminus \bigcup_{j=1}^m T_j^i\right) = X \setminus \left(\bigcup_{j=1}^m p_b^{-1}(T_j^i)\right) \subseteq Q_i.$$

By ($\dagger_7$),

$$Q_1 \cap \cdots \cap Q_{k+1} = \emptyset,$$

and so we have that

$$p_b^{-1}(R_1) \cap \cdots \cap p_b^{-1}(R_{k+1}) = \emptyset.$$

Then, since $R_i \subseteq p_b(X)$ and hence $R_i = p_b(p_b^{-1}(R_i))$ for each $i = 1, \dots, k+1$, it follows that

$$R_1 \cap \cdots \cap R_{k+1} = \emptyset.$$

Thus, the sequence $\left(\hat{y}_i, p_b\left(p_a^{-1}(\Phi_i)\right)\right)$, $i = 1, \dots, k+1$, is n-inessential in $p_b(X)$.

To conclude the proof we will show that for all $i = 1, \dots, k+1$ and $j = 1, \dots, m$, we have

$$p_{ab}^{-1}(F_j^i) \cap p_b(X) \subseteq p_b\left(p_a^{-1}\left(F_j^i\right)\right) \subseteq \widehat{Y}_j^i \subseteq p_{ab}^{-1}(W_j^i) \cap p_b(X),$$

and apply Proposition 2.11 (This means that in terms of Proposition 2.11, F_j^i corresponds to $p_{ab}^{-1}(F_j^i) \cap p_b(X)$, G_j^i to $p_b(p_a^{-1}(F_j^i))$, W_j^i to $\widehat{Y}_j^i$, and U_j^i to $p_{ab}^{-1}(W_j^i) \cap p_b(X)$). Fix i and j.

To show the left inclusion, let $x \in p_{ab}^{-1}(F_j^i) \cap p_b(X)$, and choose $y \in p_b^{-1}(x)$. Then, $p_{ab}(x) = p_{ab}(p_b(y)) = p_a(y)$. Thus, $p_a(y) \in F_j^i$, and so $y \in p_a^{-1}(F_j^i)$. Finally,

$$x = p_b(y) \in p_b(p_a^{-1}(F_j^i)),$$

proving the left inclusion. The middle inclusion follows from ($\dagger_{14}$) and the fact that $\widehat{Y}_j^i \subseteq p_b(X)$.

To show the right inclusion, we first note that by ($\dagger_{14}$), $\widehat{Y}_j^i \subseteq p_b(p_a^{-1}(W_j^i))$. Let $x \in \widehat{Y}_j^i \subseteq p_b(p_a^{-1}(W_j^i))$. There exists $y \in p_a^{-1}(W_j^i)$ such that $x = p_b(y)$. Now, $p_{ab}(x) = p_{ab} p_b(y) = p_a(y) \in W_j^i$, so $x \in p_{ab}^{-1}(W_j^i)$. Since

$x \in p_b(p_a^{-1}(W_j^i)) \subseteq p_b(X)$, then $x \in p_{ab}^{-1}(W_j^i) \cap p_b(X)$, proving the right inclusion. □

5. Corollaries

Theorem 1.1 is a corollary of Theorem 4.1 and the next fact which is Proposition 2.10 of [3]. Indeed, if (m,n)-dim$X_a \leq k$ for all $a \in A$, then by Proposition 5.1, (m,n)-dim $p_a(X_a) \leq k$. By the definition of (m,n)-dimension, condition 2 of Theorem 4.1 is satisfied.

Proposition 5.1. *Suppose that X is a space with (m,n)-dim$X \leq k$. Then for each closed subspace A of X, (m,n)-dim$A \leq k$.*

Using the next proposition (see Corollary 2.5.11., p. 140 of [1]), we can strengthen Theorem 1.1.

Proposition 5.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact Hausdorff spaces X_a and $X = \lim \mathbf{X}$. Suppose that $B \subseteq A$ is a cofinal subset of A. Then $\mathbf{Y} = \{X_a, p_{ab}, B\}$ is an inverse system of compact Hausdorff spaces. Let Y be the limit of $\mathbf{Y}$. Then the restriction $p = \pi|X$ of the projection $\pi : \prod\{X_a \,|\, a \in A\} \to \prod\{X_a \,|\, a \in B\}$ is a homeomorphism $p : X \to Y$.*

Corollary 5.3. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact Hausdorff spaces X_a and $X = \lim \mathbf{X}$. If there exists a cofinal subset $B \subseteq A$ such that for all $a \in B$, (m,n)-dim$X_a \leq k$, then (m,n)-dim$X \leq k$.* □

Corollary 5.4. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact Hausdorff spaces X_a and $X = \lim \mathbf{X}$. If there exists $a \in A$ such that (m,n)-dim$X_{a'} \leq k$ for all $a' \geq a$, then (m,n)-dim$X \leq k$.* □

References

[1] R. Engelking, General Topology, PWN–Polish Scientific Publishers, Warsaw, 1977.
[2] R. Engelking, Theory of Dimensions Finite and Infinite, Heldermann Verlag, Lemgo, 1995.
[3] V. Fedorchuk, *Finite dimensions defined by means of m-coverings*, Mat. Vesnik **64** (2012), 347–360.
[4] V. Fedorchuk, *Estimates of the dimension (m,n)-dim*, Mosc. Univ. Math. Bull. **68** (2013), 177–181.
[5] V. Fedorchuk, *Inductive (m,n)-dimensions*, Topol. Appl. **169** (2014), 120–135.
[6] V. Fedorchuk, *Valuations for dimension (m,n)-dim*, Matem. Vestnik, MUS (2013) (in Russian), in press.
[7] M. Lynam and L. R. Rubin, *Approximate inverse limits and (m,n)-dimensions*, Glasnik Mat. Ser. III **55**(**75**) (2020), 129–142.
[8] S. Mardešić and L. R. Rubin, *Approximate inverse systems of compacta and covering dimension*, Pacific J. Math. **138** (1989), 129–144.
[9] S. Mardešić and J. Segal, Shape theory, North-Holland Mathematical Library, vol. 26, North-Holland Publishing Co., Amsterdam-New York, 1982.
[10] N. Martynchuk, *Factorization theorem for the dimension (m,n)-dim*, Mosc. Univ. Math. Bull. **68** (2013), 188–191.

[11] N. Martynchuk, *Transfinite extension of dimension function* (m, n)*-dim*, Topol. Appl. **160** (2013), 2514–2522.
[12] N. Martynchuk, *Several remarks concerning* (m, n)*-dimensions*, Topol. Appl. **178** (2014), 219–229.
[13] N. Martynchuk, *Theorem on universal spaces for dimension function* (m, n)*-dim*, Mat. Zametki (2013), in press.

Inverzni sustavi kompaktnih Hausdorffovih prostora i (m, n)-dimenzija

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SAŽETAK. Godine 2012. V. Fedorchuk je koristeći m-parove i n-particije uveo pojam (m, n)-dimenzije prostora. Ona generalizira dimenziju pokrivanja; Fedorchuk je pokazao da je (m, n)-dimenzija sačuvana kod inverznih limesa kompaktnih Hausdorffovih prostora. Zasebno smo karakterizirali one aproksimativne inverzne sustave kompaktnih metričkih prostora čiji limesi imaju specificiranu (m, n)-dimenziju. Naša je karakterizacija u terminima unutrašnjih svojstava sustava. Ovdje ćemo dati paralelnu unutrašnju karakterizaciju onih inverznih sustava kompaktnih Hausdorffovih prostora čiji limesi imaju specificiranu (m, n)-dimenziju. Fedorchukov granični teorem bit će posljedica našeg rezultata.

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