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On the power values of the sum of three squares in arithmetic progression

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Abstract. In this paper, using a deep result on the existence of primitive divisors of Lehmer numbers due to Y. Bilu, G. Hanrot and P. M. Voutier, we first give an explicit formula for all positive integer solutions of the Diophantine equation $(x-d)^2+x^2+(x+d)^2 = y^n$ (*), when n is an odd prime and $d = p^r$, $p > 3$, a prime. So this improves the results of the papers of A. Koutsianas and V. Patel [19] and A. Koutsianas [18]. Secondly, under the assumption of our first result, we prove that (*) has at most one solution (x, y). Next, for a general d, we prove the following two results: (i) if every odd prime divisor q of d satisfies $q \not\equiv \pm 1 \pmod{2n}$, then (*) has only the solution $(x, y, d, n) = (21, 11, 2, 3)$, and (ii) if $n > 228000$ and $d > 8\sqrt{2}$, then all solutions (x, y) of (*) satisfy $y^n < 2^{3/2}d^3$.
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1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ be the sets of all integers, positive integers and rational numbers, respectively. Let k, n be fixed positive integers. The polynomial Diophantine equation of the form

$$1^k + 2^k + \cdots + x^k = y^n, \quad x, y \in \mathbb{N}, \quad n \geq 2 \tag{1}$$

has been studied for more than a hundred years. In 1875, a classical question of E. Lucas [23] was whether equation (1) has only the solutions $x = y = 1$ and $x = 24$, $y = 70$ for $(k, n) = (2, 2)$. In 1918, G. N. Watson [31] solved equation (1) with $(k, n) = (2, 2)$. In 1956, J. J. Schäffer [27] considered equation (1). He showed for $k \geq 1$ and $n \geq 2$ that (1) possesses at most finitely many solutions in positive integers x and y, unless

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\},$$

where, in each case, there are infinitely many such solutions. J. J. Schäffer conjectured that (1) has a unique non-trivial (i.e. $(x, y) \neq (1, 1)$) solution, namely

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$(k, n, x, y) = (2, 2, 24, 70)$. The correctness of this conjecture has been proved for some cases (see, e.g., [6, 10, 15, 16, 17, 25, 26]). But, it has not been proved completely yet.

A more general case is to consider the Diophantine equation

$$(x+1)^k + (x+2)^k + \cdots + (x+r)^k = y^n \quad x, y \in \mathbb{Z}, \quad k, n \geq 2. \tag{2}$$

In 2013, Z. Zhang and M. Bai [4] solved equation (2) with $k = 2$ and $r = x$. In 2014, the equation

$$(x-1)^k + x^k + (x+1)^k = y^n \quad x, y \in \mathbb{Z}, \quad n \geq 2, \tag{3}$$

was solved completely by Z. Zhang [32] for $k = 2, 3, 4$ (Actually, firstly, J. W. S. Cassels [13] considered equation (3) in 1985, and proved that $x = 0, 1, 2, 24$ are the only integer solutions to this equation for $k = 3$ and $n = 2$), and in 2016, M. A. Bennett, V. Patel and S. Siksek [8] extended Z. Zhang's result, completely solving equation (3) in the cases $k = 5$ and $k = 6$. In the same year, M. A. Bennett, V. Patel and S. Siksek [9] considered equation (2). They gave integral solutions to equation (2) using linear forms in logarithms, sieving and Frey curves, where $k = 3$, $2 \leq r \leq 50$, $x \geq 1$, and n is a prime.

Let $k \geq 2$ be even, and let r be a fixed non-zero integer. In 2017, V. Patel and S. Siksek [24] showed that for almost all $d \geq 2$ (in the sense of natural density), the equation

$$x^k + (x+r)^k + \cdots + (x+(d-1)r)^k = y^n, \quad x, y \in \mathbb{Z}, \quad n \geq 2,$$

has no solutions. Let $\ell \geq 2$ be a fixed integer such that ℓ is even. In the same year, the second author [28] considered the equation

$$(x+1)^k + (x+2)^k + \cdots + (\ell x)^k = y^n, \quad x, y \in \mathbb{Z} \quad n \geq 2. \tag{4}$$

He proved that equation (4) has only finitely many solutions, where $x, y \geq 1$, $k \neq 1, 3$. He also showed that equation (4) has infinitely many solutions with $n \geq 2$ and $k = 1, 3$. In 2018, A. Bérczes, I. Pink, G. Savaş and the second author [11] considered equation (4) with $\ell = 2$. They proved that equation (4) has no solutions, where $2 \leq x \leq 13$, $k \geq 1$, $\ell = 2$, $y \geq 2$ and $n \geq 3$. Recently, D. Bartoli and the second author [5] proved that all solutions of equation (4) with $x, y \geq 1, n \geq 2, k \neq 3$ and ℓ odd satisfy $\max\{x, y, n\} < C$, where C is an effectively computable constant depending only on k and ℓ. So, the remaining case for equation (4) was covered by them.

Finding perfect powers that are sums of terms in an arithmetic progression has received much interest; recent contributions can also be found in [1, 3, 7, 14].

Now we consider a generalization of equation (3). Let d be a fixed positive integer. In 2017-2019, Z. Zhang [33] A. Koutsianas and V. Patel [19] studied the integer solutions to the following equation

$$(x-d)^k + x^k + (x+d)^k = y^n, \quad x, y \in \mathbb{Z}, \quad n \geq 2, \tag{5}$$

for the cases $k = 4$ and $k = 2$, respectively. Z. Zhang gave some results on equation (5) with $k = 4$ by using a modular approach. A. Koutsianas and V. Patel [19] gave all non-trivial primitive solutions to equation (5), where $k = 2$, n is a prime and $d \le 10^4$. (According to the terminology of [19], an integer solution (x, y) of (5) is said to be primitive if $\gcd(x, y) = 1$. This is equivalent to x, y, d being pairwise coprime. A solution where $xy = 0$ is called a trivial solution). They used the characterization of primitive divisors in Lehmer sequences due to Y. F. Bilu, G. Hanrot and P. M. Voutier [12]. Then A. A. Garcia and V. Patel [2] showed that the only solutions to equation (5) with $n \ge 5$ a prime, $k = 3$, $\gcd(x, d) = 1$, and $0 < d \le 10^6$ are the trivial ones satifying $xy = 0$.

Recently, A. Koutsianas [18] studied equation (5) with $k = 2$ for an infinitely family of d, which is an extension of [19]. In [18], all solutions (x, y) to the Diophantine equation

$$(x-d)^2 + x^2 + (x+d)^2 = y^n, \quad x, y \in \mathbb{N}, \quad n \ge 2, \quad \gcd(x, y) = 1, \tag{6}$$

are given with the following table, where $d = p^r$ with $r \ge 0$, p a prime and $p \le 10^4$.

p	(x, y, r, n)
2	$(21, 11, 1, 3)$
7	$(3, 5, 1, 3)$
79	$(63, 29, 1, 3)$
223	$(345, 77, 1, 3)$
439	$(987, 149, 1, 3)$
727	$(2133, 245, 1, 3)$
1087	$(3927, 365, 1, 3)$
3109	$(627, 29, 1, 5)$
3967	$(27657, 1325, 1, 3)$
4759	$(36363, 1589, 1, 3)$
5623	$(46725, 1877, 1, 3)$
8647	$(89187, 2885, 1, 3)$

Table 1:

However, Table 1 at least omits the solution $(x, y, d, r, n) = (13, 5, 197, 1, 7)$ of (6) with $p \le 10^4$.

In this paper, extending the results in [18] and [19], we first consider the Diophantine equation (6), where

$$d = p^r \ \text{ with } \ r \in \mathbb{N}. \tag{7}$$

We prove the following two results:

Theorem 1. *Let n be an odd prime, and let d be satisfied as in* (7)*. If (x, y) is a solution to* (6)*, then $p > 3$, and there exists a constant $X_1 \in \mathbb{N}$ such that*

$$d = \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i} (-2)^i \right|. \tag{8}$$

Moreover, if (8) *holds, then the solution* (x, y) *can be expressed as*

$$x = X_1 \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i} (3X_1^2)^{(n-1)/2-i}(-2)^i \right|, \ y = 3X_1^2 + 2. \tag{9}$$

Remark 1. *Theorem 1 gives the missing solution* $(x, y, d, r, n) = (13, 5, 197, 1, 7)$ *in [18], where* $X_1 = 1$ *and* $n = 7$.

Theorem 2. *Under the assumption of Theorem 1,* (6) *has at most one solution* (x, y).

Please note that in [18], while all solutions (x, y) to (6) are given, where $d = p^r$ with $r \geq 0$, p a prime and $p \leq 10^4$, Theorem 1 gives an explicit formula to find all solutions (x, y) to (6) for all $d = p^r$ with $r \in \mathbb{N}$.

Next, for a general d, we prove the following two results:

Theorem 3. *If n is an odd prime and every odd prime divisor q of d satisfies* $q \not\equiv \pm 1 \pmod{2n}$, *then* (6) *has only the solution* $(x, y, d, n) = (21, 11, 2, 3)$.

Theorem 4. *If* $n > 228000$ *and* $d > 8\sqrt{2}$, *then all solutions* (x, y) *to* (6) *satisfy* $y^n < 2^{3/2} d^3$.

2. Proof of Theorem 1

Let D_1, D_2, k be fixed positive integers such that $\min\{D_1, D_2\} > 1$, $2 \nmid k$ and $\gcd(D_1, D_2) = \gcd(D_1 D_2, k) = 1$, and let $h(-4D_1D_2)$ denote the class number of positive binary quadratic primitive forms with discriminant $-4D_1D_2$.

Lemma 1. *If the equation*

$$D_1 X^2 + D_2 Y^2 = k^Z, \ X, Y, Z \in \mathbb{Z}, \ \gcd(X, Y) = 1, \ Z > 0$$

has solutions (X, Y, Z), *then its every solution* (X, Y, Z) *can be expressed as*

$$\begin{gathered} Z = Z_1 t, \, t \in \mathbb{N}, \ 2 \nmid t, \\ X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1 (X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2})^t, \, \lambda_1, \lambda_2 \in \{1, -1\}, \end{gathered}$$

where X_1, Y_1, Z_1 *are positive integers such that*

$$D_1 X_1^2 + D_2 Y_1^2 = k^{Z_1}, \ \gcd(X_1, Y_1) = 1$$

and $h(-4D_1D_2) \equiv 0 \pmod{2Z_1}$.

Proof. This is special case of theorems 1 and 3 of [20] for $D < 0$ and $D_1 > 1$. □

Lemma 2. *If* (6) *has solutions* (x, y), *then* $2 \nmid n$ *and its every solution* (x, y) *can be expressed as*

$$x\sqrt{3} + d\sqrt{-2} = \lambda_1 (X_1\sqrt{3} + \lambda_2 Y_1 \sqrt{-2})^n, \quad \lambda_1, \lambda_2 \in \{\pm 1\}, \tag{10}$$

$$y = 3X_1^2 + 2Y_1^2, \ X_1, Y_1 \in \mathbb{N}, \ \gcd(X_1, Y_1) = 1. \tag{11}$$

Proof. We now assume that (x, y) is a solution to (6). Then we have

$$3x^2 + 2d^2 = y^n. \tag{12}$$

Since $n > 2$ and $\gcd(x, y) = 1$, by (12), we get

$$2 \nmid x,\ 2 \nmid y,\ 3 \nmid y,\ \gcd(x, d) = 1. \tag{13}$$

Hence, we see from (12) and (13) that $\gcd(6, y) = 1$ and the equation

$$3X^2 + 2Y^2 = y^Z,\ X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0, \tag{14}$$

has a solution

$$(X, Y, Z) = (x, d, n). \tag{15}$$

Applying Lemma 1 to (14) and (15), we have

$$n = Z_1 t,\ t \in \mathbb{N},\ 2 \nmid t, \tag{16}$$

$$x\sqrt{3} + d\sqrt{-2} = \lambda_1(X_1\sqrt{3} + \lambda_2 Y_1\sqrt{-2})^t,\ \lambda_1, \lambda_2 \in \{1, -1\}, \tag{17}$$

where X_1, Y_1, Z_1 are positive integers such that

$$3X_1^2 + 2Y_1^2 = y^{Z_1},\ \gcd(X_1, Y_1) = 1, \tag{18}$$

and

$$h(-24) \equiv 0 \pmod{2Z_1}. \tag{19}$$

Further, since $h(-24) = 2$, by (19), we get $Z_1 = 1$. Hence, by (16), we have $t = n$ and $2 \nmid n$. Furthermore, by (17) and (18), we obtain (10) and (11) respectively. Thus, Lemma is proved. □

Let α, β be algebraic integers. If $(\alpha + \beta)^2$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a *Lehmer pair*. Further, let $A = (\alpha + \beta)^2$ and $C = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}(\sqrt{A} + \lambda\sqrt{B}), \quad \beta = \frac{1}{2}(\sqrt{A} - \lambda\sqrt{B}), \quad \lambda \in \{\pm 1\},$$

where $B = A - 4C$. Such (A, B) is called the parameters of Lehmer pair (α, β). Two Lehmer pairs (α_1, β_1) and (α_2, β_2) are called equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 \in \{\pm 1, \pm\sqrt{-1}\}$. Obviously, if (α_1, β_1) and (α_2, β_2) are equivalent Lehmer pairs with parameters (A_1, B_1) and (A_2, B_2), respectively, then $(A_2, B_2) = (\varepsilon A_1, \varepsilon B_1)$, where $\varepsilon \in \{\pm 1\}$. For a fixed Lehmer pair (α, β), one defines the corresponding *sequence of Lehmer numbers* by

$$L_m(\alpha, \beta) = \begin{cases} \dfrac{\alpha^m - \beta^m}{\alpha - \beta}, & \text{if } 2 \nmid m, \\ \dfrac{\alpha^m - \beta^m}{\alpha^2 - \beta^2}, & \text{if } 2|m,\ m \in \mathbb{N}. \end{cases} \tag{20}$$

Then, Lehmer numbers $L_m(\alpha, \beta)$ $(m = 1, 2, \dots)$ are nonzero integers. Further, for equivalent Lehmer pairs (α_1, β_1) and (α_2, β_2), we have $L_m(\alpha_1, \beta_1) = \pm L_m(\alpha_2, \beta_2)$ for any m. A prime q is called a *primitive divisor* of the Lehmer number $L_m(\alpha,\beta)$ ($m>1$) if $q|L_m(\alpha,\beta)$ and $q\nmid ABL_1(\alpha,\beta)\cdots L_{m-1}(\alpha,\beta)$, where (A,B) is the parameters of Lehmer pair (α,β). For a fixed positive integer m, a Lehmer pair (α,β) such that $L_m(\alpha,\beta)$ has no primitive divisor will be called an *m-defective* Lehmer pair. Further, a positive integer m is called totally *non-defective* if no Lehmer pair is m-defective.

Lemma 3 (see [30]). *Let m be such that $6<m\le 30$ and $m\neq 8,10,12$. Then up to equivalence, all parameters (A,B) $(A>0)$ of m-defective Lehmer pairs are given as follows:*

(i) $m=7$, $(A,B)=(1,-7),(1,-19),(3,-5),(5,-7),(13,-3),(14,-22)$.

(ii) $m=9$, $(A,B)=(5,-3),(7,-1),(7,-5)$.

(iii) $m=13$, $(A,B)=(1,-7)$.

(iv) $m=14$, $(A,B)=(3,-13),(5,-3),(7,-1),(7,-5),(19,-1),(22,-14)$.

(v) $m=15$, $(A,B)=(7,-1),(10,-2)$.

(vi) $m=18$, $(A,B)=(1,-7),(3,-5),(5,-7)$.

(vii) $m=24$, $(A,B)=(3,-5),(5,-3)$.

($viii$) $m=26$, $(A,B)=(7,-1)$.

(ix) $m=30$, $(A,B)=(1,-7),(2,-10)$.

Lemma 4 (see [12]). *Every positive integer m with $m>30$ is totally non-defective.*

Proof of Theorem 1 We now assume that (x,y) is a solution to (6). Then, x,y and d satisfy (12). If $p=3$, then from (7) and (12) we get $3\mid y$, which contradicts (13). So we have $p>3$.

By Lemma 2, there exist positive integers X_1 and Y_1 satisfying (10) and (11). By (10), we have

$$x=X_1\left|\sum_{i=0}^{(n-1)/2}\binom{n}{2i}(3X_1^2)^{(n-1)/2-i}(-2Y_1^2)^i\right| \tag{21}$$

and

$$d=Y_1\left|\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3X_1^2)^{(n-1)/2-i}(-2Y_1^2)^i\right|. \tag{22}$$

Since d satisfies (7), by (22), we get

$$Y_1=p^s,\ \ s\in\mathbb{Z},\ 0\le s\le r \tag{23}$$

and

$$\left|\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3X_1^2)^{(n-1)/2-i}(-2Y_1^2)^i\right| = p^{r-s}. \tag{24}$$

Let

$$\alpha = X_1\sqrt{3} + Y_1\sqrt{-2},\ \beta = X_1\sqrt{3} - Y_1\sqrt{-2}. \tag{25}$$

By (11) and (25), we have

$$\alpha + \beta = 2X_1\sqrt{3},\ \alpha - \beta = 2Y_1\sqrt{-2},\ \alpha\beta = y. \tag{26}$$

Notice that $y \geq 5$ by (11), and α/β satisfies

$$y\left(\frac{\alpha}{\beta}\right)^2 - 2(3X_1^2 - 2Y_1^2)\frac{\alpha}{\beta} + y = 0, \tag{27}$$

with $\gcd(y, 2(3X_1^2 - 2Y_1^2)) = 1$. This implies that α/β is not a root of unity. Hence, we see from (13), (25) and (26) that (α, β) is a Lehmer pair with the parameters

$$(A, B) = (12X_1^2, -8Y_1^2). \tag{28}$$

Further, let $L_m(\alpha, \beta)$ $(m = 1, 2, \dots)$ be the corresponding Lehmer numbers. By (20) and (25), we have

$$\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3X_1^2)^{(n-1)/2-i}(-2Y_1^2)^i = L_n(\alpha, \beta). \tag{29}$$

Therefore, by (24) and (29), we get

$$|L_n(\alpha, \beta)| = p^{r-s}. \tag{30}$$

If $s > 0$, by (23), (28) and (30), the Lehmer number $L_n(\alpha, \beta)$ has no primitive divisors. Therefore, since n is an odd prime, by lemmas 12 and 13, we find from (28) that $n \in \{3, 5\}$.

When $n = 3$, by (23) and (24), we have

$$9X_1^2 - 2p^{2s} = \pm p^{r-s}. \tag{31}$$

Notice that $p > 3$, $s > 0$ and $\gcd(X_1, Y_1) = \gcd(X_1, p^s) = 1$. We see from (31) that $r - s = 0$ and

$$9X_1^2 - 2p^{2s} = \pm 1. \tag{32}$$

Further, since $2 \nmid X_1$ and $9X_1^2 - 2p^{2s} \equiv 1 - 2 \equiv -1 \pmod 8$, by (32), we get

$$9X_1^2 - 2p^{2s} = -1. \tag{33}$$

But, since $(2/3) = -1$, where $(*/*)$ is the Legendre symbol, (33) is false. So, we have no solutions for $n = 3$.

When $n=5$, by (23) and (24), we have

$$45X_1^4-60X_1^2p^{2s}+4p^{4s}=\pm p^{r-s}. \tag{34}$$

If $r-s>0$, since $p>3$, then from (34) we get $p=5$ and

$$9X_1^4-12\cdot 5^{2s}X_1^2+4\cdot 5^{4s-1}=\pm 5^{r-s-1},$$

whence we obtain $r-s=1$ and

$$9X_1^4-12\cdot 5^{2s}X_1^2+4\cdot 5^{4s-1}=\pm 1. \tag{35}$$

Further, since $9X_1^4\equiv 1 \pmod 4$, the right-hand side of (35) is equal to 1. However, since $5\nmid X_1$ and $9X_1^4\equiv 9\equiv -1 \pmod 5$, the right-hand side of (35) should be equal to -1, a contradiction. So we have $r-s=0$ and

$$45X_1^4-60X_1^2p^{2s}+4p^{4s}=\pm 1. \tag{36}$$

Similarly, since $45X_1^4\equiv 1 \pmod 4$ and $4p^{4s}\equiv -1 \pmod 5$, (36) is false. This implies that we have no solutions for $n=5$.

By the above analysis, we get $s=0$. Then, by (23), we have $Y_1=1$. Therefore, by (11), (21) and (22), we obtain (8) and (9). Thus, the theorem is proved.

3. Proof of Theorem 2

For fixed d with (7) and n an odd prime, we now assume that (6) has two distinct solutions $(x,y)=(x_1,y_1)$ and (x_2,y_2). Then, by Theorem 1, we have

$$\begin{aligned} d &= \left|\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3a^2)^{(n-1)/2-i}(-2)^i\right| \\ &= \left|\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3b^2)^{(n-1)/2-i}(-2)^i\right|, \end{aligned} \tag{37}$$

$$y_1=3a^2+2,\ y_2=3b^2+2,\ a,b\in\mathbb{N},\ 2\nmid ab. \tag{38}$$

Since $(x_1,y_1)\neq(x_2,y_2)$, we have $y_1\neq y_2$. Therefore, without loss of generality, we may assume that $y_1<y_2$. Then, by (38), we get $a<b$.

Since n is an odd prime, we have $n\mid\binom{n}{2i+1}$ for $i=0,\cdots,(n-3)/2$. Hence, since $n\nmid 2^{(n-1)/2}$, we see from (37) that

$$\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3a^2)^{(n-1)/2-i}(-2)^i=\sum_{i=0}^{(n-1)/2}\binom{n}{2i+1}(3b^2)^{(n-1)/2-i}(-2)^i,$$

whence we get

$$\sum_{i=0}^{(n-3)/2}\binom{n}{2i+1}\left(\frac{(3b^2)^{(n-1)/2-i}-(3a^2)^{(n-1)/2-i}}{3b^2-3a^2}\right)(-2)^i=0. \tag{39}$$

Let $X = 3b^2$ and $Y = 3a^2$. Then (39) can be rewritten as

$$\sum_{i=0}^{(n-3)/2} \binom{n}{2i+1} \left(\frac{X^{(n-1)/2-i} - Y^{(n-1)/2-i}}{X-Y} \right) (-2)^i = 0. \tag{40}$$

By (40), we have $n > 3$ and

$$2 \left| \frac{X^{(n-1)/2} - Y^{(n-1)/2}}{X-Y} \right.. \tag{41}$$

Since $2 \nmid XY$ by (38), we see from (41) that $2 \mid (n-1)/2$. Further, let $2^y \parallel n-1$. Then we have $y \geq 2$ and

$$2^{y-1} \left\| \binom{(n-1)/2}{1} Y^{(n-3)/2} \right.. \tag{42}$$

Let $2^{r_j} \parallel j$ for $j > 1$. Since $j \geq 2^{r_j}$, we have $r_j \leq (\log j)/(\log 2) \leq j-1$. Since $X - Y \equiv 3a^2 - 3b^2 \equiv 0 \pmod{2^3}$, we get

$$\begin{aligned}
&\binom{(n-1)/2}{j} (X-Y)^{j-1} Y^{(n-1)/2-j} \\
&\quad \equiv \left(\frac{n-1}{2} \right) Y^{(n-1)/2-j} \binom{(n-3)/2}{j-1} \frac{(X-Y)^{j-1}}{j} \\
&\quad \equiv 0 \pmod{2^y}, \ j > 1.
\end{aligned} \tag{43}$$

Hence, since

$$\frac{X^{(n-1)/2} - Y^{(n-1)/2}}{X-Y} = \sum_{j=1}^{(n-1)/2} \binom{(n-1)/2}{j} (X-Y)^{j-1} Y^{(n-1)/2-j},$$

from (42) and (43) we obtain that

$$2^{y-1} \left\| \frac{X^{(n-1)/2} - Y^{(n-1)/2}}{X-Y} \right. = \frac{(3b^2)^{(n-1)/2} - (3a^2)^{(n-1)/2}}{3b^2 - 3a^2}. \tag{44}$$

On the other hand, let $2^{\delta_i} \parallel 2i$ for $i \geq 1$. Then we have

$$\delta_i \leq \frac{\log(2i)}{\log 2} \leq i, \ i \geq 1. \tag{45}$$

By (45), we get

$$\binom{n}{2i+1} (-2)^i \equiv n(n-1) \binom{n-2}{2i-1} \frac{(-2)^i}{2i(2i+1)} \equiv 0 \pmod{2^y}, \ i \geq 1. \tag{46}$$

Therefore, since $2 \nmid n$, we find from (44) and (46) that (39) is false. It implies that, under the assumption of Theorem 1, (6) has at most one solution (x, y). The theorem is proved.

4. Proof of Theorem 3

Lemma 5 (see [22]). *If n is an odd prime and r is a prime divisor of the Lehmer number $L_n(\alpha, \beta)$, then $r \equiv \pm 1 \pmod{2n}$.*

Proof of Theorem 3 By Lemma 2, if (x, y) is a solution to (6), then x, y and d satisfy (10) and (11). Let α, β be defined as in (25). Then (α, β) is a Lehmer pair with the parameters (28). Further, let $L_m(\alpha, \beta)$ $(m = 1, 2, \cdots)$ be the corresponding Lehmer numbers. By (22) and (29), we have

$$d = Y_1 |L_n(\alpha, \beta)|. \tag{47}$$

Since n is an odd prime and every odd prime divisor q of d satisfies $q \not\equiv \pm 1 \pmod{n}$, by Lemma 5, from (47) we get

$$|L_n(\alpha, \beta)| = 1, \tag{48}$$

and

$$Y_1 = d. \tag{49}$$

We see from (48) that the Lehmer number $L_n(\alpha, \beta)$ has no primitive divisors. Therefore, using the same method as in the proof of Theorem 1, by lemmas 3 and 4, we can deduce from (48) that $n \in \{3, 5\}$.

When $n = 3$, by (29), (48) and (49), we have

$$9X_1^2 - 2d^2 = \pm 1. \tag{50}$$

Since $n = 3$ and every odd prime divisor q of d satisfies $q \not\equiv \pm 1 \pmod{3}$, q can only be equal to 3. However, by (50), it is impossible. Hence, d must be a power of 2. Then (50) reduces to the equation

$$X^2 + 1 = 2^{2k+1},\ X = 3X_1,\ k \geq 0, \tag{51}$$

or

$$X^2 - 1 = 2^{2k+1},\ X = 3X_1,\ k \geq 0. \tag{52}$$

By [21], we see that (51) has no solution. Since $\gcd(X + 1, X - 1) = 2$, from (52) we get $X - 1 = 2$ and $k = 1$. It follows that the equation has only the solution $(X, k) = (3, 1)$. Therefore, it is easy to get $X_1 = 1$ and $d = 2$. Thus, (6) has only the solution $(x, y, d, n) = (21, 11, 2, 3)$ in this case.

When $n = 5$, by (29), (48) and (49), we have

$$45X_1^4 - 60X_1^2 d^2 + 4d^4 = \pm 1. \tag{53}$$

But, since $2 \nmid X_1$, $45X_1^4 \equiv 1 \pmod{4}$, $5 \nmid d$ and $4d^4 \equiv -1 \pmod{5}$, (53) is false. The theorem is proved.

5. Proof of Theorem 4

For any algebraic number θ of degree ℓ over $\mathbb{Q}$, let $h(\theta)$ be the absolute logarithmic height of θ by the formula

$$h(\theta)=\frac{1}{\ell}\left(\log|a|+\sum_{j=1}^{\ell}\log\max\left\{1,|\theta^{(j)}|\right\}\right),$$

where a is the leading coefficient of the minimal polynomial of θ over $\mathbb{Z}$ and $\theta^{(j)}$ $(j=1,\cdots,\ell)$ are all the conjugates of θ. Further, let $\log\theta$ be any determination of the logarithm of θ.

Lemma 6 (Appendix to [12]). *Let θ be a complex algebraic number with $|\theta|=1$, and θ is not a root of unity. Let b_1, b_2 be positive integers, and let $\Lambda=b_1\log\theta-b_2\pi\sqrt{-1}$. Then we have*

$$\log|\Lambda|>-(9.03H^2+0.23)(Dh(\theta)+25.84)-2H-2\log H-0.7D+2.07,$$

where $D=[\mathbb{Q}(\theta):\mathbb{Q}]/2$, $H=D(\log B-0.96)+4.49$, $B=\max\{13,b_1,b_2\}$.

Proof of Theorem 4 By Lemma 2, if (x,y) is a solution to (6), then

$$d=\frac{1}{2\sqrt{2}}|\alpha^n-\beta^n|, \tag{54}$$

where α,β are defined as in (25). By (11) and (25), we have

$$|\alpha|=|\beta|=\sqrt{y}. \tag{55}$$

Let $\theta=\alpha/\beta$. By (55) and (27), it is a complex algebraic number with $|\theta|=1$, θ is not a root of unity and

$$h(\theta)=\frac{1}{2}\log y. \tag{56}$$

By (54) and (55), we have

$$d=\frac{1}{2\sqrt{2}}|\beta^n|\left|\left(\frac{\alpha}{\beta}\right)^n-1\right|=\frac{1}{2\sqrt{2}}y^{n/2}|\theta^n-1|. \tag{57}$$

It is well known that for any complex number z, we have either $|e^z-1|\geq\frac{1}{2}$ or $|e^z-1|\geq\frac{2}{\pi}|z-t\pi\sqrt{-1}|$ for some integers t (see [29]). Put $z=n\log\theta$. We get either

$$|\theta^n-1|\geq\frac{1}{2}, \tag{58}$$

or

$$|\theta^n-1|\geq\frac{2}{\pi}|n\log\theta-t\pi\sqrt{-1}|,\ t\in\mathbb{N},\ t\leq n. \tag{59}$$

If (58) holds, since $d > 8\sqrt{2}$, then from (57) we obtain $y^n \le 32d^2 < 2^{3/2}d^3$ and the theorem is true. So we just have to worry about the case (59).

Let

$$\Lambda = n \log \theta - t\pi\sqrt{-1}. \tag{60}$$

By (57), (59) and (60), we have

$$d \ge \frac{y^{n/2}}{\pi\sqrt{2}}|\Lambda|. \tag{61}$$

If $y^n \ge 2^{3/2}d^3$, then from (61) we get

$$\pi \ge y^{n/6}|\Lambda|,$$

whence we obtain

$$\log \pi \ge \frac{n}{6}\log y + \log|\Lambda|. \tag{62}$$

Notice that $[\mathbb{Q}(\theta) : \mathbb{Q}] = 2$, $n \ge t$ and $n > 228000$. Applying Lemma 6 to (60), by (56), we have

$$\log|\Lambda| > -(9.03H^2 + 0.23)(\frac{1}{2}\log y + 25.84) - 2H - 2\log H + 1.37, \tag{63}$$

where

$$H = \log n + 3.53. \tag{64}$$

A combination of (62) and (63) yields

$$(9.03H^2 + 0.23)\left(0.5 + \frac{25.84}{\log y}\right) + \frac{2H + 2\log H}{\log y} > \frac{n}{6}. \tag{65}$$

Further, by (11), we have $y \ge 5$. Hence, by (64) and (65), we get

$$\begin{aligned} &99.36(9.03(\log n + 3.53)^2 + 0.23) + 7.50(\log n + 3.53 \\ &+ \log(\log n + 3.53)) = 99.36(9.03H^2 + 0.23) \\ &+ 7.50(H + \log H) > n. \end{aligned} \tag{66}$$

However, by (66), we calculate that $n < 228000$, a contradiction. Thus, if $n > 228000$ and $d > 8\sqrt{2}$, then $y^n < 2^{3/2}d^3$. The theorem is proved.

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