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A note on the Fibonacci m-step sequences modulo q

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Abstract. We briefly discuss a congruence relation of the subsequence of the Fibonacci m-step numbers. Then we use the obtained result to establish many identities regarding the Tribonacci numbers mod q with indices in arithmetic progression and connect them with the existing results. Finally, we discuss arbitrary linear recurrence sequences.

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1. Introduction

Tribonacci sequence is a sequence of integer numbers that was introduced formally in 1963 by Feinberg [5]. It has received a lot of attention since then, including generalizations to Tetranacci or, more general, to Fibonacci m-step numbers (or m-nacci numbers) [7]. For $n \geq 3$, the classical Tribonacci sequence $\{T_n\}_{n\geq 0}$ satisfies the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with initial values $T_0 = T_1 = 0$ and $T_2 = 1$ [11], and it will be called *the regular Tribonacci sequence* for the purpose of this article (see [10] for the values of the sequence).

There are many known properties of the Tribonacci numbers (see [1, 4, 9, 11]) that also generalize to Fibonacci m-step numbers. In [6], the goal is to consider indices forming arithmetic progression and to find the partial sum of such subsequence. While the closed (and somehow complicated) formula for that can exist, it is also interesting to find the residue class modulo q of the sum. Such an approach was considered in part in a short remark by Atanassova [2], who proved that

$$T_n + T_{n+13} + T_{n+26} \equiv 0 \pmod 3. \tag{1}$$

In a more general case, in the classic paper of Waddill [12], the following theorem is presented:

Theorem 1. *Suppose p is a prime number and K is the period of $(T_n \mod p)_{n\in\mathbb{N}}$. If $0 < k < K$ and $T_k = T_{k+1} \equiv 0 \pmod p$, then $K = 3k$ and $T_n + T_{n+k} + T_{n+2k} \equiv 0 \pmod p$ for all n.*

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The number K defined in the above theorem is called the *Pisano period* (or the *Pisano number*) of the sequence $(T_n \mod p)_{n\in\mathbb{N}}$.

Let $(F_n^{(m)}(a_0,\dots,a_{m-1}))_{n\in\mathbb{N}}$ denote the sequence of Fibonacci m-step numbers with initial values $F_i^{(m)} = a_i$ for $i = 0,\dots,m-1$. Here, the recurrence is given by

$$F_{n+1}^{(m)}(a_0,\dots,a_{m-1}) = \sum_{k=n-m+1}^{n} F_k^{(m)}(a_0,\dots,a_{m-1}).$$

We let $F_n^{(m)}$ denote the *regular Fibonacci m-step sequence*, that is, the sequence with initial values $a_0 = \cdots = a_{m-2} = 0, a_{m-1} = 1$. If $m \le 4$, then we use the convenience $a_0 = a$, $a_1 = b$, $a_2 = c$ and $a_3 = d$. If $m = 3$ ($m = 4$, respectively), then we use a standard notation for the Tribonacci numbers $T_n = F_n^{(3)}$ (the Tetranacci numbers $Q_n = F_n^{(4)}$, respectively). We extend the definition of the Pisano period to Fibonacci m-step sequences modulo arbitrary positive integer q and call it the period of the sequence $(F_n^{(m)}(a_0,\dots,a_{m-1}) \mod q)_{n\in\mathbb{N}}$. In this note, the symbol p always stands for some prime number.

In the paper of Waddill [12], a different initial condition for T_n is set, that is, $T_0 = 0$ and $T_1 = T_2 = 1$. In the paper of Atanassova [2], three different initial conditions are provided for the Tribonacci sequence and for each of those cases congruence (1) is established. Keeping the spirit of considering different initial conditions, one can ask how important these conditions are when establishing (1) or a similar formula. In this note, we investigate that problem. In particular, we will show the relation between

$$\sum_{j=0}^{\ell-1} F_{n+kj}^{(m)} \equiv 0 \pmod{q} \tag{2}$$

and

$$\sum_{j=0}^{\ell-1} F_{n+kj}^{(m)}(a_0,\dots,a_{m-1}) \equiv 0 \pmod{q}. \tag{3}$$

Here, $q > 1$ (not necessarily prime), $k > 0$ (called the *spread)* and $\ell > 1$ (called the *length*); in particular, the indices form arithmetic progression of length ℓ with the difference k.

2. Positive relations

In order to investigate the relation between (2) and (3) for various initial conditions, we first establish the relation between the sequences themselves. In particular, we show in the following lemma that any Fibonacci m-step sequence can be represented using the regular Fibonacci m-step sequence. For technical reasons we extend the sequence to negative indices using the relation

$$F_{n-m+1}^{(m)}(a_0,\dots,a_{m-1}) = F_{n+1}^{(m)}(a_0,\dots,a_{m-1}) - \sum_{k=n-m+2}^{n} F_k^{(m)}(a_0,\dots,a_{m-1}).$$

Then, for example, for $m = 4$ we have

$$Q_{-1}(a,b,c,d)=Q_3(a,b,c,d)-Q_2(a,b,c,d)-Q_1(a,b,c,d)-Q_0(a,b,c,d)=d-c-b-a,$$
$$Q_{-2}(a,b,c,d)=Q_2(a,b,c,d)-Q_1(a,b,c,d)-Q_0(a,b,c,d)-Q_{-1}(a,b,c,d)=2c-d,$$
$$Q_{-3}(a,b,c,d)=Q_1(a,b,c,d)-Q_0(a,b,c,d)-Q_{-1}(a,b,c,d)-Q_{-2}(a,b,c,d)=2b-c.$$

Note that in the case of regular Fibonacci m-step numbers we also have

$$F_{-1}^{(m)} = 1, \quad F_{-2}^{(m)} = -1, \quad F_{-3}^{(m)} = \cdots = F_{-m}^{(m)} = 0. \tag{4}$$

Lemma 1. *We have*

$$F_n^{(m)}(a_0,\ldots,a_{m-1}) = a_{m-1}F_n^{(m)} + \sum_{j=0}^{m-2} a_j \left[\sum_{k=0}^{j} F_{n-1-k}^{(m)}\right]. \tag{5}$$

Proof. The proof goes by induction on n.

First, we show that the formula holds for $n = 0,\ldots,m-1$. Indeed, since $F_{m-1}^{(m)} = 1$ and $F_0^{(m)} = \cdots = F_{m-2}^{(m)} = 0$, we have

$$\begin{aligned}
a_{m-1} &= a_{m-1}\cdot 1 + \sum_{j=0}^{m-2} a_j \cdot 0 \\
&= a_{m-1}F_{m-1}^{(m)} + \sum_{j=0}^{m-2} a_j \left[\sum_{k=0}^{j} F_{m-2-k}^{(m)}\right], \\
a_{m-2} &= a_{m-1}\cdot 0 + a_{m-2}\cdot 1 + \sum_{j=0}^{m-3} a_j \cdot 0 \\
&= a_{m-1}F_{m-2}^{(m)} + a_{m-2}F_{-1}^{(m)} + \sum_{j=0}^{m-3} a_j \left[\sum_{k=0}^{j} F_{m-3-k}^{(m)}\right] \\
&= a_{m-1}F_{m-2}^{(m)} + \sum_{j=0}^{m-2} a_j \left[\sum_{k=0}^{j} F_{m-3-k}^{(m)}\right].
\end{aligned}$$

Then for $2 < i \le m$ and using (4),

$$\begin{aligned}
\sum_{j=0}^{m-i-1} a_j \left[\sum_{k=0}^{j} F_{m-i-1-k}^{(m)}\right] &= \sum_{j=0}^{m-i-1} a_j \cdot 0 = 0, \\
a_{m-i} \left[\sum_{k=0}^{m-i} F_{m-i-1-k}^{(m)}\right] &= a_{m-i}F_{-1}^{(m)} = a_{m-i}, \\
\sum_{j=m-i+1}^{m-2} a_j \left[\sum_{k=0}^{j} F_{m-i-1-k}^{(m)}\right] &= \sum_{j=m-i+1}^{m-2} a_j (F_{-1}^{(m)} + F_{-2}^{(m)}) = 0.
\end{aligned}$$

This implies

$$a_{m-i} = a_{m-1}F^{(m)}_{m-i} + \sum_{j=0}^{m-2} a_j \left[\sum_{k=0}^{j} F^{(m)}_{m-i-1-k}\right]$$

and completes the first step of the induction.

We now move to the second step. Suppose that (5) holds for some fixed $n \geq m-1$. Then

$$\begin{aligned}
F^{(m)}_{n+1}(a_0,\dots,a_{m-1}) &= \sum_{k=n-m+1}^{n} F^{(m)}_k(a_0,\dots,a_{m-1}) \\
&= \sum_{k=n-m+1}^{n} a_{m-1}F^{(m)}_k + \sum_{k=n-m+1}^{n}\sum_{j=0}^{m-2} a_j \left[\sum_{i=0}^{j} F^{(m)}_{k-i}\right] \\
&= a_{m-1}F^{(m)}_{n+1} + \sum_{j=0}^{m-2} a_j \left[\sum_{i=0}^{j}\sum_{k=n-m+1}^{n} F^{(m)}_{k-i}\right] \\
&= a_{m-1}F^{(m)}_{n+1} + \sum_{j=0}^{m-2} a_j \left[\sum_{i=0}^{j} F^{(m)}_{n+1-i}\right].
\end{aligned}$$

This completes the proof. □

The formula obtained above resembles (in some way) the formula for the n-th power of t, where t is a unique real solution of $x^m = x^{m-1} + \cdots + x + 1$ (the characteristic polynomial of regular Fibonacci m-step recurrence). For more details regarding the case $m = 3$, see [3].

We are ready to show that if the congruence (2) holds, then (3) holds as well.

Theorem 2. *Suppose*

$$\sum_{j=0}^{\ell-1} F^{(m)}_{n+jk} \equiv 0 \pmod q$$

holds for some $q > 1$, fixed spread k and length ℓ and arbitrary n. Then for all n,

$$\sum_{j=0}^{\ell-1} F^{(m)}_{n+jk}(a_0,\dots,a_{m-1}) \equiv 0 \pmod q, \tag{6}$$

provided $(a_0,\dots,a_{m-1}) \neq (0,\dots,0) \in \mathbb{Z}^m$.

Proof. The proof is a direct application of Lemma 1 to the left-hand side of the formula (6). □

Remark 1. *Note that if $a_0 = \cdots = a_{m-1}$, then the relation* (6) *holds trivially. Thus this case is not considered as a part of the theorem.*

Corollary 1 (see [2])**.** *For any $n \in \mathbb{N}$,*

$$\begin{aligned}
T_n(0,0,1) + T_{n+13}(0,0,1) + T_{n+26}(0,0,1) &\equiv 0 \pmod 3, \\
T_n(0,1,1) + T_{n+13}(0,1,1) + T_{n+26}(0,1,1) &\equiv 0 \pmod 3, \\
T_n(1,1,1) + T_{n+13}(1,1,1) + T_{n+26}(1,1,1) &\equiv 0 \pmod 3.
\end{aligned}$$

Proof. The first identity can be directly verified. Then the remaining ones follow from Theorem 2. □

As a showcase, we now establish some example congruences valid for the Tribonacci numbers.

Corollary 2. *For any $(a,b,c) \neq (0,0,0)$ (in each case the inequality is considered up to the respective residue class), we have*

$$\sum_{j=0}^{3} T_{n+42j}(a,b,c) \equiv 0 \pmod{13}, \qquad \sum_{j=0}^{3} T_{n+176j}(a,b,c) \equiv 0 \pmod{2113},$$
$$\sum_{j=0}^{3} T_{n+865j}(a,b,c) \equiv 0 \pmod{13841}, \qquad \sum_{j=0}^{4} T_{n+31j}(a,b,c) \equiv 0 \pmod{5},$$
$$\sum_{j=0}^{4} T_{n+242j}(a,b,c) \equiv 0 \pmod{3631}, \qquad \sum_{j=0}^{5} T_{n+8j}(a,b,c) \equiv 0 \pmod{7},$$
$$\sum_{j=0}^{5} T_{n+86j}(a,b,c) \equiv 0 \pmod{3613}, \qquad \sum_{j=0}^{6} T_{n+63j}(a,b,c) \equiv 0 \pmod{883},$$
$$\sum_{j=0}^{6} T_{n+938j}(a,b,c) \equiv 0 \pmod{19699}, \qquad \sum_{j=0}^{66} T_{n+98j}(a,b,c) \equiv 0 \pmod{19699}.$$

Proof. By Theorem 2 it is sufficient to check whether any of the relations hold for $(a,b,c) = (0,0,1)$. These can be easily verified by means of a computer. □

Recall that the sequence $(T(0,1,1) \mod p)_{n\in\mathbb{N}}$ is considered in the paper of Waddill [12]. We conclude this section with the following observation which shows that Waddill's choice of initial conditions also gives the result in the spirit of Theorem 2.

Corollary 3. *Suppose*

$$\sum_{j=0}^{\ell-1} T_{n+jk}(0,1,1) \equiv 0 \pmod{q}$$

holds for some $q > 1$, fixed spread k and length ℓ and arbitrary n. Then for all $n \in \mathbb{N}$,

$$\sum_{j=0}^{\ell-1} T_{n+jk}(a,b,c) \equiv 0 \pmod{q}$$

provided $(a,b,c) \neq (0,0,0)$.

Proof. It follows from the fact that $T_n(0,1,1) = T_{n+1}(0,0,1)$ holds for any integer n. □

3. Discussion of negative relations

It turns out that the converse of Theorem 2 does not hold. In this case, however, we do not know if there is any argument valid for all $m \geq 2$. Since the counterexample requires finding for each admissible value of m three integers q, k and ℓ, the only viable solution for us was to solve it case-by-case by means of a computer. We thus give two examples along with some comments related to them.

Proposition 1. *The converse of Theorem 2 does not hold in the case $m = 3$.*

Proof. Take $(a,b,c) = (3,1,3)$, $\ell = 10$, $k = 1$ and $q = 11$. Then

$$T_n(a,b,c) + \cdots + T_{n+9}(a,b,c) \equiv 0 \pmod{11}$$

for any n. Indeed, the sequence $(T_n(a,b,c) \mod 11)_{n\in\mathbb{N}}$ is periodic with the period

$$(3,1,3,7,0,10,6,5,10,10),$$

and these numbers add up to $0 \pmod{11}$. On the other hand,

$$T_0 + \cdots + T_9 = 96 \not\equiv 0 \pmod{11}.$$

□

Remark 2. *The example presented in the above proposition was found looking for the initial condition such that the Pisano number of $(T_n(a,b,c) \mod 11)_{n\in\mathbb{N}}$ is strictly smaller than the Pisano number of $(T_n \mod 11)_{n\in\mathbb{N}}$. In the case $(a,b,c) = (3,1,3)$, the Pisano number is 10, while for a regular Tribonacci sequence the Pisano number is 110. Other examples that are suitable for the proposition to hold were found by means of a computer. Below we present the ones with $a = 0$.*

b	1	2	3	4	5	6	7	8	9	10
c	5	10	4	9	3	8	2	7	1	6

In total, we have found 120 triples (a,b,c) forming a sequence with Pisano period 10 that sum up to $0 \mod 11$. It is also interesting to note that for each a and b with $(a,b) \neq (0,0)$, there is a unique c such that $(T_n(a,b,c) \mod 11)_{n\in\mathbb{N}}$ has Pisano period 10.

Proposition 2. *The converse of Theorem 2 does not hold in the case $m = 4$.*

Proof. Consider $q = 7$, $a = 2$, $b = 3$, $c = 1$, $d = 3$. Then the Pisano period of $(Q_n(0,0,0,1) \mod 7)_{n\in\mathbb{N}}$ equals 342, while the Pisano period of $(Q_n(2,3,1,3) \mod 7)_{n\in\mathbb{N}}$ equals 171. It is now easy to check that the sum

$$\sum_{k=0}^{170} Q_{n+k}(2,3,1,3) \equiv 0 \pmod{7}$$

for any $n \in \mathbb{N}$, but

$$\sum_{k=0}^{170} Q_k \equiv 3 \pmod{7}.$$

□

Remark 3. *Similarly to the previous case, for each (a, b, c, d) with $(a, b, c) \neq (0, 0, 0)$, there is a unique d such that $(Q_n(a, b, c, d))_{n\in\mathbb{N}}$ has the property*

$$\sum_{k=0}^{170} Q_{n+k}(a, b, c, d) \equiv 0 \pmod 7.$$

In total, we have found 342 such quadruples.

4. Linear recurrence sequences

In this section, we consider integer sequences $(U_n)_{n\in\mathbb{N}}$ defined by the recurrence

$$U_{n+1} = c_0 U_n + c_1 U_{n-1} + \cdots + c_{m-1} U_{n-m+1}, \tag{7}$$

with initial conditions

$$U_0 = a_0,\ U_1 = a_1,\ \ldots,\ U_{m-1} = a_{m-1}. \tag{8}$$

To avoid ambiguity we introduce the following notation. Fix $m > 0$ and let $c = (c_0, \ldots, c_{m-1}) \in \mathbb{Z}^m$ and $a = (a_0, \ldots, a_{m-1}) \in \mathbb{Z}^m$. Fixing the vectors a and c in $\mathbb{Z}^m$, we let $U_n^{(m)}(a, c)$ denote the sequence U_n, which is a solution of (7) with initial values (8).

We now investigate a generalization of Theorem 2 to sequences $(U_n^{(m)}(a, c))_{n\in\mathbb{N}}$. We are interested in a result similar to Lemma 1, that is, to decompose an arbitrary linear recurrence to the sum of regular Fibonacci m-step numbers.

It turns out that it is no longer possible to decompose $U_n^{(m)}(a, c)$ as described and maintain any relation between the two respective congruences. Namely, if we consider two congruences: (2) and

$$\sum_{j=0}^{\ell-1} U_{n+kj}^{(m)}(a, c) \equiv 0 \pmod q, \tag{9}$$

then there is no relation between them.

Proposition 3. *The relation* (2) *does not imply* (9).

Proof. One can check that (see Remark 2)

$$\sum_{j=0}^{109} T_{n+j} \equiv 0 \pmod{11}.$$

On the other hand,

$$\sum_{j=0}^{109} U_j^{(3)}((0, 0, 1), (3, 1, 5)) \equiv 3 \pmod{11}.$$

□

Proposition 4. *The relation* (9) *does not imply* (2).

Proof. One can check that

$$\sum_{j=0}^{54} U_{n+j}^{(3)}((1,1,1),(3,1,1)) \equiv 0 \pmod{11}$$

holds for arbitrary n. Indeed, the sequence $(U_{n+j}^{(3)}((1,1,1),(3,1,1)) \mod 11)_{n\in\mathbb{N}}$ is periodic with the Pisano period equal to 55:

$$\begin{array}{l}(1,\ 1,\ 1,5,6,\ 2,\ \ 6,\ 4,9,4,\ 3,\\ \ 0,\ \ 7,\ 2,2,4,\ 5,\ 10,6,0,5,10,\\ \ 2,\ 10,9,6,4,\ 5,\ \ 3,\ 7,7,9,\ 8,\\ \ 7,\ \ 5,\ 8,3,0,\ 0,\ \ 3,\ 9,8,3,\ 4,\\ \ 1,\ 10,2,6,8,10,\ 0,\ 7,9,1,\ 8).\end{array}$$

Furthermore, the sum of the period reduces to $0 \pmod{11}$ and so does any shift of that period.

On the other hand,

$$\sum_{j=0}^{54} T_j \equiv 8 \pmod{11}.$$

$\square$

Nevertheless, we note the following result, which is a generalization of the observation provided in [8]. Let $e_1,\dots,e_m$ denote the canonical basis of $\mathbb{R}^m$.

Lemma 2. *Let $U_n^{(m)}(a,c)$ be the solution to* (7) *with initial values* (8). *Then*

$$U_n^{(m)}(a,c) = \sum_{j=0}^{m-1} a_j U_n^{(m)}(e_j,c).$$

Proof. It follows from the definition of the sequences. The argument is similar to the arguments used in Section 3. $\square$

5. Concluding remarks

Our consideration shows that whenever we deal with any congruence related to the sum of the form (3) for a given set of initial values $a_0,\dots,a_{m-1}$, it is worth checking whether the same congruence holds for $a_0 = \cdots = a_{m-2} = 0$ and $a_{m-1} = 1$. With that we can establish many congruences related to Fibonacci m-step numbers.

We also point out that while discussing counterexamples, we were able to find $p^{m-1}-1$ unique initial values in Remark 2 and Remark 3 ($120 = 11^{3-1}-1$ and $342 = 7^{4-1}-1$, respectively), provided at least one counterexample exists. These numbers suggest a more general formula for the number of initial values, which we leave as an open question.

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