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Existence and uniqueness of a periodic solution to a certain third-order neutral functional differential equation

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Abstract. In this paper, by applying Mawhin's continuation theorem of coincidence degree theory, some sufficient conditions for the existence and uniqueness of an ω-periodic solution for the following third-order neutral functional differential equation are established:

$$\frac{d^3}{dt^3}\Big(x(t) - d(t)x\big(t-\delta(t)\big)\Big) + a(t)\ddot{x}(t) + b(t)f\big(t,\dot{x}(t)\big) + \sum_{i=1}^{n} c_i(t)g\big(t, x(t-\tau_i(t))\big) = e(t).$$

Moreover, we present an example and a graph to demonstrate the validity of an analytical conclusion.
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1. Introduction

Neutral functional differential equations provide good models in many fields including biology, mechanics, physics, medicine, chemistry, ecology, aerospace, and economics, see [1, 7, 9, 12]. Recently, the study of periodic solutions for neutral differential equations has attracted the attention of many researchers, see e.g., [4, 5, 11, 14, 15, 16, 17]. Several methods were developed to obtain periodic solutions to differential equations. One of the main tools is Mawhin's continuation theorem, which is become very powerful technique in the existence of periodic solutions to nonlinear differential equations.

In [3, 4, 6, 10, 13, 14], the authors discussed the properties of the neutral operator $(A_1x)(t) = x(t) - cx(t-\delta)$, where c and δ are constants, $|c| \neq 1$. Lu and Ge [6] investigated an extension of A_1, namely, the neutral operator $(A_2x)(t) = x(t) - \sum_{i=1}^{n} c_i x(t-\delta_i)$, and obtained the existence of periodic solutions for the corresponding neutral differential equation. In [8, 16], the researchers discussed the properties of the neutral operator with a variable delay $(A_3x)(t) = x(t) - cx(t-\delta(t))$.

In this paper, we consider the neutral operator $(Ax)(t) = x(t) - d(t)x(t-\delta(t))$, where $|d(t)| \neq 1,\ d, \delta \in C^3(\mathbb{R}, \mathbb{R})$ are ω-periodic functions for some $\omega > 0$. It is

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noted that A is a natural generalization of the operators A_1, A_2 and A_3. A class of neutral differential equation with A possesses a more complicated nonlinearity than a neutral differential equation with A_1, A_2 or A_3. Because neutral operators A_1 and A_2 are homogeneous, that is $\dfrac{d}{dt}\big(A_i x\big)(t) = (A_i \dot{x})(t)$ for $i = 1, 2$, the neutral operator A is in general inhomogeneous.

Now we display some papers which discussed the existence and uniqueness of periodic solutions for differential equations with the neutral operator, as follows:

In 2011, Ren et al. [8] established sufficient conditions for the existence of periodic solutions to a second-order neutral differential equation of the form:

$$\frac{d^2}{dt^2}\Big(x(t) - cx\big(t - \delta(t)\big)\Big) = f(t, \dot{x}(t)) + g(t, x(t - \tau(t))) + e(t),$$

where $\tau, e \in C_\omega$ and $\int_0^\omega e(t)dt = 0$; f and g are continuous functions defined on $\mathbb{R}^2$ and periodic in t, with $f(t, .) = f(t + \omega, .)$, $g(t, .) = g(t + \omega, .)$, $f(t, u) \geq 0$, or $f(t, u) \leq 0$ for all $(t, u) \in \mathbb{R}^2$.

In 2014, Xin et al. [15] established sufficient conditions for the existence of positive periodic solutions for a generalized third-order neutral differential equation:

$$\frac{d^3}{dt^3}\Big(x(t) - c(t)x\big(t - \delta(t)\big)\Big) = a(t)x(t) - \lambda b(t)f(t, x(t - \tau(t))),$$

where λ is a positive parameter; $\delta(t)$ is a variable delay, $c, \delta \in C^1(\mathbb{R}, \mathbb{R})$ and δ is an ω-periodic function for some $\omega > 0$, $f \in C(\mathbb{R}, [0, \infty))$, and $f(x) > 0$ for $x > 0$; $a, b \in C(\mathbb{R}, (0, \infty))$, $\tau \in C(\mathbb{R}, \mathbb{R})$, with $a(t), b(t), \tau(t)$ are ω-periodic functions.

In 2017, Na Wang [10] investigated the existence and uniqueness of periodic solution for the second-order neutral functional differential equation with delays:

$$\frac{d^2}{dt^2}\Big(u(t) - ku\big(t - \tau\big)\Big) = f(\dot{u}(t))\dot{u}(t) + \alpha(t)g(u(t)) + \sum_{i=1}^{n} \beta_i(t)h(u(t - \tau_i(t))) + p(t).$$

where $f, g, h \in C(\mathbb{R}, \mathbb{R})$, $p, \tau_i(t)$ $(i = 1, 2, ..., n)$ are continuous periodic functions defined on $\mathbb{R}$ with period $\omega > 0$; $\alpha(t), \beta_i(t)$ $(i = 1, 2, ..., n)$ are continuous periodic functions defined on $\mathbb{R}$ that have the same sign, and k, $\tau \in \mathbb{R}$ are constants such that $|k| \neq 1$.

In this paper, we consider a third-order generalized neutral differential equation as follows:

$$\begin{aligned} \frac{d^3}{dt^3}\bigg(x(t) - d(t)x\big(t - \delta(t)\big)\bigg) &+ a(t)\ddot{x} + b(t)f\big(t, \dot{x}(t)\big) \\ &+ \sum_{i=1}^{n} c_i(t)g\big(t, x(t - \tau_i(t))\big) = e(t), \end{aligned} \tag{1}$$

where $|d(t)| \neq 1$, $d, \delta \in C^3(\mathbb{R}, \mathbb{R})$ are ω-periodic functions for some $\omega > 0$, $\dot{\delta}(t) < 1$ for all $t \in [0, \omega]$; a, b, c_i, e $(i = 1, 2, ..., n)$ are continuous periodic functions defined on $\mathbb{R}$ with period $\omega > 0$, such that a, b, c_i have the same sign and $\int_0^\omega e(t)dt = 0$;

$\tau_i \in C^1(\mathbb{R},\mathbb{R})$ is a periodic function with $\dot{\tau}_i < 1$ $(i = 1, 2, ...n)$; f, g are continuous functions defined on $\mathbb{R}^2$ and periodic in the first argument, that is, $f(t,.) = f(t+\omega,.)$, $g(t,.) = g(t+\omega,.)$ and $f(t,0) = 0$.

2. Analysis of the generalized neutral operator

Let $C_\omega = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ with the norm $\|x\|_\infty = \max_{t\in[0,\omega]} |x(t)|$. Then $(C_\omega, \|.\|_\infty)$ is a Banach space. Define operators $A, B : C_\omega \to C_\omega$ by

$$(Ax)(t) = x(t) - d(t)x(t - \delta(t)), \quad (Bx)(t) = d(t)x(t - \delta(t)),$$

where $|d(t)| \neq 1$, $d, \delta \in C^3(\mathbb{R},\mathbb{R})$ are ω-periodic functions for some $\omega > 0$.

For convenience, we introduce the following notations:

$$\begin{aligned}
&d_0 = \min_{t\in[0,\omega]} |d(t)|, \quad d_\infty = \max_{t\in[0,\omega]} \left|d(t)\right|, \quad d_1 = \max_{t\in[0,\omega]} \left|\dot{d}(t)\right|, \quad d_2 = \max_{t\in[0,\omega]} \left|\ddot{d}(t)\right|,\\
&d_3 = \max_{t\in[0,\omega]} \left|\dddot{d}(t)\right|, \quad \delta_1 = \max_{t\in[0,\omega]} \left|\dot{\delta}(t)\right|, \quad \delta_2 = \max_{t\in[0,\omega]} \left|\ddot{\delta}(t)\right|, \quad \delta_3 = \max_{t\in[0,\omega]} \left|\dddot{\delta}(t)\right| \qquad (2)\\
&\tau = \max_{t\in[0,\omega]} |\tau_i(t)|, \quad c = \max_{t\in[0,\omega]} \left|c_i(t)\right|, \quad i = 1, 2, ..., n.
\end{aligned}$$

The following lemmas are needed in the proof of our results:

Lemma 1 (see [16]). *If $|d(t)| < 1$, then the operator A has a continuous inverse A^{-1} on the space C_ω, satisfying*

$$\left|(A^{-1}x)(t)\right| \leq \frac{\|x\|_\infty}{1 - d_\infty}, \quad \text{for } d_\infty < 1, \ \forall\, x \in C_\omega.$$

Lemma 2 (see [5]). *If $d(t) \in C_\omega$, $\delta(t) \in C^1_\omega := \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+\omega) = x(t),\ t \in \mathbb{R}\}$ and $\delta'(t) < 1$, $\forall t \in [0,\omega]$. Then $d(\mu(t)) \in C_\omega$, where $\mu(t)$ is the inverse function of $t - \delta(t)$.*

Lemma 3 (see [16]). *If $|d(t)| > 1$ and $\delta'(t) < 1$, then the operator A has a continuous inverse A^{-1} on C_ω, satisfying*

$$\left|(A^{-1}x)(t)\right| \leq \frac{\|x\|_\infty}{d_0 - 1}, \quad \text{for } d_0 > 1, \ \forall\, x \in C_\omega.$$

Here, we recall Mawhin's continuation theorem of coincidence degree theory, which our study is based upon. Let X and Y be real Banach spaces and $L : D(L) \to Y$ a Fredholm operator with index zero, where $D(L) \subset X$ denotes the domain of L. This means that $\mathrm{Im}L$ is closed in Y and $\dim \mathrm{Ker}L = (\dim Y/\mathrm{Im}L) < +\infty$.

Consider subspaces X_1, Y_1 of X, Y, respectively, such that $X = \mathrm{Ker}L \oplus X_1$ and $Y = \mathrm{Im}L \oplus Y_1$. Let $P : X \to \mathrm{Ker}L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $\mathrm{Ker}\, L \cap \left(D(L) \cap X_1\right) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible, and L_P^{-1} denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is called L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $L_P^{-1}(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 1 ([2, Mawhin's Continuation Theorem])**.** *Let X, Y be Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact operator on $\overline{\Omega}$, where Ω is an open bounded subset in X. Moreover, assume that all of the following conditions hold:*

(*i*) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$ ($\partial\Omega$ denotes the boundary of Ω);*

(*ii*) *$QNx \neq 0$, for all $x \in \partial\Omega \cap KerL$;*

(*iii*) *$\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$, where $J : ImQ \to KerL$ is an isomorphism.*

Then, $Lx = Nx$ has at least one periodic solution in $\overline{\Omega} \cap D(L)$.

First, we rewrite (1) as a system:

$$\begin{aligned} \frac{d}{dt}\big(Ax_1\big)(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t), \\ \dot{x}_3(t) &= -a(t)\ddot{x}_1(t) - b(t)f(t, \dot{x}_1(t)) - \sum_{i=1}^{n} c_i(t)g(t, x_1(t - \tau_i(t))) + e(t). \end{aligned} \tag{3}$$

Clearly, if $x(t) = \big(x_1(t), x_2(t), x_3(t)\big)^\top$ is a periodic solution for (3), then $x_1(t)$ must be a periodic solution for (1). Therefore, the problem of finding an ω-periodic solution for (1) is reduced to finding one for (3).

Set $X = Y = \big\{x(t) = \big(x_1(t), x_2(t), x_3(t)\big)^\top \in C(\mathbb{R}, \mathbb{R}^3) : x(t + \omega) = x(t)\big\}$ with the norm $\|x\|_\infty = \max\big\{\|x_1\|_\infty, \|x_2\|_\infty, \|x_3\|_\infty\big\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \big\{x(t) \in C^1(\mathbb{R}, \mathbb{R}^3) : x(t + \omega) = x(t)\big\} \subset X \to Y,$$

by

$$(Lx)(t) = \begin{pmatrix} \frac{d}{dt}\big(Ax_1\big)(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix},$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} x_2(t) \\ x_3(t) \\ -a(t)\ddot{x}_1(t) - b(t)f(t, \dot{x}_1(t)) - \sum_{i=1}^{n} c_i(t)g(t, x_1(t - \tau_i(t))) + e(t) \end{pmatrix}. \tag{4}$$

Then (3) can be transformed to the abstract equation $Lx = Nx$. From the definition of L, we can see that

$$\mathrm{Ker}L \cong \mathbb{R}^3, \qquad \mathrm{Im}L = \left\{ y \in Y : \int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

We observe that $\dim(\ker L) = \dim(Y/\mathrm{Im}L) = 3$, which implies $\dim(\mathrm{Ker}L) = \mathrm{codim}(\mathrm{Im}L)$. Therefore, L is a Fredholm operator with index zero.

Let $P : X \to \mathrm{Ker}L$ and $Q : Y \to \mathrm{Im}Q \subset \mathbb{R}^3$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} \quad \text{and} \quad Qy = \frac{1}{\omega}\int_0^\omega \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix} ds;$$

then $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$.

Setting $L_P = L|_{(D(L)\cap \mathrm{Ker}P)} : (D(L) \cap \mathrm{Ker}P) \to \mathrm{Im}L$ and the inverse of L_P is $L_P^{-1} : \mathrm{Im}L \to (D(L) \cap \mathrm{Ker}P)$ defined by

$$[L_P^{-1}y](t) = \begin{pmatrix} A^{-1}\int_0^t y_1(s)ds \\ \\ \int_0^t y_2(s)ds \\ \\ \int_0^t y_3(s)ds \end{pmatrix}.$$

From (4), we have:

$$(QNx)(t) = \frac{1}{\omega}\int_0^\omega \begin{pmatrix} x_2(t) \\ x_3(t) \\ -a(t)\ddot{x}_1(t) - b(t)f(t,\dot{x}_1(t)) - \sum_{i=1}^n c_i(t)g(t, x_1(t-\tau_i(t))) + e(t) \end{pmatrix} dt. \tag{5}$$

Thus, it is easy to see that QN and $L_P^{-1}(I-Q)N$ are continuous. Hence $QN(\overline{\Omega})$ is bounded and then $L_P^{-1}(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means that N is L-compact on $\overline{\Omega}$.

3. Existence result

In this section, we will study the existence of aperiodic solution for (1).

Theorem 2. *Suppose that there exist positive constants α, β, m and D such that the following conditions satisfied:*

(H_1) $\left|f(t,u)\right| \le \alpha|u| + \beta \quad \forall\ t, u \in \mathbb{R}$;

(H_2) $\left|g(t,x_1) - g(t,x_2)\right| \le m\left|x_1 - x_2\right| \quad \forall\ t, x_1, x_2 \in \mathbb{R}$;

(H_3) *There exist positive constants M_1, M_2 such that one of the following conditions hold:*

(i) If $d(t) < 1$, *and* $0 < \dfrac{M_2}{1 - d_\infty - M_1} < 1$,

(ii) If $d(t) > 1$, *and* $0 < \dfrac{M_2}{d_0 - 1 - M_1} < 1$,

where $M_1 = \frac{1}{4}\omega^2 d_2 + \omega d_1 + \omega d_1\delta_1 + \frac{1}{2}\omega d_\infty \delta_2 + d_\infty \delta_1^2 + 2d_\infty\delta_1$,

and $M_2 = \omega\|a\|_\infty + \dfrac{1}{2}\alpha\omega^2\|b\|_\infty + \dfrac{1}{4}cm\omega^3 \displaystyle\sum_{i=1}^n \left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty$;

(H$_4$) $x\left\{a(t)v+b(t)f(t,u)+\sum_{i=1}^{n}c_i(t)g(t,x)\right\}\neq 0,\ \forall\ t,x,u,v\in\mathbb{R}$ *and* $|x|>D$.

Then (1) *has at least one* ω*-periodic solution.*

Proof. Consider the operator equation $Lx=\lambda Nx$ with a parameter $\lambda\in(0,1]$. Define $\Omega_1=\{x\in C_\omega : Lx=\lambda Nx\}$. Then $x=(x_1,x_2,x_3)\in\Omega_1$ satisfies:

$$\begin{aligned}\frac{d}{dt}\big(Ax_1\big)(t)&=\lambda x_2(t),\\ \dot{x}_2(t)&=\lambda x_3(t),\\ \dot{x}_3(t)&=\lambda\Big(-a(t)\ddot{x}_1(t)-b(t)f(t,\dot{x}_1(t))-\sum_{i=1}^{n}c_i(t)g(t,x_1(t-\tau_i(t)))+e(t)\Big).\end{aligned}\tag{6}$$

Substituting $x_3(t)=\dfrac{1}{\lambda^2}\dfrac{d^2}{dt^2}\Big[(Ax_1)(t)\Big]$ into the third equation of (6), we get:

$$\begin{aligned}&\frac{d^3}{dt^3}\Big[(Ax_1)(t)\Big]\\ &=\lambda^3\Big(-a(t)\ddot{x}_1(t)-b(t)f(t,\dot{x}_1(t))-\sum_{i=1}^{n}c_i(t)g(t,x_1(t-\tau_i(t)))+e(t)\Big).\end{aligned}\tag{7}$$

By integrating both sides of (7) from 0 to ω, we obtain:

$$\int_0^\omega\Big(a(t)\ddot{x}_1(t)+b(t)f(t,\dot{x}_1(t))+\sum_{i=1}^{n}c_i(t)g(t,x_1(t-\tau_i(t)))\Big)dt=0.$$

By using the integral mean value theorem, this implies that there exists a point $\zeta\in[0,\omega]$ such that:

$$a(\zeta)\ddot{x}_1(\zeta)+b(\zeta)f(\zeta,\dot{x}_1(\zeta))+\sum_{i=1}^{n}c_i(\zeta)g(\zeta,x_1(\zeta-\tau_i(\zeta)))=0.$$

In view of (H$_4$), we find that $|x_1(\zeta-\tau(\zeta))|\leq D$.

Let $\zeta-\tau(\zeta)=n\omega+t_0$, where $t_0\in[0,\omega]$ and n is an integer. Then we have $|x_1(\zeta-\tau(\zeta))|=|x_1(t_0)|$. Thus for $t\in[t_0,t_0+\omega]$, we obtain:

$$|x_1(t)|=\Big|x_1(t_0)+\int_{t_0}^{t}\dot{x}_1(s)ds\Big|\leq D+\int_{t_0}^{t}|\dot{x}_1(s)|ds.$$

Since $x_1(t)$ is ω-periodic, then for $t\in[t_0,t_0+\omega]$, we get:

$$|x_1(t)|=\Big|x_1(t_0+\omega)+\int_{t_0+\omega}^{t}\dot{x}_1(s)ds\Big|\leq D+\int_{t}^{t_0+\omega}|\dot{x}_1(s)|ds.$$

Combining the above two inequalities, we obtain:

$$|x_1(t)|\leq D+\frac{1}{2}\int_0^\omega|\dot{x}_1(s)|ds.$$

Thus

$$\|x_1\|_\infty = \max_{t\in[0,\omega]}|x_1(t)| \le D + \frac{1}{2}\int_0^\omega |\dot{x}_1(s)|ds \le D + \frac{1}{2}\omega\|\dot{x}_1\|_\infty. \tag{8}$$

Furthermore, since $x_1(0) = x_1(\omega)$, there exists a point $t_1 \in [0,\omega]$ such that $\dot{x}_1(t_1) = 0$. Then we have:

$$|\dot{x}_1(t)| = \left|\dot{x}_1(t_1) + \int_{t_1}^t \ddot{x}_1(s)ds\right| \le \int_{t_1}^t |\ddot{x}_1(s)|ds, \quad t \in [t_1, t_1+\omega].$$

Moreover, since $\dot{x}_1(t)$ is ω-periodic, then for $t \in [t_1, t_1+\omega]$, we get:

$$|\dot{x}_1(t)| = \left|\dot{x}_1(t_1+\omega) + \int_{t_1+\omega}^t \ddot{x}_1(s)ds\right| \le \int_t^{t_1+\omega} |\ddot{x}_1(s)|ds.$$

By combining the previous two inequalities, we obtain:

$$|\dot{x}_1(t)| \le \frac{1}{2}\int_0^\omega |\ddot{x}_1(s)|ds.$$

Thus

$$\|\dot{x}_1\|_\infty = \max_{t\in[0,\omega]}|\dot{x}_1(t)| \le \frac{1}{2}\int_0^\omega |\ddot{x}_1(s)|ds \le \frac{1}{2}\omega\|\ddot{x}_1\|_\infty. \tag{9}$$

Then, by substituting (9) in (8), we get:

$$\|x_1\|_\infty \le D + \frac{1}{2}\omega\|\dot{x}_1\|_\infty \le D + \frac{1}{4}\omega^2\|\ddot{x}_1\|_\infty. \tag{10}$$

Now, by the definition of operator $(Ax_1)(t) = x_1(t) - d(t)x_1(t-\delta(t))$, we get:

$$\frac{d}{dt}\big(Ax_1\big)(t) \ = \dot{x}_1(t) - \dot{d}(t)x_1(t-\delta(t)) - d(t)\dot{x}_1(t-\delta(t))(1-\dot{\delta}(t)).$$

Then from (2) and (8), we have:

$$\begin{aligned}\left|\frac{d}{dt}\big(Ax_1\big)(t)\right| &\le |\dot{x}_1(t)| + |\dot{d}(t)||x_1(t-\delta(t))| + |d(t)||\dot{x}_1(t-\delta(t))||(1-\dot{\delta}_1(t))| \\ &\le \|\dot{x}_1\|_\infty + d_1\|x_1\|_\infty + d_\infty\|\dot{x}_1\|_\infty(1+\delta_1) \\ &\le d_1 D + \left[1 + \frac{1}{2}d_1\omega + d_\infty(1+\delta_1)\right]\|\dot{x}_1\|_\infty.\end{aligned} \tag{11}$$

We also find:

$$\begin{aligned}\frac{d^2}{dt^2}\Big[\big(Ax_1\big)(t)\Big] &= \frac{d}{dt}\Big[\dot{x}_1(t) - \dot{d}(t)x_1(t-\delta(t)) - d(t)\dot{x}_1(t-\delta(t))(1-\dot{\delta}(t))\Big] \\ &= \big(A\ddot{x}\big)(t) - \ddot{d}(t)x_1(t-\delta(t)) - \Big[2\dot{d}(t) - 2\dot{d}(t)\dot{\delta}(t) - d(t)\ddot{\delta}(t)\Big]\dot{x}_1(t-\delta(t)) \\ &\quad - \Big[d(t)\dot{\delta}^2(t) - 2d(t)\dot{\delta}(t)\Big]\ddot{x}_1(t-\delta(t)).\end{aligned}$$

Thus

$$\big(A\ddot{x}_1\big)(t) = \frac{d^2}{dt^2}\Big[\big(Ax_1\big)(t)\Big] + \ddot{d}(t)x_1(t-\delta(t)) + \Big[2\dot{d}(t) - 2\dot{d}(t)\dot{\delta}(t) - d(t)\ddot{\delta}(t)\Big]\dot{x}_1(t-\delta(t)) + \Big[d(t)\dot{\delta}^2(t) - 2d(t)\dot{\delta}(t)\Big]\ddot{x}_1(t-\delta(t)).$$

Hence, from (2) we get:

$$\begin{aligned} \big|\big(A\ddot{x}_1\big)(t)\big| &\le \left|\frac{d^2}{dt^2}\Big[\big(Ax_1\big)(t)\Big]\right| + \big|\ddot{d}(t)\big|\big|x_1(t-\delta(t))\big| \\ &\quad + \Big[2\big|\dot{d}(t)\big| + 2\big|\dot{d}(t)\big|\big|\dot{\delta}(t)\big| + \big|d(t)\big|\big|\ddot{\delta}(t)\big|\Big]\big|\dot{x}_1(t-\delta(t))\big| \\ &\quad + \Big[\big|d(t)\big|\big|\dot{\delta}^2(t)\big| + 2\big|d(t)\big|\big|\dot{\delta}(t)\big|\Big]\big|\ddot{x}_1(t-\delta(t))\big| \\ &\le \|x_3\|_\infty + d_2\|x_1\|_\infty + \big(2d_1 + 2d_1\delta_1 + d_\infty\delta_2\big)\|\dot{x}_1\|_\infty \\ &\quad + \big(d_\infty\delta_1^2 + 2d_\infty\delta_1\big)\|\ddot{x}_1\|_\infty. \end{aligned} \tag{12}$$

Now, we need to prove that $\|x_1\|_\infty, \|x_2\|_\infty, \|x_3\|_\infty$ are bounded. So, we consider two cases:

- **Case (I)**: If $d(t) < 1$, from (12) and by applying Lemma 1, we obtain:

$$\begin{aligned} \|\ddot{x}_1\|_\infty &= \max_{t\in[0,\omega]}\big|A^{-1}A\ddot{x}_1(t)\big| \\ &\le \frac{1}{1-d_\infty}\max_{t\in[0,\omega]}\big|A\ddot{x}_1(t)\big| \\ &\le \frac{1}{1-d_\infty}\Big[\|x_3\|_\infty + d_2\|x_1\|_\infty + \big(2d_1 + 2d_1\delta_1 + d_\infty\delta_2\big)\|\dot{x}_1\|_\infty \\ &\quad + \big(d_\infty\delta_1^2 + 2d_\infty\delta_1\big)\|\ddot{x}_1\|_\infty\Big]. \end{aligned} \tag{13}$$

By substituting (9) and (10) into (13), we have:

$$\begin{aligned} \|\ddot{x}_1\|_\infty &\le \frac{1}{1-d_\infty}\Big[\|x_3\|_\infty + d_2\big(D + \frac{1}{4}\omega^2\|\ddot{x}_1\|_\infty\big) + \frac{1}{2}\omega\big(2d_1 + 2d_1\delta_1 + d_\infty\delta_2\big)\|\ddot{x}_1\|_\infty \\ &\quad + \big(d_\infty\delta_1^2 + 2d_\infty\delta_1\big)\|\ddot{x}_1\|_\infty\Big] \\ &\le \frac{1}{1-d_\infty}\Big[\|x_3\|_\infty + d_2D + M_1\|\ddot{x}_1\|_\infty\Big], \end{aligned}$$

where $M_1 = \frac{1}{4}\omega^2 d_2 + \omega d_1 + \omega d_1\delta_1 + \frac{1}{2}\omega d_\infty\delta_2 + d_\infty\delta_1^2 + 2d_\infty\delta_1$.

Since $1 - d_\infty - M_1 > 0$ by $(\mathrm{H}_3)(i)$, then we get:

$$\|\ddot{x}_1\|_\infty \le \frac{\|x_3\|_\infty + d_2D}{1 - d_\infty - M_1}. \tag{14}$$

On the other hand, since $x_2(0) = x_2(\omega)$, then there exists a point $t_2 \in [0, \omega]$ such that $\dot{x}_2(t_2) = x_3(t_2) = 0$. Thus we obtain:

$$|x_3(t)| = \big|x_3(t_2) + \int_{t_2}^{t} \dot{x}_3(s)ds\big| \leq \int_{t_2}^{t} |\dot{x}_3(s)|ds, \quad t \in [t_2, t_2 + \omega].$$

Furthermore, since $x_3(t)$ is ω-periodic, then for $t \in [t_2, t_2 + \omega]$, we get:

$$|x_3(t)| = \big|x_3(t_2 + \omega) + \int_{t_2+\omega}^{t} \dot{x}_3(s)ds\big| \leq \int_{t}^{t_2+\omega} |\dot{x}_3(s)|ds.$$

Hence, from the above two inequalities and by using (6), we find:

$$\begin{aligned}
|x_3(t)| \; &\leq \int_0^\omega |\dot{x}_3(s)|ds \\
&\leq \lambda \int_0^\omega \Big| - a(t)\ddot{x}_1(t) - b(t)f(t, \dot{x}_1(t)) - \sum_{i=1}^{n} c_i(t)g(t, x_1(t - \tau_i(t))) + e(t)\Big|dt \\
&\leq \int_0^\omega \big|a(t)\big||\ddot{x}_1(t)|dt + \int_0^\omega |b(t)||f(t, \dot{x}_1(t))|dt \\
&\quad + \int_0^\omega \sum_{i=1}^{n} |c_i(t)||g(t, x_1(t - \tau_i(t)))|dt + \int_0^\omega |e(t)|dt.
\end{aligned}$$

From (H_1) and (H_2), we obtain:

$$\begin{aligned}
|x_3(t)| \; &\leq \int_0^\omega \big|a(t)\big||\ddot{x}_1(t)|dt + \int_0^\omega |b(t)|\big(\alpha|\dot{x}_1(t)| + \beta\big)dt \\
&\quad + \int_0^\omega \sum_{i=1}^{n} |c_i(t)|m|x_1(t - \tau_i(t))|dt + \int_0^\omega \sum_{i=1}^{n} |c_i(t)||g(t, 0)|dt + \int_0^\omega |e(t)|dt \\
&\leq \omega\|a\|_\infty\|\ddot{x}_1\|_\infty + \alpha\omega\|b\|_\infty\|\dot{x}_1\|_\infty + \beta\omega\|b\|_\infty + cm\sum_{i=1}^{n}\int_0^\omega |x_1(t - \tau_i(t))|dt \\
&\quad + nc\omega \max\big\{|g(t, 0)| : 0 \leq t \leq \omega\big\} + \omega\|e\|_\infty. \qquad (15)
\end{aligned}$$

Put $s = t - \tau_i(t)$. Then $dt = \dfrac{ds}{1 - \dot{\tau}_i(\gamma_i(s))}$ since $\gamma_i(t)$ is the inverse of $t - \tau_i(t)$. Thus:

$$\begin{aligned}
\int_0^\omega \big|x_1(t - \tau_i(t))\big|dt \; &= \int_{-\tau_i(0)}^{\omega - \tau_i(\omega)} \left|\frac{x_1(s)ds}{1 - \dot{\tau}_i(\gamma_i(s))}\right| \leq \left\|\frac{1}{1 - \dot{\tau}_i}\right\|_\infty \int_0^\omega |x_1(s)|ds \\
&\leq \omega\|x_1\|_\infty\left\|\frac{1}{1 - \dot{\tau}_i}\right\|_\infty. \qquad (16)
\end{aligned}$$

Substituting (16) in (15), and from (9) and (10), we get:

$$
\begin{aligned}
|x_3(t)| &\le \omega\|a\|_\infty\|\ddot{x}_1\|_\infty + \alpha\omega\|b\|_\infty\|\dot{x}_1\|_\infty + \beta\omega\|b\|_\infty + cm\omega\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty\|x_1\|_\infty \\
&\quad + nc\omega\max\big\{|g(t,0)| : 0 < t < \omega\big\} + \omega\|e\|_\infty \\
&\le \omega\|a\|_\infty\|\ddot{x}_1\|_\infty + \frac{1}{2}\alpha\omega^2\|b\|_\infty\|\ddot{x}_1\|_\infty + \beta\omega\|b\|_\infty + cm\omega\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty \\
&\quad \big(D + \frac{1}{4}\omega^2\|\ddot{x}_1\|_\infty\big) + nc\omega\max\big\{|g(t,0)| : 0 \le t \le \omega\big\} + \omega\|e\|_\infty \\
&\le \beta\omega\|b\|_\infty + cm\omega D\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty + nc\omega\max\big\{|g(t,0)| : 0 \le t \le \omega\big\} \\
&\quad + \omega\|e\|_\infty + \left(\omega\|a\|_\infty + \frac{1}{2}\omega^2\alpha\|b\|_\infty + \frac{1}{4}cm\omega^3\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty\right)\|\ddot{x}_1\|_\infty \\
&\le M_3 + M_2\|\ddot{x}\|_\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
M_2 &= \omega\|a\|_\infty + \frac{1}{2}\alpha\omega^2\|b\|_\infty + \frac{1}{4}cm\omega^3\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty, \\
M_3 &= \beta\omega\|b\|_\infty + cm\omega D\sum_{i=1}^{n}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty + nc\omega\max\big\{|g(t,0)| : 0 \le t \le \omega\big\} + \omega\|e\|_\infty.
\end{aligned}
$$

Hence,

$$\|x_3\|_\infty = \max_{t\in[0,\omega]}|x_3(t)| \le M_3 + M_2\|\ddot{x}_1\|_\infty.$$

Thus, from (14) we obtain:

$$\|x_3\|_\infty \le M_3 + \frac{M_2\|x_3\|_\infty + d_2DM_2}{1-d_\infty-M_1},$$

Since $\frac{M_2}{1-d_\infty-M_1} < 1$ by $(\mathrm{H}_3)(i)$, then there exists a positive constant K_3 (independent of λ) such that:

$$\|x_3\|_\infty \le K_3,\ \ K_3 = \frac{M_3\big(1-d_\infty-M_1\big) + d_2DM_2}{1-d_\infty-M_1-M_2}. \tag{17}$$

Substituting (17) in (14), we obtain:

$$\|\ddot{x}_1\|_\infty \le \frac{K_3 + d_2D}{1-d_\infty-M_1}. \tag{18}$$

From (10) and (18), we find that there exists a positive constant K_1 such that:

$$\|x_1\|_\infty \le K_1,\ \ K_1 = D + \frac{\omega^2\big(K_3+d_2D\big)}{4\big(1-d_\infty-M_1\big)}.$$

Furthermore, by substituting (18) into (9), we have:

$$\|\dot{x}_1\|_\infty \le \frac{1}{2}\omega\|\ddot{x}_1\|_\infty \le \frac{\omega\big(K_3+d_2D\big)}{2\big(1-d_\infty-M_1\big)}. \tag{19}$$

Since $x_2(t)=\dfrac{d}{dt}\big[\big(Ax_1\big)(t)\big]$, then from (11) and (19) we find that:

$$\begin{aligned}\|x_2\|_\infty = \max_{t\in[0,\omega]}|x_2(t)| &= \max_{t\in[0,\omega]}\left|\frac{d}{dt}\big[(Ax_1)(t)\big]\right|\\ &\le d_1D+\left[1+\frac{1}{2}d_1\omega+d_\infty(1+\delta_1)\right]\|\dot{x}_1\|_\infty\\ &\le d_1D+\left[1+\frac{1}{2}d_1\omega+d_\infty(1+\delta_1)\right]\frac{\omega\big(K_3+d_2D\big)}{2\big(1-d_\infty-M_1\big)}.\end{aligned}$$

Hence, it is easy to see that there exists a positive constant K_2 such that:

$$\|x_2\|_\infty \le K_2,\quad K_2=d_1D+\left[1+\frac{1}{2}d_1\omega+d_\infty(1+\delta_1)\right]\frac{\omega\big(K_3+d_2D\big)}{2\big(1-d_\infty-M_1\big)}.$$

Thus, from $Lx=\lambda Nx$ we get:

$$\|x_1\|_\infty \le K_1,\quad \|x_2\|_\infty\le K_2,\quad \|x_3\|_\infty \le K_3.$$

- **Case (II)**: If $d(t)>1$, from (12) and by applying Lemma 3, we get:

$$\begin{aligned}\|\ddot{x}_1\|_\infty &= \max_{t\in[0,\omega]}\big|A^{-1}A\ddot{x}_1(t)\big|\\ &\le \frac{1}{d_0-1}\big|A\ddot{x}_1(t)\big|\\ &\le \frac{1}{d_0-1}\Big[\|x_3\|_\infty+d_2\|x_1\|_\infty+\big(d_\infty\delta_2+2d_1+2d_1\delta_1\big)\|\dot{x}_1\|_\infty\\ &\quad+d_\infty\big(2\delta_1+\delta_1^2\big)\|\ddot{x}_1\|_\infty\Big]\\ &\le \frac{1}{d_0-1}\Big[\|x_3\|_\infty+d_2D+M_1\|\ddot{x}_1\|_\infty\Big].\end{aligned}$$

Since $d_0-1-M_1>0$, then by $\mathrm{H}_3(ii)$ we obtain:

$$\|\ddot{x}_1\|_\infty \le \frac{\|x_3\|_\infty+d_2D}{d_0-1-M_1}.$$

By following the same strategy and notation as in Case (I), we can get:

$$\|x_1\|_\infty \le K_1^{'},\ \ K_1^{'}=D+\frac{\omega^2\big(K_3^{'}+d_2D\big)}{4\big(d_0-1-M_1\big)},$$

$$\|x_2\|_\infty \leq K_2^{'},\ K_2^{'} = d_1 D + \left[1 + \frac{1}{2} d_1 \omega + d_\infty (1 + \delta_1)\right] \frac{\omega\big(K_3^{'} + d_2 D\big)}{2\big(d_0 - 1 - M_1\big)},$$

$$\|x_3\|_\infty \leq K_3^{'},\ K_3^{'} = \frac{M_3\big(d_0 - 1 - M_1\big) + d_2 D M_2}{d_0 - 1 - M_1 - M_2}.$$

Hence, from $Lx = \lambda Nx$, we obtain:

$$\|x_1\|_\infty \leq K_1^{'}, \quad \|x_2\|_\infty \leq K_2^{'}, \quad \|x_3\|_\infty \leq K_3^{'}.$$

Let $K_4 = \max\big\{K_1, K_2, K_3\big\} + 1$, $\Omega_2 = \big\{x(t) = \big(x_1(t), x_2(t), x_3(t)\big)^\top : \|x\|_\infty < K_4\big\}$. Then we see that x belongs to the interior of Ω_2, which yields a contradiction with $x \in \partial\Omega_2$; this contradiction came from $Lx = \lambda Nx$. Hence, condition (i) of Theorem 1 is satisfied.

Moreover, $\forall x \in \partial\Omega_2 \cap \mathrm{Ker}L$, and by using (5), if $QNx = 0$, then $x_2(t) = x_3(t) = 0$, $x_1(t) = K_4$ or $x_1(t) = -K_4$,. But if $x_1(t) = K_4$, we have

$$\int_0^\omega \sum_{i=1}^n c_i(t) g(t, K_4) dt = 0,$$

which implies by integral mean value theorem that there exists a point $\eta \in [0, \omega]$ such that $g(\eta, K_4) = 0$. From (H_4), we have $K_4 \leq D$, which yields a contradiction. Similarly, if $x_1(t) = -K_4$. Therefore, we also have $QNx \neq 0$, that is $\forall x \in \partial\Omega_2 \cap \mathrm{Ker}L$, $x \notin \mathrm{Ker}Q = \mathrm{Im}L$, so condition (ii) of Theorem 1 is satisfied.

Now, we will show that condition (iii) holds. Since $\mathrm{Im}Q$ and $\mathrm{Ker}L$ can be identified, we can define the isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ as follows:

$$J(x_1, x_2, x_3)^\top = (-x_3, x_1, x_2)^\top.$$

Then

$$JQNx = \frac{1}{\omega} \int_0^\omega \begin{pmatrix} a(t)\ddot{x}_1 + b(t) f(t, \dot{x}_1(t)) + \sum_{i=1}^n c_i(t) g(t, x_1(t - \tau_i(t))) - e(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} dt.$$

Thus, we shall show that $\deg\big(JQN, \Omega_2 \cap \mathrm{Ker}L, 0\big) \neq 0$. In order to do that, we use the homotopy invariance of the degree.

Define a continuous function $H(x, \mu) : \big(\overline{\Omega}_2 \cap \mathrm{Ker}L\big) \times [0, 1] \to \mathbb{R}$ by

$$H(x, \mu) = (1 - \mu)x + \mu JQNx, \qquad \mu \in [0, 1].$$

Then for all $(x, \mu) \in \big(\partial\Omega_2 \cap \mathrm{Ker}L\big) \times (0, 1)$

$$H(x, \mu) = \begin{pmatrix} (1-\mu)x_1 + \frac{\mu}{\omega} \int_0^\omega \big[a(t)\ddot{x}_1 + b(t) f(t, \dot{x}_1) + \sum_{i=1}^n c_i g(t, x_1(t - \tau_i)) - e(t)\big] dt \\ \big\{(1-\mu) + \mu\big\} x_2(t) \\ \big\{(1 - \mu) + \mu\big\} x_3(t) \end{pmatrix}.$$

Thus $H(x, 0) = Ix$ and $H(x, 1) = JQNx$, where Ix is the identity. Therefore, $H(x, \mu)$ is a homotopy from identity I to JQN.

Since $\int_0^\omega e(t)dt = 0$, and from (H$_4$), it is obvious that $x^\top H(x,\mu) \neq 0$, for all $(x,\mu) \in \big(\partial\Omega_2 \cap \mathrm{Ker}L\big) \times [0,1]$. Thus, by using the homotopy invariance theorem, we get:

$$\deg\{JQN, \Omega_2 \cap \mathrm{Ker}L, 0\} = \deg\{I, \Omega_2 \cap \mathrm{Ker}L, 0\} \neq 0.$$

So condition (iii) of Theorem 1 is satisfied. Then by applying Theorem 1, we can conclude that equation $Lx = Nx$ has at least one solution $x = (x_1, x_2, x_3)^\top$ on $\overline{\Omega}_2 \cap D(L)$, that is, (1) has an ω-periodic solution $x_1(t)$.

Similarly, let $K_4^{'} = \max\big\{K_1^{'}, K_2^{'}, K_3^{'}\big\}+1$, $\Omega_3 = \big\{x = (x_1, x_2, x_3)^\top : \|x\|_\infty < K_4^{'}\big\}$.

By following the same strategy as in Case (I), we find that all conditions of Theorem 1 hold, so (1) has an ω-periodic solution. $\square$

4. Uniqueness result

In this section, if in (1) $f(t,\dot{x}(t)) = \dot{x}(t),\ a(t) = \|a\|_\infty,\ b(t) = \|b\|_\infty$, then we have the following uniqueness result.

Theorem 3. *Assume that all conditions of Theorem 2 hold, A is a linear operator and $g(t,x)$ is a strictly monotone decreasing function of x such that:*

(H$_5$) (i) $0 < \dfrac{M_4}{1-d_\infty} < 1$, *if* $|d(t)| < 1$,

(ii) $0 < \dfrac{M_4}{d_0 - 1} < 1$, *if* $|d(t)| > 1$,

where

$$\begin{aligned} M_4 = d_\infty(1+d_\infty) &+ \frac{\omega^3}{8}\Big[mc\sum_{i=1}^{n}\Big\|\frac{1}{1-\dot{\tau}_i}\Big\| + d_3\Big] \\ &+ \frac{\omega^2}{4}\Big[3d_2 + 3d_2\delta_1 + 3d_1\delta_2 + d_\infty\delta_3 + \|b\|_\infty\Big] \\ &+ \frac{\omega}{2}\Big[\|a\|_\infty + 3d_1(1+\delta_1)^2 + 3d_\infty\delta_2(1+\delta_1)\Big] + d_\infty\delta_1\Big[3 + 3\delta_1 + \delta_1^2\Big]. \end{aligned}$$

Then (1) *has a unique ω-periodic solution.*

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are two ω-periodic solutions of (1); then we get:

$$\begin{aligned} &\frac{d^3}{dt^3}\Big[Au_1(t) - Au_2(t)\Big] + \|a\|_\infty\Big[\ddot{u}_1(t) - \ddot{u}_2(t)\Big] + \|b\|_\infty\Big[\dot{u}_1(t) - \dot{u}_2(t)\Big] \\ &\quad + \sum_{i=1}^{n} c_i(t)\Big[g(t, u_1(t-\tau_i(t))) - g(t, u_2(t-\tau_i(t)))\Big] = 0. \end{aligned}$$

Let $z(t) = u_1(t) - u_2(t)$. We obtain:

$$\begin{aligned} &\frac{d^3}{dt^3}\Big[\Big(Az(t)\Big)\Big] + \|a\|_\infty\ddot{z}(t) + \|b\|_\infty\dot{z}(t) \\ &\quad + \sum_{i=1}^{n} c_i(t)\Big[g(t, u_1(t-\tau_i(t))) - g(t, u_2(t-\tau_i(t)))\Big] = 0. \end{aligned} \tag{20}$$

Integrating (20) from 0 to ω, we get:

$$\int_0^{\omega} \sum_{i=1}^{n} c_i(t)\Big[g(t,u_1(t-\tau_i(t))) - g(t,u_2(t-\tau_i(t)))\Big] = 0.$$

By using the mean value theorem, we find that there exists $t_3 \in [0,\omega]$ such that:

$$g(t_3,u_1(t_3-\tau(t_3))) - g(t_3,u_2(t_3-\tau(t_3))) = 0. \tag{21}$$

Let $t_3-\tau(t_3) = t_3^* + m\omega$, where $t_3^* \in [0,\omega]$, and m is an integer. Since $g(t,x)$ is a strictly monotone decreasing function of u, and from (21) and (H_2) we get:

$$z(t_3^*) = u_1(t_3^*) - u_2(t_3^*) = u_1(t_3-\tau(t_3)) - u_2(t_3-\tau(t_3)) = 0.$$

Therefore,

$$|z(t)| = \big|z(t_3^*) + \int_{t_3^*}^{t} \dot{z}(s)ds\big| \le \int_{t_3^*}^{t} |\dot{z}(s)|ds, \qquad t \in [t_3^*, t_3^*+\omega].$$

Again, since $z(t)$ is ω-periodic, then we get:

$$|z(t)| = \big|z(t_3^*+\omega) + \int_{t_3^*+\omega}^{t} \dot{z}(s)ds\big| \le \int_{t}^{t_3^*+\omega} |\dot{z}(s)|ds, \qquad t \in [t_3^*, t_3^*+\omega].$$

Combining the two above inequalities, we obtain:

$$|z(t)| \le \frac{1}{2}\int_0^{\omega} |\dot{z}(s)|ds,$$

which implies that

$$\|z\|_\infty = \frac{1}{2}\omega\|\dot{z}\|_\infty. \tag{22}$$

Furthermore, since $z(0) = z(\omega)$, then there exists a point $t_4 \in [0,\omega]$ such that $\dot{z}(t_4) = 0$; thus

$$|\dot{z}(t)| = \big|\dot{z}(t_4) + \int_{t_4}^{t} \ddot{z}(s)ds\big| \le \int_{t_4}^{t} |\ddot{z}(s)|ds.$$

Since $\dot{z}(t)$ is an ω-periodic function, then

$$|\dot{z}(t)| = \big|\dot{z}(t_4+\omega) + \int_{t_4+\omega}^{t} \ddot{z}(s)ds\big| \le \int_{t}^{t_4+\omega} |\ddot{z}(s)|ds.$$

Combining the previous two inequalities, we get:

$$|\dot{z}(t)| \le \frac{1}{2}\int_0^{\omega} |\ddot{z}(s)|ds,$$

which implies that

$$\|\dot{z}\|_\infty = \frac{1}{2}\omega\|\ddot{z}\|_\infty. \tag{23}$$

Similarly, we find:

$$\|\ddot{z}\|_\infty = \frac{1}{2}\omega\|\dddot{z}\|_\infty. \tag{24}$$

Now, by definition of operator A, we find that $(Az)(t) = z(t) - d(t)z(t-\delta(t))$, and

$$\begin{aligned}\frac{d^3}{dt^3}\Big[(Az)(t)\Big] = A\dddot{z}(t) - &\dddot{d}(t)z(t-\delta(t))\\ &-\Big[3\ddot{d}(t)(1-\dot{\delta}(t)) - 3\dot{d}(t)\ddot{\delta}(t) - d(t)\dddot{\delta}(t)\Big]\dot{z}(t-\delta(t))\\ &-\Big[3\dot{d}(t)(1-\dot{\delta}(t))^2 - 3d(t)(1-\dot{\delta}(t))\ddot{\delta}(t)\Big]\ddot{z}(t-\delta(t))\\ &-\Big[-3d(t)\dot{\delta}(t) + 3d(t)\dot{\delta}^2(t) - d(t)\dot{\delta}^3(t)\Big]\dddot{z}(t-\delta(t)).\end{aligned} \tag{25}$$

Substituting of (25) in (20), then multiplying by $\dddot{z}(t)$, we get:

$$\begin{aligned}&\big(A\dddot{z}(t)\big)\dddot{z}(t)\\ &\quad=\dddot{d}(t)z(t-\delta(t))\dddot{z}(t)\\ &\qquad+\Big[3\ddot{d}(t)(1-\dot{\delta}(t)) - 3\dot{d}(t)\ddot{\delta}(t) - d(t)\dddot{\delta}(t)\Big]\dot{z}(t-\delta(t))\dddot{z}(t)\\ &\qquad+\Big[3\dot{d}(t)(1-\dot{\delta}(t))^2 - 3d(t)(1-\dot{\delta}(t))\ddot{\delta}(t)\Big]\ddot{z}(t-\delta(t))\dddot{z}(t)\\ &\qquad+\Big[-3d(t)\dot{\delta}(t) + 3d(t)\dot{\delta}^2(t) - d(t)\dot{\delta}^3(t)\Big]\dddot{z}(t-\delta(t))\dddot{z}(t) - \|a\|_\infty\ddot{z}(t)\dddot{z}(t)\\ &\qquad-\|b\|_\infty\dot{z}(t)\dddot{z}(t) - \sum_{i=1}^n c_i(t)\Big[g(t,u_1(t-\tau_i(t))) - g(t,u_2(t-\tau_i(t)))\Big]\dddot{z}(t).\end{aligned}$$

Hence, from (2) and (H_2), we obtain:

$$\begin{aligned}&\big|\big(A\dddot{z}(t)\big)\big|\big|\dddot{z}(t)\big|\\ &\quad\le d_3\big|z(t-\delta(t))\big|\big|\dddot{z}(t)\big| + \Big[3d_2(1+\delta_1) + 3d_1\delta_2 + d_\infty\delta_3\Big]\big|\dot{z}(t-\delta(t))\big|\big|\dddot{z}(t)\big|\\ &\qquad+\Big[3d_1(1+\delta_1)^2 + 3d_\infty(1+\delta_1)\delta_2\Big]\big|\ddot{z}(t-\delta(t))\big|\big|\dddot{z}(t)\big|\\ &\qquad+\Big[3d_\infty\delta_1 + 3d_\infty\delta_1^2 + d_\infty\delta_1^3\Big]\big|\dddot{z}(t-\delta(t))\big|\big|\dddot{z}(t)\big|\\ &\qquad+\|a\|_\infty|\ddot{z}(t)||\dddot{z}(t)| + \|b\|_\infty|\dot{z}(t)||\dddot{z}(t)| + \sum_{i=1}^n cm|z(t-\tau_i(t))||\dddot{z}(t)|.\end{aligned} \tag{26}$$

Integrating (26) from 0 to ω, we have:

$$\begin{aligned}\int_0^\omega |\big(A\dddot{z}(t)\big)||\dddot{z}(t)|dt \le{}& \omega\Big[d_3 + \sum_{i=1}^n cm\Big\|\frac{1}{1-\dot{\tau}_i}\Big\|_\infty\Big]\|z\|_\infty\|\dddot{z}\|_\infty\\ &+\omega\Big[3d_2(1+\delta_1) + 3d_1\delta_2 + d_\infty\delta_3 + \|b\|_\infty\Big]\|\dot{z}\|_\infty\|\dddot{z}\|_\infty\\ &+\omega\Big[\|a\|_\infty + 3d_1(1+\delta_1)^2 + 3d_\infty(1+\delta_1)\delta_2\Big]\|\ddot{z}\|_\infty\|\dddot{z}\|_\infty\\ &+\omega d_\infty\delta_1\Big[3 + 3\delta_1 + \delta_1^2\Big]\|\dddot{z}\|_\infty^2.\end{aligned}$$

Thus, from (22), (23), and (24), we get:

$$\begin{aligned}\int_0^{\omega} |\big(A \dddot{z}(t)\big)||\dddot{z}(t)|dt \le \frac{\omega^4}{8}\Big[d_3 + \sum_{i=1}^{n} cm\Big\|\frac{1}{1-\dot{\tau}_i}\Big\|_{\infty}\Big]\|\dddot{z}\|_{\infty}^2 \\ + \frac{\omega^3}{4}\Big[3d_2(1+\delta_1) + 3d_1\delta_2 + d_{\infty}\delta_3 + \|b\|_{\infty}\Big]\|\dddot{z}\|_{\infty}^2 \\ + \frac{\omega^2}{2}\Big[\|a\|_{\infty} + 3d_1(1+\delta_1)^2 + 3d_{\infty}(1+\delta_1)\delta_2\Big]\|\dddot{z}\|_{\infty}^2 \\ + \omega d_{\infty}\delta_1\Big[3 + 3\delta_1 + \delta_1^2\Big]\|\dddot{z}\|_{\infty}^2.\end{aligned} \tag{27}$$

But

$$\begin{aligned}\int_0^{\omega} \big|\big(A \dddot{z}(t)\big)\big|\big|\dddot{z}(t)\big|dt &= \int_0^{\omega} \big|\big(A \dddot{z}(t)\big)\big|^2 dt + \int_0^{\omega} \big|\big(A \dddot{z}(t)\big)\big|\big|d(t)\dddot{z}(t-\delta(t))\big|dt \\ &\le \omega\Big[\big\|A \dddot{z}\big\|_{\infty}^2 + d_{\infty}(1+d_{\infty})\|\dddot{z}\|_{\infty}^2\Big].\end{aligned} \tag{28}$$

Substituting (28) in (27), we obtain:

$$\begin{aligned}\big\|A \dddot{z}\big\|_{\infty}^2 \le \Big\{ & d_{\infty}(1+d_{\infty}) + \frac{\omega^3}{8}\Big[d_3 + \sum_{i=1}^{n} cm\Big\|\frac{1}{1-\dot{\tau}_i}\Big\|_{\infty}\Big] \\ &+ \frac{\omega^2}{4}\Big[3d_2(1+\delta_1) + 3d_1\delta_2 + d_{\infty}\delta_3 + \|b\|_{\infty}\Big] \\ &+ \frac{\omega}{2}\Big[\|a\|_{\infty} + 3d_1(1+\delta_1)^2 + 3d_{\infty}(1+\delta_1)\delta_2\Big] + d_{\infty}\delta_1\Big[3+3\delta_1+\delta_1^2\Big]\Big\}\|\dddot{z}\|_{\infty}^2 \\ \le\ & M_4\|\dddot{z}\|_{\infty}^2.\end{aligned} \tag{29}$$

Now, we have two cases:

- **Case (I)**: If $|d(t)| < 1$, then from Lemma 1, we get:

$$\|\dddot{z}\|_{\infty}^2 = \max_{t\in[0,\omega]}\big|\dddot{z}(t)|^2 = \max_{t\in[0,\omega]}\big|A^{-1}A \dddot{z}(t)|^2 \le \frac{1}{1-d_{\infty}}\|A \dddot{z}\|_{\infty}^2. \tag{30}$$

Thus, by substituting (30) in (29) and from (H_5), we obtain:

$$\|A \dddot{z}\|_{\infty}^2 \le \frac{M_4}{1-d_{\infty}}\|A \dddot{z}\|_{\infty}^2 \implies \Big(1 - \frac{M_4}{1-d_{\infty}}\Big)\|A \dddot{z}\|_{\infty}^2 \le 0,$$

which implies that $\|A \dddot{z}\|_{\infty}^2 = 0$.

- **Case (II)**: If $|d(t)| > 1$, then from Lemma 3, we get:

$$\|\dddot{z}\|_{\infty}^2 = \max_{t\in[0,\omega]}\big|\dddot{z}(t)|^2 = \max_{t\in[0,\omega]}\big|A^{-1}A \dddot{z}(t)|^2 \le \frac{1}{d_0-1}\|A \dddot{z}\|_{\infty}^2. \tag{31}$$

Hence, by substituting (31) in (29) and from (H_5), we have:

$$\|A \dddot{z}\|_{\infty}^2 \le \frac{M_4}{d_0-1}\|A \dddot{z}\|_{\infty}^2 \implies \Big(1 - \frac{M_4}{d_0-1}\Big)\|A \dddot{z}\|_{\infty}^2 \le 0,$$

which implies that $\|A\dddot{z}\|_\infty^2 = 0$.

Since $Az(t)$, $A\dot{z}(t)$, $A\ddot{z}(t)$, $A\dddot{z}(t)$ are ω-periodic and continuous functions, we get:

$$Az(t) \equiv A\dot{z}(t) \equiv A\ddot{z}(t) \equiv A\dddot{z}(t) \equiv 0,$$

that is,

$$z(t) \equiv \dot{z}(t) \equiv \ddot{z}(t) \equiv \dddot{z}(t) \equiv 0.$$

Thus $u_1(t) \equiv u_2(t)$ for all $t \in R$, and the proof of Theorem 3 is completed. $\square$

5. Example

Consider the following third-order neutral nonlinear differential equation:

$$\begin{aligned}&\frac{d^3}{dt^3}\Big(x(t) - \frac{1}{512}\sin(4t)\ x\Big(t - \frac{1}{256}\cos^2(4t)\Big)\Big) + \frac{1}{32}\cos(4t)\ \ddot{x}(t) + \frac{1}{16}\cos(4t)\ \dot{x}(t)\\ &\quad + \sum_{i=1}^{n}\frac{1}{2i}\sin(4t)\ \Big(\frac{2}{\pi}x\Big(t - i\sin(4t)\Big)\Big) = 4\cos(4t).\end{aligned}$$

For $n = 2$,

$$\begin{aligned}&d(t) = \frac{1}{512}\sin(4t),\quad \delta(t) = \frac{1}{256}\cos^2(4t),\quad a(t) = \frac{1}{32}\cos(4t),\quad b(t) = \frac{1}{16}\cos(4t),\\ &c_1(t) = \frac{1}{2}\sin(4t),\quad c_2(t) = \frac{1}{4}\sin(4t),\quad f(t,\dot{x}(t)) = \dot{x}(t),\quad \tau_1(t) = \sin(4t),\\ &\tau_2(t) = 2\sin(4t),\quad e(t) = 4\cos(4t).\end{aligned}$$

It is clear that $\omega = \dfrac{\pi}{2}$ and

$$|f(t,\dot{x})| = |\dot{x}| \ \text{ with }\ \alpha = 1,\quad \beta = 0,\quad f(t,0) = 0,\quad \int_0^{\frac{\pi}{2}} e(t)dt = 0,\quad m = \frac{2}{\pi}.$$

We also find:

$$d_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|d(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{512}\sin(4t)\right| = \frac{1}{512},$$

$$d_1 = \max_{t\in[0,\frac{\pi}{2}]}\left|\dot{d}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{128}\cos(4t)\right| = \frac{1}{128},$$

$$d_2 = \max_{t\in[0,\frac{\pi}{2}]}\left|\ddot{d}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{-1}{32}\sin(4t)\right| = \frac{1}{32},$$

$$d_3 = \max_{t\in[0,\frac{\pi}{2}]}\left|\dddot{d}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{-1}{8}\cos(4t)\right| = \frac{1}{8},$$

$$\delta_1 = \max_{t\in[0,\frac{\pi}{2}]}\left|\dot{\delta}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{-1}{64}\sin(8t)\right| = \frac{1}{64},$$

$$\delta_2 = \max_{t\in[0,\frac{\pi}{2}]}\left|\ddot{\delta}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{-1}{8}\cos(8t)\right| = \frac{1}{8},$$

$$\delta_3 = \max_{t\in[0,\frac{\pi}{2}]}\left|\dddot{\delta}(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\sin(8t)\right| = 1,$$

$$\|a\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|a(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{32}\cos(4t)\right| = \frac{1}{32},$$

$$\|b\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|b(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{16}\cos(4t)\right| = \frac{1}{16},$$

$$\|c_1\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|c_1(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{2}\sin(4t)\right| = \frac{1}{2},$$

$$\|c_2\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|c_1(t)\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{4}\sin(4t)\right| = \frac{1}{4},$$

$$c = \max\{\|c_1\|_\infty, \|c_2\|_\infty\} = \frac{1}{2},$$

and

$$\left\|\frac{1}{1-\dot{\tau}_1}\right\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{1-\dot{\tau}_1}\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{1-4\cos(4t)}\right| = \frac{1}{3},$$

$$\left\|\frac{1}{1-\dot{\tau}_2}\right\|_\infty = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{1-\dot{\tau}_2}\right| = \max_{t\in[0,\frac{\pi}{2}]}\left|\frac{1}{1-8\cos(4t)}\right| = \frac{1}{7},$$

$$\sum_{i=1}^{2}\left\|\frac{1}{1-\dot{\tau}_i}\right\|_\infty = \left\|\frac{1}{1-\dot{\tau}_1}\right\|_\infty + \left\|\frac{1}{1-\dot{\tau}_2}\right\|_\infty = \frac{1}{3}+\frac{1}{7} \approx 0.47619,$$

$$M_1 \approx 0.03199, \quad 1-d_\infty - M_1 \approx 0.96606 > 0, \quad M_2 \approx 0.27307, \quad M_4 \approx 0.2227.$$

Clearly, $d_\infty < 1$ and $0 < \dfrac{M_2}{1-d_\infty-M_1} \approx 0.28266 < 1$, and $0 < \dfrac{M_4}{1-d_\infty} \approx 0.2231 < 1$. Thus, all the hypotheses of Theorem 2 and Theorem 3 hold.

It has also been proven that the example fulfills the hypotheses of Theorem 2 and Theorem 3 using Matlab-Simulink. The periodicity of a solution for the considered differential equation is shown by the following graph:

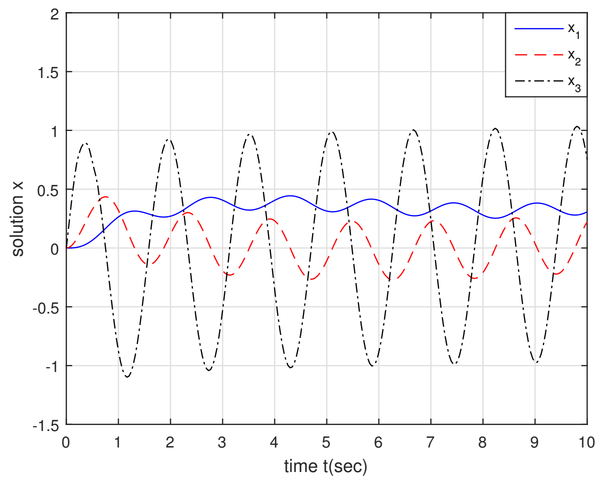

Figure 1: *Trajectory* $x(t)$

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