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A limit formula for real Richardson orbits

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Abstract. Let $G_{\mathbb{R}}$ be a real, semisimple, linear and connected Lie group. Let K denote the complexification of a maximal compact group of $G_{\mathbb{R}}$. Assume that $G_{\mathbb{R}}$ has a compact Cartan subgroup. We prove a formula which computes the Liouville measure on a real nilpotent Richardson orbit obtained by the Sekiguchi correspondence from a K-nilpotent Richardson orbit as a limit of differentiated measures on regular elliptic orbits.
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1. Introduction

Let G be a complex, semisimple, linear group, and let $G_{\mathbb{R}}$ a real form of G. Denote by $K_{\mathbb{R}}$ a maximal compact subgroup, and by K its complexification. In a series of papers [11, 12, 13], Schmid and Vilonen studied microlocal properties of equivariant sheaves on the flag variety. They proved that the natural correspondence between K-equivariant and $G_{\mathbb{R}}$-eqivariant sheaves, called the Matsuki correspondence for sheaves, descends to the nilpotent cone, where it induces the Sekiguchi correspondence between nilpotent K and $G_{\mathbb{R}}$-orbits. This correspondence identifies two important invariants of an irreducible representation: the associated cycle and the wave front cycle. Sometimes it is easier to compute one of the invariants, and then obtain the other through the Sekiguchi correspondence. For example, one can use this approach to determine the wave front cycle of a Vogan-Zuckerman module.

In this paper we would like to apply these ideas to study asymptotic properties of invariant eigendistributions on the Lie algebra of $G_{\mathbb{R}}$. In more detail, let X be the flag variety of G, let Y be a generalized flag variety, and let $\pi : X \longrightarrow Y$ be the natural projection. The main motivation for the present paper is to transfer the limit formula for invariant measures on singular elliptic orbits obtained in [2] to a limit formula for invariant measures on regular elliptic orbits. To prove such a formula we have to work in the setting of a flag variety, while in loc.cit. the limit formula was proved in the setting of a generalized flag variety. The lift of the limit formula from Y to X is accomplished by comparing the Matsuki correspondence for sheaves on Y and X, computing the pull-back under π of the characteristic cycle of a standard sheaf on an open $G_{\mathbb{R}}$-orbit on Y and using the identity for characteristic cycles from [5]. This identity corresponds to a character identity which relates the

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character of a Vogan-Zuckerman module and the signed average of a discrete series character over the Weyl group of a Levi factor of a θ-stable parabolic subgroup of G. We use the theory of Schmid and Vilonen to describe the leading term of the asymptotic expansion of the invariant eigendistribution on the Lie algebra of $G_{\mathbb{R}}$ associated with the characteristic cycle $CC(j_!\mathbb{C}_{\pi^{-1}Z})$, where $Z \subset Y$ is an open $G_{\mathbb{R}}$-orbit, and $j : \pi^{-1}Z \to X$ is the embedding. We prove that the Fourier transform of the leading term is supported on a single nilpotent $G_{\mathbb{R}}$-orbit $\mathcal{O}$ which we identify as a Richardson orbit for the group $G_{\mathbb{R}}$ [14]. The limit formula for the computation of the canonical measure on $\mathcal{O}$ by differentiation of canonical measures on regular elliptic orbits is proved by the method from [10], [4], which relies on Rossmann theory of Weyl group representations.

2. Preliminaries

Suppose $G_{\mathbb{R}}$ is a real, connected, linear, semisimple Lie group. We embed $G_{\mathbb{R}}$ into the complexification G, and denote by $\tau : G \longrightarrow G$ a conjugation on G having $G_{\mathbb{R}}$ as the set of fixed points. Next, we choose a Cartan involution $\theta : G_{\mathbb{R}} \longrightarrow G_{\mathbb{R}}$, and denote by $K_{\mathbb{R}}$ the set of fixed points. Extend θ to G so that it commutes with τ, and denote by K the set of fixed points. We denote by $U_{\mathbb{R}}$ the set of fixed points of $\theta\tau$ on G. Write $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{u}_{\mathbb{R}}$ for the Lie algebras of G, K, $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $U_{\mathbb{R}}$, respectively, and denote the involutions on $\mathfrak{g}$ induced by θ, τ by the same letters. In addition, let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}\,, \ \ \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the Cartan decompositions defined by θ. We shall assume that $G_{\mathbb{R}}$ has a compact Cartan subgroup

$$T_{\mathbb{R}} \subset K_{\mathbb{R}}.$$

Denote the Lie algebra of $T_{\mathbb{R}}$ by

$$\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}.$$

Next, we introduce the notation related to the geometry of the flag variety. Write X for the flag variety of Borel subalgebras of $\mathfrak{g}$. Let $n = \dim_{\mathbb{C}} X$. Given $x \in X$, we denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra, by $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ the nilpotent radical, and by $B_x \subset G$ the Borel subgroup, which stabilizes x via the adjoint action. All the quotients $\mathfrak{b}_x/\mathfrak{n}_x$ can be identified canonically, hence we can define the universal Cartan algebra by the condition

$$\mathfrak{h} \simeq \mathfrak{b}_x/\mathfrak{n}_x \ \text{ for any } \ x \in X.$$

Let $\mathfrak{c} \subset \mathfrak{b}_x$ be a Cartan subalgebra. Then we have a canonical isomorphism $\tau_x : \mathfrak{c} \to \mathfrak{b}_x/\mathfrak{n}_x \simeq \mathfrak{h}$. We denote by $\tau_x^* : \mathfrak{h}^* \to \mathfrak{c}^*$ the dual isomorphism. Then $\Delta = \tau_x^{*-1}(\Delta(\mathfrak{g}, \mathfrak{c}))$ is independent on the choice of the pair $(\mathfrak{c}, x)$, and is called the universal root system. Set $\Delta_x^+ = \Delta(\mathfrak{g}/\mathfrak{b}_x, \mathfrak{c})$. A positive root system in Δ is defined by the condition

$$\Delta^+ = \tau_x^{*-1}(\Delta_x^+).$$

The triple $(\mathfrak{h}^*, \Delta, \Delta^+)$ is called the universal Cartan triple. Given $\lambda \in \Delta$, and a pair $(\mathfrak{c}, x)$ as above, we write $\lambda_x = \tau_x^*(\lambda)$. The universal Weyl group W is defined as the Weyl group of the root system Δ. Denote by $\rho \in \mathfrak{h}^*$ half the sum of positive roots, and by $\mathfrak{h}'^*$ the set of regular elements. Note that $\mathfrak{h}^*$ comes equipped with the W-invariant symmetric bilinear form $(\cdot,\cdot)$ whose specialization at $x \in X$ coincides with the Killing form.

Let us recall the definition of the moment map and of the twisted moment map. Denote by T^*X the cotangent bundle of the variety X. Given $x \in X$, denote by $\mathfrak{b}_x^\perp \subset \mathfrak{g}^*$ the space of linear forms vanishing on $\mathfrak{b}_x$. We use the identification

$$T^*X \cong \left\{(x,\xi) : x \in X,\ \xi \in \mathfrak{b}_x^\perp\right\}$$

to consider T^*X as a submanifold of $X \times \mathfrak{g}^*$. The moment map is defined by

$$\mu : T^*X \longrightarrow \mathfrak{g}^*, \qquad \mu(x,\xi) = \xi.$$

Denote by $\mathcal{N}^*$ the cone of nilpotent elements in $\mathfrak{g}^*$. Note that $\mu(T^*X) = \mathcal{N}^*$. The definition of the twisted moment map is due to Rossmann [10]. Note that any $x \in X$ is fixed by a unique maximal torus $C_\mathbb{R} \subset U_\mathbb{R}$. We can use the decomposition $\mathfrak{g} = \mathfrak{c} + [\mathfrak{c},\mathfrak{g}]$ to view $\mathfrak{c}^*$ as a subspace of $\mathfrak{g}^*$. Now we define the twisted moment map by

$$\mu_\lambda : T^*X \longrightarrow \mathfrak{g}^*,\ \ \mu_\lambda(x,\xi) = \lambda_x + \mu(x,\xi), \quad \xi \in \mathfrak{b}_x^\perp.$$

If λ is regular, one can show that μ_λ is a $U_\mathbb{R}$-equivariant, real algebraic isomorphism of T^*X with complex coadjoint orbit $Ad^*(G)\lambda_x$. Note that $Ad^*(G)\lambda_x$ is independent of $x \in X$. We shall write $G \cdot \lambda = Ad^*(G)\lambda_x$.

Let $\mathcal{V}$ be a coadjoint G-orbit in $\mathfrak{g}^*$ or a coadjoint $G_\mathbb{R}$-orbit in $i\mathfrak{g}_\mathbb{R}^*$. To treat both cases simultaneously write $M = G$ or $M = G_\mathbb{R}$ and denote by $\mathfrak{m}$ the Lie algebra of M. The space

$$\mathfrak{m} \cdot \xi = \{\mathrm{ad}^*(x)(\xi) :\ x \in \mathfrak{m}\}$$

identifies with tangent space $T_\xi\mathcal{V}$ of $\mathcal{V}$ at ξ, and we define an M-equivariant 2-form $\sigma_\mathcal{V}$ on $\mathcal{V}$ by the formula

$$\sigma_{\mathcal{V},\xi}(x\cdot\xi, y\cdot\xi) = \xi[x,y]\,,\ \ x,y \in \mathfrak{m}.$$

In case $M = G_\mathbb{R}$, the form $-i\sigma_\mathcal{V}$ is real valued and we use the form

$$(-i\sigma_\mathcal{V})^k\,,\ \ 2k = \dim_\mathbb{R}\mathcal{V}$$

to orient $\mathcal{V}$. In this case we define the measure $m_\mathcal{V}$ by the formula

$$dm_\mathcal{V} = \frac{1}{(2\pi i)^k k!}\sigma_\mathcal{V}^k,$$

and call it the Liouville measure. When $\mathcal{V} = M\cdot\lambda$, $\lambda \in \mathfrak{c}^*$, $\mathfrak{c} \subset \mathfrak{g}$ a Cartan subalgebra, we shall write $\sigma_\mathcal{V} = \sigma_\lambda$ and $m_\mathcal{V} = m_\lambda$.

Let $\lambda \in \mathfrak{h}^*$. Then a $U_\mathbb{R}$-equivariant 2-form τ_λ on X is defined at x by

$$\tau_\lambda(a_x, b_x) = \tau_x^*(\lambda)([a,b]).$$

Here a_x and b_x denote the tangent vectors induced by a, $b \in \mathfrak{u}_\mathbb{R}$ by differentiation of the $U_\mathbb{R}$-action.

Denote by $\pi_X : T^*X \longrightarrow X$ the natural projection, and by σ the canonical symplectic form on T^*X. Following [12], we shall relate invariant distributions on the Lie algebra and integrals of certain differential forms over the semi-algebraic cycles in T^*X. The Fourier transform of a test function $\phi \in C_c^\infty(\mathfrak{g}_\mathbb{R})$ will be defined by

$$\hat{\phi}(\xi) = \int_{\mathfrak{g}_\mathbb{R}} e^{\xi(x)}\phi(x)dx\,,\ \ \xi \in \mathfrak{g}^*,$$

without the usual i in the exponential. Here dx denotes a suitably normalized Lebesgue measure on $\mathfrak{g}_\mathbb{R}$. Let us denote by $T^*_{G_\mathbb{R}}X$ the union of conormal bundles to the $G_\mathbb{R}$-orbits on X. One can prove that for a test function $\phi \in C_c^\infty(\mathfrak{g}_\mathbb{R})$ and $\lambda \in \mathfrak{h}^*$ the integral

$$\Theta(\Gamma,\lambda)(\phi) = \int_\Gamma \mu_\lambda^*(\hat{\phi})(-\sigma + \pi_X^*\tau_\lambda)^n$$

converges and depends holomorphically on λ. Moreover, $\Theta(\Gamma,\lambda)$ is a $G_\mathbb{R}$-invariant distribution on $\mathfrak{g}_\mathbb{R}$.

Denote by $\mathcal{N}$ the set of nilpotent elements in $\mathfrak{g}$. There exists a natural bijection between the sets of K-orbits in $\mathcal{N} \cap \mathfrak{p}$ and $G_\mathbb{R}$-orbits in $\mathcal{N} \cap i\mathfrak{g}_\mathbb{R}$, called the Sekiguchi correspondence. We remark that our parametrization of the Sekiguchi correspondence will be identical to [13], 6.7.

Let us choose $h_0 \in i\mathfrak{t}_\mathbb{R}$, and define the parabolic subalgebra $\mathfrak{q}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l}$ and $\mathfrak{u}$ are specified by the conditions

$$\Delta(\mathfrak{l},\mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}) : \alpha(h_0) = 0\}\,,\ \ \Delta(\mathfrak{u},\mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g},\mathfrak{t}) : \alpha(h_0) > 0\}\,.$$

Then we have

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{u} + \tau\mathfrak{u},\ \ \theta\mathfrak{u} = \mathfrak{u},\ \ \text{and}\ \ \mathfrak{l} = \mathfrak{q} \cap \tau\mathfrak{q}.$$

In particular, $\mathfrak{l}$ is the complexification of a reductive, θ-stable subalgebra $\mathfrak{l}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$. Let $Q \subset G$ be the parabolic subgroup corresponding to $\mathfrak{q}$. Set $Y = G/\tau Q$, and denote by $y_0 \in Y$ the point determined by τQ. Let

$$S = G_\mathbb{R} \cdot y_0 \subset Y \ \text{and}\ Z = K \cdot y_0.$$

A short computation shows that $\dim_\mathbb{R}(\mathfrak{g}_\mathbb{R}/\mathfrak{l}_\mathbb{R}) = \dim_\mathbb{R}(\mathfrak{g}/\tau\mathfrak{q})$, hence the orbit S is open in Y. On the other hand, the orbit Z is associated with S by the Matsuki correspondence [8] on Y and it is closed. Let us choose a positive root system $\Delta^+(\mathfrak{l},\mathfrak{t})$ and define $x_0 \in X$ by the condition

$$\Delta(\mathfrak{b}_{x_0},\mathfrak{t}) = -\Delta^+(\mathfrak{l},\mathfrak{t}) \cup \Delta(\overline{\mathfrak{u}},\mathfrak{t}). \tag{1}$$

We remark that the orbit

$$S_0 = G_\mathbb{R} \cdot x_0$$

is open in X.

Now we shall review several results related to Rossmann's construction of Weyl group representations on the homology groups of conormal varieties for the action of $G_\mathbb{R}$ (K) on the flag variety X [10]. These results will be used in the proof of the main theorem.

Let $A = K$ or $A = G_{\mathbb{R}}$ and let $\mathfrak{a}$ be the Lie algebra of A. When

$$U = \mathfrak{a}^{\perp} \cap \mathcal{N}^*,\ U = \overline{\mathcal{O}},\ U = \mathcal{O}, U = \{\nu\}\,,$$

Rossmann defines W-module structure on homology groups

$$H_*(\mu^{-1}(U), \mathbb{C}).$$

Here $\mathcal{O}$ is an A-orbit and $\nu \in \mathcal{N}^*$. In the first case we have

$$\mu^{-1}(\mathfrak{a}^{\perp} \cap \mathcal{N}^*) = T_A^* X\,.$$

Denote by

$$C_G(\nu) \ \text{ resp. } \ C_A(\nu)$$

the group of connected components of the centralizer of ν in G resp. A. Let

$$d = d(\nu) = \dim_{\mathbb{C}} \mu^{-1}(\nu).$$

Then we have natural homommorphisms of W-modules

$$\begin{aligned} H_{2n}(\mu^{-1}(\overline{\mathcal{O}}), \mathbb{C}) &\longrightarrow H_{2n}(\mu^{-1}(\mathcal{O}), \mathbb{C}) \\ &\simeq\ \ H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_A(\nu)} \\ &\longrightarrow H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)} \end{aligned}$$

Recall that the W-module $H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)}$ is irreducible. This is the Springer representation associated with orbit $G \cdot \nu$, and the corresponding character will be denoted by χ_ν.

Denote by $\mathcal{H}_d(\mathfrak{h}^*)$ ($\mathcal{H}_d(\mathfrak{h})$) the space of harmonic polynomials on $\mathfrak{h}^*$ ($\mathfrak{h}$) of degree d. The map

$$H_{2d}(X, \mathbb{C}) \longrightarrow \mathcal{H}_d(\mathfrak{h}^*)\,,\ \ \gamma \mapsto b(\gamma) = \frac{1}{(2\pi i)^d d!} \int_\gamma \tau_\lambda^d$$

is an isomorphism of W-modules called the Borel isomorphism. Here the W-action on $H_{2d}(X, \mathbb{C})$ is induced by the natural W-action on X. On the other hand, we have a natural homomorphism

$$H_{2d}(\mu^{-1}(\nu), \mathbb{C}) \longrightarrow H_{2d}(X, \mathbb{C}),$$

defined by the embedding $\mu^{-1}(\nu) \longrightarrow X \times \{\nu\}$. Rossmann shows this is a nonzero W-module homomorphism which factors through the projection

$$H_{2d}(\mu^{-1}(\nu), \mathbb{C}) \longrightarrow H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)}.$$

It is known that χ_ν appears exactly once in $\mathcal{H}_d(\mathfrak{h}^*)$. We denote the corresponding subspace by $\mathcal{H}_d(\mathfrak{h}^*)_\nu$. Now we compose previous mappings to obtain a surjective homomorphism of W-modules

$$H_{2n}(\mu^{-1}(\overline{\mathcal{O}}), \mathbb{C}) \longrightarrow \mathcal{H}_d(\mathfrak{h}^*)_\nu\,,\ \ \Gamma \mapsto p_\Gamma. \tag{2}$$

Rossmann's definition of W-action on $H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C})$ implies the following W-equivariance formula for distributions $\Theta(\Gamma, \lambda)$:

$$\Theta(w\Gamma, \lambda) = \Theta(\Gamma, w^{-1}\lambda)\,, \quad w \in W,\ \lambda \in \mathfrak{h}^*.$$

Our goal is to study the asymptotic behaviour of distributions $\Theta(\Gamma, \lambda)$ when $\lambda \in \mathfrak{h}^*$ approaches zero. We recall additional facts needed for this analysis. Denote by $\Theta_{\mathcal{O}}$ the Fourier transform of the Liouville measure $m_{\mathcal{O}}$. In more detail,

$$\Theta_{\mathcal{O}}(\phi) = \frac{1}{(2\pi i)^k k!}\int_{\mathcal{O}} \hat{\phi}\sigma_{\mathcal{O}}^k\,, \quad 2k = \dim_{\mathbb{R}} \mathcal{O},\ \phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}}).$$

Let $\Gamma \in H_{2n}(\mu^{-1}(\overline{\mathcal{O}}), \mathbb{C})$. Rossmann proves [10] the following formula relating distributions $\Theta(\Gamma, \lambda)$ and $\Theta_{\mathcal{O}}$:

$$\Theta(\Gamma, \lambda) = p_\Gamma(\lambda)\Theta_{\mathcal{O}} + o(\lambda^{n-k}).$$

The term $o(\lambda^{n-k})$ can be described as follows. For any $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$, $o(\lambda^{n-k})(\phi)$ is a holomorphic function of λ and

$$\lim_{t \to 0} \frac{o((t\lambda)^{n-k})(\phi)}{t^{n-k}} = 0.$$

Finally, we shall mention two results from [4] that will be used in the proof of the main theorem. Denote by $\mathbb{C}[\mathfrak{h}]$ resp. $\mathbb{C}[\mathfrak{h}^*]$ the algebra of polynomial functions on $\mathfrak{h}$ resp. $\mathfrak{h}^*$ and by $D(\mathfrak{h}^*)$ the algebra of differential operators on $\mathfrak{h}^*$ with constant coefficients. Then we have a natural isomorphism of algebras

$$\mathbb{C}[\mathfrak{h}] \cong D(\mathfrak{h}^*), \ \ p \mapsto p(\partial),\, p \in \mathbb{C}[\mathfrak{h}].$$

Let $r \in \mathcal{H}_d(\mathfrak{h}^*)_\nu$. By [4], Lemma 3.2 we can find $p \in \mathcal{H}_d(\mathfrak{h})_\nu$ such that

$$p(\partial)r \neq 0. \tag{3}$$

Furthermore, if $\Gamma \in H_{2n}(T^*_{\mathbb{R}}X, \mathbb{C})$, $\lambda \in \mathfrak{h}^*$, $p \in \mathbb{C}[\mathfrak{h}]$ and $w \in W$, then by [4], Lemma 3.3 we have:

$$\lim_{\lambda \to 0} p(\partial)\Theta(\Gamma, \lambda) \tag{4}$$

exists as a distribution on $\mathfrak{g}_{\mathbb{R}}$, and

$$\lim_{\lambda \to 0} w^{-1}p(\partial)\Theta(\Gamma, \lambda) = \lim_{\lambda \to 0} p(\partial)\Theta(w\Gamma, \lambda). \tag{5}$$

3. A limit formula

Following [13], we shall recall the construction of the eqiuvalence of categories of K-equivariant and of $G_{\mathbb{R}}$-equivariant sheaves on the flag variety. We shall explain that in an analogous way we can obtain an equivalence of categories of K-equivariant and of $G_{\mathbb{R}}$-equivariant sheaves on the generalized flag variety Y.

Let $Z = X$ or $Z = Y$. Define maps

$$Z \xleftarrow{a_Z} G_{\mathbb{R}} \times Z \xrightarrow{q_Z} G_{\mathbb{R}}/K_{\mathbb{R}} \times Z \xrightarrow{p_Z} Z$$

by $a_Z(g,z) = g^{-1}z$, $q_Z(g,z) = (gK_{\mathbb{R}}, z)$ and $p_Z(gK_{\mathbb{R}}, z) = z$. These maps are $G_{\mathbb{R}} \times K_{\mathbb{R}}$-equivariant for the following actions: $(g,k)\cdot z = k \cdot z$, $(g,k)\cdot(g',z) = (gg'k^{-1}, z)$, $(g,k)\cdot(g'K_{\mathbb{R}}, z) = (gg'K_{\mathbb{R}}, g\cdot z)$ and $(g,k)\cdot z = g \cdot z$. The action of $G_{\mathbb{R}}$ on Z is trivial, hence we may regard an object $\mathcal{F}$ from $D^b_K(Z)$ (after applying the functor $For^K_{K_{\mathbb{R}})}$) as an object from $D^b_{G_{\mathbb{R}}\times K_{\mathbb{R}}}(Z)$. On the other hand, the action of $K_{\mathbb{R}}$ on $G_{\mathbb{R}} \times Z$ is free, hence there exists a unique object $\mathcal{F}'$ in $D^b_{G_{\mathbb{R}}}(G_{\mathbb{R}}/K_{\mathbb{R}} \times Z)$ such that $a^*_Z(\mathcal{F}) \simeq q^*_Z(\mathcal{F}')$. We remark that in loc.cit. the condition $a^!_Z(\mathcal{F}) \simeq q^!_Z(\mathcal{F}')$ is used to pick $\mathcal{F}'$. These two conditions are equivalent if we orient Z as a complex manifold and $G_{\mathbb{R}}/K_{\mathbb{R}} \simeq \mathfrak{p}_{\mathbb{R}}$ as a differentiable manifold [13], 4.2. Finally, we define

$$\gamma_Z : D^b_K(Z) \longrightarrow D^b_{G_{\mathbb{R}}}(Z), \;\; \gamma_Z(\mathcal{F}) = Rp_{Z!}(\mathcal{F}').$$

Proposition 1. *The following diagram is commutative:*

$$\begin{array}{ccc} D^b_K(Y) & \xrightarrow{\gamma_Y} & D^b_{G_{\mathbb{R}}}(Y) \\ {\scriptstyle \pi^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi^*} \\ D^b_K(X) & \xrightarrow{\gamma_X} & D^b_{G_{\mathbb{R}}}(X). \end{array}$$

Proof. We cosider a sequence of Cartesian squares:

$$\begin{array}{ccccccc} X & \xleftarrow{a_X} & G_{\mathbb{R}} \times X & \xrightarrow{q_X} & G_{\mathbb{R}}/K_{\mathbb{R}} \times X & \xrightarrow{p_X} & X \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle id\times\pi}\big\downarrow & & {\scriptstyle id\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Y & \xleftarrow{a_Y} & G_{\mathbb{R}} \times Y & \xrightarrow{q_Y} & G_{\mathbb{R}}/K_{\mathbb{R}} \times Y & \xrightarrow{p_Y} & Y, \end{array}$$

and pick an object $\mathcal{F}$ in $D^b_K(Y)$. Let $\mathcal{F}'$ be an object in $D^b_{G_{\mathbb{R}}}(G_{\mathbb{R}}/K_{\mathbb{R}} \times Y)$ such that $a^*_Y(\mathcal{F}) \simeq q^*_Y(\mathcal{F}')$. The commutativity of the the first and second diagram from left to right implies

$$q^*_X((id \times \pi)^*(\mathcal{F}')) \simeq a^*_X(\pi^*(\mathcal{F})).$$

Now we apply the base change formula to the third diagram to deduce

$$\gamma_X(\pi^*\mathcal{F}) = Rp_{X!}((id\times\pi)^*\mathcal{F}') = \pi^* Rp_{Y!}(\mathcal{F}').$$

The proposition follows. □

Let us recall that the orbit $Z = K \cdot y_0$ is closed, and the orbit $S = G_{\mathbb{R}} \cdot y_0$ is open in the generalized flag variety Y. Let

$$i' : Z \longrightarrow Y,\; j' : S \longrightarrow Y,\; i : \pi^{-1}(Z) \longrightarrow X \text{ and } j : \pi^{-1}(S) \longrightarrow X$$

denote the embeddings. Our goal is to relate sheaves $i_*(\mathbb{C}_{\gamma^{-1}(Z)})$ and $j_!(\mathbb{C}_{\gamma^{-1}(S)})$. We begin by comparing standard sheaves on Z and S.

Lemma 1. *The action of* γ_Y *on* $i'_*(\mathbb{C}_Z)$ *is given by the formula*

$$\gamma_Y(i'_*(\mathbb{C}_Z)) = j'_!(\mathbb{C}_S)[2\,codim_{\mathbb{C}} Z].$$

Proof. In the setting of a flag variety the formula is proved in [9]. For a generalized flag variety there is also a geometric interpretation of the Matsuki duality between K and $G_{\mathbb{R}}$-orbits which is derived from the properties of the gradient flow defined by squared norm of the moment map [7]. Formulas for the action of γ_Y on standard sheaves can be deduced analogously as in the setting of the flag variety [9] using the properties of the gradient flow on the generalized flag variety. □

Proposition 2. *The action of* γ_X *on* $i_*(\mathbb{C}_{\gamma^{-1}(Z)})$ *is given by the following formula:*

$$\gamma_X(i_*(\mathbb{C}_{\gamma^{-1}(Z)})) = j_!(\mathbb{C}_{\gamma^{-1}(S)})[2\,codim_{\mathbb{C}} Z].$$

Proof. First, we remark that i' and i are closed embeddings, hence $i'_* = i'_!$ and $i_* = i_!$. Now we apply the base change formula to the Cartesian diagram

$$\begin{tikzcd} \pi^{-1}(Z) \arrow[r, "i"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "i'"] & Y \end{tikzcd}$$

to deduce $i_*(\mathbb{C}_{\pi^{-1}(Z)}) = \pi^* i'_!(\mathbb{C}_Z)$. Using this formula and the Cartesian square

$$\begin{tikzcd} \pi^{-1}(S) \arrow[r, "j"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "j'"] & Y \end{tikzcd}$$

we obtain

$$\begin{aligned} \gamma_X(i_*(\mathbb{C}_{\gamma^{-1}(Z)})) &= \gamma_X(\pi^* i'_!(\mathbb{C}_Z)) \\ &= \pi^* \gamma_Y(i'_!(\mathbb{C}_Z)) \\ &= \pi^* j'_!(\mathbb{C}_S)[2\mathrm{codim}_{\mathbb{C}} Z] \\ &= j_!(\mathbb{C}_{\gamma^{-1}(S)})[2\mathrm{codim}_{\mathbb{C}} Z]. \end{aligned}$$

□

Schmid and Vilonen define a homomorphism [13], 3.7,

$$\Phi : H_{2n}(T^*_K X, \mathbb{C}) \longrightarrow H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C}),$$

which makes the following diagram commutative:

$$\begin{tikzcd} D_K(X) \arrow[r, "\gamma"] \arrow[d, "CC"'] & D_{G_{\mathbb{R}}}(X) \arrow[d, "CC"] \\ H_{2n}(T^*_K X, \mathbb{C}) \arrow[r, "\Phi"] & H_{2n}(T^*_{G_{\mathbb{R}}} X, \mathbb{C}). \end{tikzcd}$$

Put $\mathcal{N}_k = \cup\{G\cdot x : x\in\mathcal{N},\ \dim_{\mathbb{C}} G\cdot x = 2k\}$ and $\overline{\mathcal{N}}_k = \cup_{l\le k}\mathcal{N}_l$. The map Φ descends to the nilpotent cone by integrating cycles over the fibres of the moment map. At this point we shall use the identification $\mathfrak{g}^* \cong \mathfrak{g}$ defined by the Killing form. In particular, we view the moment map as the map

$$\mu : T^*X \longrightarrow \mathfrak{g}.$$

Let $\mathcal{V}' \subset \mathcal{N}_k \cap \mathfrak{p}$ be a K-orbit, and $\mathcal{O}' \subset \mathcal{N}\cap i\mathfrak{g}_{\mathbb{R}}$ a $G_{\mathbb{R}}$-orbit. We orient $\mathcal{V}'$ by the complex structure, and $\mathcal{O}'$ by the Liouville form. We denote the corresponding cycles by $[\mathcal{V}']$ and $[\mathcal{O}']$. Schmid and Vilonen introduce the maps

$$(\mathrm{gr}\mu_*)_{K,\lambda} : H_{2n}(T^*_K X,\mathbb{C}) \longrightarrow \bigoplus_{k\ge 0} H_{2k}(\mathcal{N}_k\cap\mathfrak{p},\mathbb{C}),$$

$$(\mathrm{gr}\mu_*)_{G_{\mathbb{R}},\lambda} : H_{2n}(T^*_{G_{\mathbb{R}}} X,\mathbb{C}) \longrightarrow \bigoplus_{k\ge 0} H_{2k}(\mathcal{N}_k\cap i\mathfrak{g}_{\mathbb{R}},\mathbb{C}),$$

and define a homomorphism [13], 5.10,

$$\phi : \bigoplus_{k\ge 0} H_{2k}(\mathcal{N}_k\cap\mathfrak{p},\mathbb{C}) \longrightarrow \bigoplus_{k\ge 0} H_{2k}(\mathcal{N}_k\cap i\mathfrak{g}_{\mathbb{R}},\mathbb{C}),$$

such that

$$(\mathrm{gr}\mu_*)_{G_{\mathbb{R}},\lambda}\circ\Phi = \phi\circ(\mathrm{gr}\mu_*)_{K,\lambda}.$$

Moreover, they compute ϕ on the invariant part of the homology [13], 6.3, and show that

$$\phi([\mathcal{V}']) = [\mathcal{O}'],$$

if a K-orbit $\mathcal{V}'$ and a $G_{\mathbb{R}}$-orbit $\mathcal{O}'$ are related by the Sekiguchi correspondence.

The next theorem describes multiplicities of the orbits in the cycles obtained by descent of $C\in H_{2n}(T^*_K X,\mathbb{C})$ and $\Phi(C)\in H_{2n}(T^*_{G_{\mathbb{R}}} X,\mathbb{C})$ to the nilpotent cone. The theorem was explained in more detail in [6].

Theorem 1. *Let $C\in H_{2n}(T^*_K X,\mathbb{C})$, and let $k = k(C)$ be the minimal integer such that $C\in H_{2n}(T^*_K X\cap\mu^{-1}(\overline{\mathcal{N}}_k),\mathbb{C})$. Let us write*

$$\mathcal{N}_k\cap\mathfrak{p} = \mathcal{V}_1\cup\cdots\cup\mathcal{V}_l,$$

where $\mathcal{V}_1,\cdots,\mathcal{V}_l$ are K-orbits. Let us denote by $\mathcal{O}_i$ the $G_{\mathbb{R}}$-orbit related to the orbit $\mathcal{V}_i$ by the Sekiguchi correspondence. Let us consider the restriction $C^0 = C|_{T^*_K X\cap\mu^{-1}(\mathcal{N}_k)}$. Then we have

$$C^0 = \sum_{j=1}^{l} C_{\mathcal{V}_i}, \qquad C_{\mathcal{V}_i}\in H_{2n}(\mu^{-1}\mathcal{V}_i,\mathbb{C}).$$

Now we draw attention to (6) *and denote by $p(\mathcal{C},\mathcal{V}_i)$ the polynomial in $\mathcal{H}_{(n-k)}(\mathfrak{h}^\star)$ associated with cycle $C_{\mathcal{V}_i}$. Then*

$$(gr\mu_*)_{K,\lambda}(C) = \sum_{i=1}^{l} p(C,\mathcal{V}_i)(\lambda)[\mathcal{V}_i]\ \textit{and}$$

$$\Theta(\Phi(C),\lambda)=\sum_{i=1}^{l}p(C,\mathcal{V}_i)(\lambda)\Theta_{\mathcal{O}_i}+o(\lambda^{n-k}).$$

We shall apply this theorem to the sheaves $i_*(\mathbb{C}_{\pi^{-1}(Z)})$ and $j_!(\mathbb{C}_{\pi^{-1}(S)})$. Let $\mathcal{V}\subset\mathcal{N}\cap\mathfrak{p}$ be the K-orbit such that

$$\overline{\mathcal{V}}=K\cdot(\overline{\mathfrak{u}}\cap\mathfrak{p}). \tag{6}$$

Let

$$\mathcal{O}\subset i\mathfrak{g}_{\mathbb{R}}\cap\mathcal{N} \tag{7}$$

be the $G_{\mathbb{R}}$-orbit related to $\mathcal{O}$ by the Sekiguchi correspondence. To simplify notation set

$$C_Z=CC(i_*(\mathbb{C}_{\pi^{-1}(Z)}))\text{ and }C_S=CC(j_!(\mathbb{C}_{\pi^{-1}(S)})).$$

Theorem 2. *Let $\lambda\in\mathfrak{h}^*$ and let $k=\dim_{\mathbb{C}}K\cdot(\overline{\mathfrak{u}}\cap\mathfrak{p}))$. Then*

$$\Theta(C_S,\lambda)=p(C_Z,\mathcal{V})(\lambda)\Theta_{\mathcal{O}}+o(\lambda^{n-k}),$$

where $p(C_Z,\mathcal{V})\neq 0$.

Proof. We remark that $\pi^{-1}Z$ is closed in X, hence $C_Z=[T^*_{\pi^{-1}(Z)}X]$, where the conormal bundle is oriented by its complex structure. Descending to the nilpotent cone we obtain $\mu(T^*_{\pi^{-1}(Z)}X)=K\cdot(\overline{\mathfrak{u}}\cap\mathfrak{p}))$. The definition of $\mathcal{V}$ and Theorem 1 imply now the formula

$$\Theta(C_S,\lambda)=p(C_Z,\mathcal{V})(\lambda)\Theta_{\mathcal{O}}+o(\lambda^{n-k}).$$

It remains to show $p(C_Z,\mathcal{V})\neq 0$. To prove this statement we have to work with $\mathcal{D}$-modules. Define $\mathcal{D}_X=\mathcal{D}_\rho$-module $\mathcal{M}=i_+(\mathcal{O}_{\pi^{-1}(Z)})$. Here $\mathcal{O}$ denotes the sheaf of regular functions and i_+ the direct image functor in the category of $\mathcal{D}$-modules. Let $M=\Gamma(X,\mathcal{M})$. Then M is a $(\mathfrak{g},K)$-module with trivial infinitesimal character. Moreover, M is an irreducible $(\mathfrak{g},K)$-module, since $\mathcal{M}$ is irreducible as a $\mathcal{D}_\rho$-module. Denote by $Ass(M)$ the associated cycle of M. Then [13], 7.5 implies

$$(gr\mu_*)_\rho(C_Z)=Ass(M).$$

Now the irreducibility of M and Theorem 3.4 imply $p(C_Z,\mathcal{V})(\rho)\neq 0$. □

Recall that $\lambda_{x_0}\in i\mathfrak{t}^*_{\mathbb{R}}$ is dominant if

$$(\lambda_{x_0},\alpha)>0\ \text{ for any }\ \alpha\in\Delta(\mathfrak{g}/\mathfrak{b}_{x_0},\mathfrak{t}).$$

Set $p=\dim_{\mathbb{C}}\mathfrak{n}\cap\mathfrak{p}$. The next theorem is a variant of Rossmann's character formula for a standard sheaf associated with an open $G_{\mathbb{R}}$-orbit on X proved by Schmid and Vilonen [12]. In loc.cit. the formula stated in the theorem below was proved for standard sheaves defined by the direct image functor for an open embedding. An analogous formula for the proper direct image functor was discussed in [3]. In fact, using the result about orientation of $CC(Rj_{0!}\mathbb{C}_{S_0})$ from loc.cit., and an analogous argument as in [12], §7, §8, we can prove the following theorem.

Theorem 3. *Let $G_{\mathbb{R}}$ be a connected, linear, semisimple Lie group. Consider the Borel subalgebra $\mathfrak{b}_{x_0}$ defined in* (4) *and the open orbit $S_0 = Ad(G_{\mathbb{R}}) \cdot x_0$. Let $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$. Denote by $\hat{\phi}$ the Fourier transform of ϕ. Then for $\lambda \in \mathfrak{h}^{*\prime}$ such that λ_{x_0} is dominant we have*

$$\int_{CC(Rj_{0!}\mathbb{C}_{S_0})} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^m) = (-1)^p \int_{G_{\mathbb{R}}\cdot\lambda_{x_0}} \hat{\phi}\sigma_\lambda^m.$$

Theorem 4. *Let $\mathfrak{b}_{x_0}$ be the Borel subalgebra defined in* (4)*, and let $C \subset i\mathfrak{t}_{\mathbb{R}}$ be the corresponding positive chamber. Let $\mathcal{V}$ be the K-orbit defined in* (13) *and $\mathcal{O}$ the $G_{\mathbb{R}}$-orbit defined in* (14)*. Let $k = \dim_{\mathbb{C}} \mathcal{V}$. Denote by χ_ν the Springer character defined by the complex orbit $G\cdot\mathcal{V}$. Then there is a harmonic polynomial $p \in \mathcal{H}_{n-k}(\mathfrak{h}^*)_\nu$ and a non-zero constant κ such that the following formula holds:*

$$\lim_{\lambda\to 0(C)} \sum_{w\in W(L_{\mathbb{R}},T_{\mathbb{R}})} (-1)^{l(w)} wp(\partial) m_\lambda = \kappa m_{\mathcal{O}}.$$

Proof. We begin by recalling the identity for $CC(j_!(\mathbb{C}_{\pi^{-1}(S)}))$ from [5] :

$$|W(L_{\mathbb{R}},T_{\mathbb{R}})| CC(j_!(\mathbb{C}_{\pi^{-1}(S)})) = \sum_{w\in W(L_{\mathbb{R}},T_{\mathbb{R}})} (-1)^{l(w)} w \cdot CC(j_{S_0!}\mathbb{C}_{S_0}).$$

This implies further the identity between invariant eigendistributions on $\mathfrak{g}_{\mathbb{R}}$:

$$|W(L_{\mathbb{R}},T_{\mathbb{R}})| \Theta(CC(j_!(\mathbb{C}_{\pi^{-1}(S)}), \lambda) = \sum_{w\in W(L_{\mathbb{R}},T_{\mathbb{R}})} (-1)^{l(w)} \Theta(w \cdot CC(j_{S_0!}\mathbb{C}_{S_0}), \lambda).$$

To simplify notation denote $r(\lambda) = p(C_Z, \mathcal{V})(\lambda)$. By Theorem 2 we can write:

$$|W(L_{\mathbb{R}},T_{\mathbb{R}})| r(\lambda)\Theta_{\mathcal{O}} + o(\lambda^{n-k}) = \sum_{w\in W(L_{\mathbb{R}},T_{\mathbb{R}})} (-1)^{l(w)} \Theta(w \cdot CC(j_{S_0!}\mathbb{C}_{S_0}), \lambda).$$

By (9) we can find a harmonic polynomial $p \in \mathcal{H}_{n-k}(\mathfrak{h})_\nu$, $\nu \in G \cdot \mathcal{V}$, such that $\partial(p)r \neq 0$. We act on the previous identity by $\partial(p)$, and apply (10) and (11) to conclude

$$\lim_{\lambda\to 0(C_{x_0})} \sum_{w\in W(L_{\mathbb{R}},T_{\mathbb{R}})} (-1)^{l(w)} wp(\partial) \Theta(CC(j_{S_0!}\mathbb{C}_{S_0}), \lambda) = |W(L_{\mathbb{R}},T_{\mathbb{R}})| \eta \Theta_{\mathcal{O}},$$

where $\eta \neq 0$. To complete the proof it will suffice to use Theorem 3, and take the inverse Fourier transform. □

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