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(R, S) conjugate solution to coupled Sylvester complex matrix equations with conjugate of two unknowns

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ABSTRACT

In this work, we are concerned with (R, S) – conjugate solutions to coupled Sylvester complex matrix equations with conjugate of two unknowns. When the considered two matrix equations are consistent, it is demonstrated that the solutions can be obtained by utilizing this iterative algorithm for any initial arbitrary (R, S) – conjugate matrices V_1, W_1. A necessary and sufficient condition is established to guarantee that the proposed method converges to the (R, S) – conjugate solutions. Finally, two numerical examples are provided to demonstrate the efficiency of the described iterative technique.

1. Introduction

Many scholars have given complex matrix equations a lot of thought. Chang et al. [1] gives the expression of (R, S) – conjugate solution to the system of matrix equations $AX = C, XB = D$. Trench [2] investigated the system of linear equations $Az = w$ for R – conjugate matrices; further, as an extension of R – conjugate matrix, Trench [3] defined (R, S) – conjugate matrix and min $\| Az - w \|$ for (R, S) – conjugate matrices where z, w are known column vectors. Dong et al. [4] presented the Hermitian R – conjugate solution to the system of complex matrix equations $AX = C, XB = D$. Ramadan and El-Danaf [5] introduced an iterative method for obtaining the generalized bisymmetric solution to the coupled Sylvester matrix equations $(AV + BW, MV + NW) = (EVF + C, GVH + D)$. Bayoumi and Ramadan [9] presented the Hermitian R – conjugate solution to a generalized coupled Sylvester-conjugate matrix equations. Bayoumi [10] proposed an iterative algorithm for solving a generalized coupled Sylvester – conjugate matrix equations over Hamiltonian matrices. Li et al. [11] presented two algorithms for solving the matrix $AXB = C$ for (R, S)-symmetric matrices X based on the idea of the traditional conjugate gradient method and conjugate gradient Least Squares method.

Trench [12] defined a special class of matrices called (R, S)-symmetric matrices (R, S)-skew symmetric matrices. Balani and Hajarian [13] used the generalized accelerated overrelaxation method and generalized

conjugate gradient methods for presenting iteration methods to solve linear systems of equations. Dehghan and Shirilord [14] used the accelerated double-step scale splitting iteration method for solving a class of complex matrix equations. Bayoumi [15] presented two relaxed gradient-based algorithms to find the Hermitian and skew-Hermitian solutions to the linear matrix equation $AXB + CXD = F$. Ramadan et al. [16] used Sylvester block sum and block matrix Kronecker map to offer an explicit solution of a system of Sylvester matrix equations.

This paper is sorted out as follows: In section 2, we present several notations and lemmas that will play vital roles in the sequel section. In section 3, we provide an iterative algorithm for solving the coupled Sylvester complex matrix equation with conjugate of two unknown over (R, S) – conjugate matrices and we give the convergence properties of these iterative algorithms. In section 4, Two numerical examples are presented to support the theoretical results of the proposed iterative algorithm.

2. Preliminaries

The set of all $m \times n$ complex matrices is denoted by $\mathbb{C}^{m \times n}$, and we utilize $\bar{A}, A^T, tr(A), A^H$, Re(a) and A to denote the conjugate, transpose, the trace, conjugate transpose, the real part of the number a and the Frobenius norm of a matrix A, respectively. $A \otimes B = (a_{ij}B)$

denotes the Kronecker product of two matrices A and B. For a matrix $X = [x_1 x_2 \cdots x_n] \in \mathbb{C}^{m \times n}$, $vec(X)$ is the column stretching operation of X, and defined as $vec(X) = [x_1^T x_2^T \cdots x_n^T]^T$. A well-known feature of the Kronecker product for matrices A, B and C with appropriate dimension, is $vec(ABC) = (C^T \otimes A)vec(B)$. Furthermore, it is obvious that $A = \| vec(A) \|_2$ for any matrix A.

Lemma 2.1 [6]: *For the matrix equation $AXB = F$ where $A \in \mathbb{C}^{m \times r}, B \in \mathbb{C}^{s \times n}$ and $F \in \mathbb{C}^{m \times n}$ are known matrices and $X \in \mathbb{C}^{r \times s}$ is the matrix to be determined, an iterative algorithm is constructed as $X(k+1) = X(k) + \mu A^H (F - AX(k)B)B^H$ with $0 < \mu < \frac{2}{\|A\|_2^2 \|B\|_2^2}$.*

If this matrix equation has a unique solution X^*, then the iterative solution $X(k)$ converges to the unique solution X^* ∎

Definition 2.1 [7]: : In the space $\mathbb{C}^{m \times n}$ over the field $\mathbb{R}$, an inner product space can be described as $A, B = Re[tr(A^H B)]$

The Frobenius norm of C is denoted by $\| C \|$, that is $\| C \| = \sqrt{tr(C^H C)}$. The matrices $B, C \in \mathbb{C}^{m \times n}$ are called orthogonal if $\langle B, C \rangle = 0$. ∎

Definition 2.2 [3]: Let R, S be an $n \times n$ symmetric orthogonal matrices, that is, $R^T = R, R^2 = I, S^T = S, S^2 = I$, a matrix $A \in \mathbb{C}^{n \times n}$ is termed (R, S) – conjugate matrix if $RAS = \bar{A}$. Let $\mathbb{RSC}^{n \times n}$ represent the set of all (R, S) – conjugate matrices that is, $\mathbb{RSC}^{n \times n} = \{A : RAS = \bar{A}\}$ where R, S be an $n \times n$ symmetric orthogonal matrices. ∎

Lemma 2.1: *For any $X \in \mathbb{C}^{n \times n}$, then $X + R\bar{X}S \in \mathbb{RSC}^{n \times n}$ where R, S are $n \times n$ symmetric orthogonal matrices.*

Proof: By definition of $\mathbb{RSC}^{n \times n}$

$$R[X + R\bar{X}S]S = RXS + \bar{X} = \overline{[X + R\bar{X}S]} \qquad ∎$$

Lemma 2.2: *For any $X \in \mathbb{C}^{n \times n}, P \in \mathbb{RSC}^{n \times n}$, then $\langle P, \frac{X + R\bar{X}S}{2} \rangle = \langle P, X \rangle$.*

Proof:

$$\langle P, \frac{X + R\bar{X}S}{2} \rangle = \frac{1}{2}\langle P, X \rangle + \frac{1}{2}\langle P, R\bar{X}S \rangle$$
$$= \frac{1}{2}\langle P, X + \frac{1}{2}RPS, \bar{X} \rangle = \frac{1}{2}\langle P, X \rangle + \frac{1}{2}\langle \bar{P}, \bar{X} \rangle$$
$$= \frac{1}{2}\langle P, X \rangle + \frac{1}{2}\langle P, X \rangle = \langle P, X \rangle \qquad ∎$$

3. The main results

3.1. The iterative algorithm

In this section, the following coupled Sylvester complex matrix equations with conjugate of two unknowns are considered

$$\begin{cases} A_{11}VB_{11} + C_{11}WD_{11} + A_{12}\bar{V}B_{12} + C_{12}\bar{W}D_{12} = E_1 \\ A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\bar{V}B_{22} + C_{22}\bar{W}D_{22} = E_2 \end{cases}$$
$$(1)$$

where $A_{11}, A_{12}, C_{11}, C_{12}, A_{21}, A_{22}, C_{21}, C_{22} \in \mathbb{C}^{m \times n}$, $B_{11}, B_{12}, D_{11}, D_{12}, B_{21}, B_{22}, D_{21}, D_{22}$ $\in \mathbb{C}^{n \times r}$ and $E_1, E_2 \in \mathbb{C}^{m \times r}$ are given matrices, while $V, W \in \mathbb{RSC}^{n \times n}$ are matrices to be determined.

Denote

$$g_1(V, W) = A_{11}VB_{11} + C_{11}WD_{11}$$
$$+ A_{12}\bar{V}B_{12} + C_{12}\bar{W}D_{12}$$
$$g_2(V, W) = A_{21}VB_{21} + C_{21}WD_{21}$$
$$+ A_{22}\bar{V}B_{22} + C_{22}\bar{W}D_{22}.$$

Lemma 3.1.1: The coupled Sylvester matrix equation (1) has (R, S) – conjugate solutions if and only if the following system of matrix equations

$$\begin{cases} A_{11}VB_{11} + C_{11}WD_{11} + A_{12}\bar{V}B_{12} + C_{12}\bar{W}D_{12} = E_1, \\ A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\bar{V}B_{22} + C_{22}\bar{W}D_{22} = E_2, \\ \overline{A_{11}}RVS\overline{B_{11}} + \overline{C_{11}}RWS\overline{D_{11}} + \overline{A_{12}}VB_{12} + \overline{C_{12}}WD_{12} = \overline{E_1}, \\ \overline{A_{21}}RVS\overline{B_{21}} + \overline{C_{21}}RWS\overline{D_{21}} + \overline{A_{22}}VB_{22} + \overline{C_{22}}WD_{22} = \overline{E_2}. \end{cases}$$
$$(2)$$

is consistent. ∎

Proof: If the coupled Sylvester matrix equation (1) has (R, S) – conjugate solutions $V^*, W^* \in \mathbb{RSC}^{n \times n}$ i.e. $RV^*S = \overline{V^*}$ and $RW^*S = \overline{W^*}$, it is obvious that V^*, W^* are also solutions of equation (2). Conversely assume that the system of matrix equation (2) has solutions $V_1, W_1 \in \mathbb{C}^{n \times n}$, let $V^* = \frac{1}{2}[V_1 + R\overline{V_1}S], W^* = \frac{1}{2}[W_1 + R\overline{W_1}S]$, Therefore $V^*, W^* \in \mathbb{RSC}^{n \times n}$ and

$$A_{11}V^*B_{11} + C_{11}W^*D_{11} + A_{12}\overline{V^*}B_{12} + C_{12}\overline{W^*}D_{12}$$
$$= \frac{1}{2}[A_{11}V_1B_{11} + A_{11}R\overline{V_1}SB_{11} + C_{11}W_1D_{11}$$
$$+ C_{11}R\overline{W_1}SD_{11} + A_{12}\overline{V_1}B_{12} + A_{12}RV_1SB_{12}$$
$$+ C_{12}\overline{W_1}D_{12} + C_{12}RW_1SD_{12}],$$
$$= \frac{1}{2}[A_{11}V_1B_{11} + C_{11}W_1D_{11} + A_{12}\overline{V_1}B_{12}$$
$$+ C_{12}\overline{W_1}D_{12} + A_{11}R\overline{V_1}SB_{11} + C_{11}R\overline{W_1}SD_{11}$$
$$+ A_{12}RV_1SB_{12} + C_{12}RW_1SD_{12}]$$
$$= \frac{1}{2}[E_1 + E_1] = E_1$$

$$A_{21}V^*B_{21} + C_{21}W^*D_{21} + A_{22}\overline{V^*}B_{22} + C_{22}\overline{W^*}D_{22}$$
$$= \frac{1}{2}[A_{21}V_1B_{21} + A_{21}R\overline{V_1}SB_{21} + C_{21}W_1D_{21}$$
$$+ C_{21}R\overline{W_1}SD_{21} + A_{22}\overline{V_1}B_{22} + A_{22}RV_1SB_{22}$$
$$+ C_{22}\overline{W_1}D_{22} + C_{22}RW_1SD_{22}],$$
$$= \frac{1}{2}[A_{21}V_1B_{21} + C_{21}W_1D_{21} + A_{22}\overline{V_1}B_{22}$$
$$+ C_{22}\overline{W_1}D_{22} + A_{21}R\overline{V_1}SB_{21} + C_{21}R\overline{W_1}SD_{21}$$

$$+ A_{22}RV_1SB_{22} + C_{22}RW_1SD_{22}]$$

$$= \frac{1}{2}[E_2 + E_2] = E_2 \qquad \blacksquare$$

Thus, V^*, W^* are the solutions to the coupled Sylvester matrix equation (1). So the solvability of the coupled Sylvester matrix equation (1) is equivalent to that of matrix equation (2).

Let us rewrite the matrix equation (2) into the equivalent system $Sz = b$, let

$$S = \begin{bmatrix} B_{11}^H \otimes A_{11} & D_{11}^H \otimes C_{11} \\ +B_{12}^H S \otimes A_{12}R & +D_{12}^H S \otimes C_{12}R \\ B_{21}^H \otimes A_{21} & D_{21}^H \otimes C_{21} \\ +B_{22}^H S \otimes A_{22}R & +D_{22}^H S \otimes C_{22}R \\ \bar{B}_{11}^H S \otimes \overline{A_{11}}R & \bar{D}_{11}^H S \otimes \overline{C_{11}}R \\ +\bar{B}_{12}^H \otimes \overline{A_{12}} & +\bar{D}_{12}^H \otimes \overline{C_{12}} \\ \bar{B}_{21}^H S \otimes \overline{A_{21}}R & \bar{D}_{21}^H S \otimes \overline{C_{21}}R \\ +\bar{B}_{22}^H \otimes \overline{A_{22}} & +\bar{D}_{22}^H \otimes \overline{C_{22}} \end{bmatrix},$$

$$b = \begin{bmatrix} vec(E_1) \\ vec(E_2) \\ vec(\overline{E_1}) \\ vec(\overline{E_2}) \end{bmatrix}, z = \begin{bmatrix} vec(V) \\ vec(W) \end{bmatrix}.$$

The following is a well-known theorem:

Theorem 3.1.1 [8]: The system of matrix equation (2) has a unique (R, S) – conjugate solutions if $rank(S, b) = rank(S)$.

We now present the (R, S) – conjugate iterative algorithm to solve the coupled Sylvester matrix equation (1) over (R, S) – conjugate matrices. $\blacksquare$

Algorithm I: 1. *Input matrices* $A_{11}, C_{11}, A_{12}, C_{12}, A_{21}, C_{21}, A_{22}, C_{22} \in \mathbb{C}^{m \times n}$, $E_1, E_2 \in \mathbb{C}^{m \times r}$ *and* $B_{11}, D_{11}, B_{12}, D_{12}, B_{21}, D_{21}, B_{22}, D_{22} \in \mathbb{C}^{n \times r}$.

2. *Chosen arbitrary initial* (R, S) *– conjugate matrices* $V_1, W_1 \in \mathbb{RSC}^{n \times n}$ *where* R, S *be an* $n \times n$ *symmetric orthogonal matrices.*

3. *Calculate*

$V(k + 1)$

$$= V(k) + \frac{\mu}{2}[A_{11}^H r_1(k)B_{11}^H + \bar{A}_{12}^H \overline{r_1(k)}\bar{B}_{12}^H$$

$$+ A_{21}^H r_2(k)B_{21}^H + \bar{A}_{22}^H \overline{r_2(k)}\bar{B}_{22}^H$$

$$+ R\overline{A_{11}^H r_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{r_1(k)}\bar{B}_{12}^H}S$$

$$+ R\overline{A_{21}^H r_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{r_2(k)}\bar{B}_{22}^H}S],$$

$W(k + 1)$

$$= W(k) + \frac{\mu}{2}[C_{11}^H r_1(k)D_{11}^H + \bar{C}_{12}^H \overline{r_1(k)}\bar{D}_{12}^H$$

$$+ C_{21}^H r_2(k)D_{21}^H + \bar{C}_{22}^H \overline{r_2(k)}\bar{D}_{22}^H$$

$$+ R\overline{C_{11}^H r_1(k)D_{11}^H}S + R\overline{\bar{C}_{12}^H \overline{r_1(k)}\bar{D}_{12}^H}S$$

$$+ R\overline{C_{21}^H r_2(k)D_{21}^H}S + R\overline{\bar{C}_{22}^H \overline{r_2(k)}\bar{D}_{22}^H}S].$$

where

$$r_1(k) = E_1 - g_1(V(k), W(k)),$$
$$r_2(k) = E_2 - g_2(V(k), W(k))$$

4. *If* $r_1(k + 1) = 0$, $r_2(k + 1) = 0$, *then stop and* V_k, W_k *are the solution; otherwise, put* $k = k + 1$ *and go to STEP 3.*

$\blacksquare$

3.2. Convergence analysis

In this subsection, we present convergence properties of the suggested algorithm I.

Theorem 3.2.1: *If the coupled Sylvester matrix equation* (1) *has a unique* (R, S) *– conjugate solutions pair* $[V^*, W^*]$, *then the iterative solution pair* $[V(k), W(k)]$ *given by algorithm I, converges to* $[V^*, W^*]$ *for any initial* (R, S) *– conjugate matrices pair* $[V(1), W(1)]$ *if the parameter* μ *satisfies the inequality*

$$0 < \mu < \frac{2}{H} \qquad (3)$$

with

$$H = \| A_{11} \|^2 \| B_{11} \|^2 + \| A_{12} \|^2 \| B_{12} \|^2$$

$$+ \| A_{21} \|^2 \| B_{21} \|^2 + \| A_{22} \|^2 \| B_{22} \|^2$$

$$+ \| C_{11} \|^2 \| D_{11} \|^2 + \| C_{12} \|^2 \| D_{12} \|^2$$

$$+ \| C_{21} \|^2 \| D_{21} \|^2 + \| C_{22} \|^2 \| D_{22} \|^2$$

$\blacksquare$

Proof: We first construct the estimate error matrices as

$$\xi_1(k) = V(k) - V^* \text{ and } \xi_2(k) = W(k) - W^*$$

$$\text{for } k = 1, 2, \cdots.$$

$\blacksquare$

Since $V(k), W(k), V^*, W^* \in \mathbb{RSC}^{n \times n}$, we have

$$R\,\xi_1(k)\,S = R\,V(k)\,S - RV^*S = \overline{V(k)} - \overline{V^*}$$

$$= \overline{V(k) - V^*} = \overline{\xi_1(k)},$$

$$R\xi_2(k)S = R\,W(k)S - RW^*S = \overline{W(k)} - \overline{W^*}$$

$$= \overline{W(k) - W^*} = \overline{\xi_2(k)}$$

These demonstrate that $\xi_1(k), \xi_2(k) \in \mathbb{RSC}^{n \times n}$. Denote

$$Z_1(k) = A_{11}\xi_1(k)B_{11} + C_{11}\xi_2(k)D_{11} + A_{12}\overline{\xi_1(k)}B_{12}$$

$$+ C_{12}\overline{\xi_2(k)}D_{12} \qquad (4)$$

$$Z_2(k) = A_{21}\xi_1(k)B_{21} + C_{21}\xi_2(k)D_{21} + A_{22}\overline{\xi_1(k)}B_{22}$$
$$+ C_{22}\overline{\xi_2(k)}D_{22} \qquad (5)$$

Utilizing the aforementioned error matrices and algorithm I, we can obtain

$$\xi_1(k+1) = \xi_1(k) - \frac{\mu}{2}[A_{11}^H Z_1(k)B_{11}^H + \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H$$
$$+ A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H$$
$$+ R\overline{A_{11}^H Z_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S$$
$$+ R\overline{A_{21}^H Z_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S]$$
$$(6)$$

$$\xi_2(k+1) = \xi_2(k) - \frac{\mu}{2}[C_{11}^H Z_1(k)D_{11}^H + \bar{C}_{12}^H \overline{Z_1(k)}\bar{D}_{12}^H$$
$$+ C_{21}^H Z_2(k)D_{21}^H + \bar{C}_{22}^H \overline{Z_2(k)}\bar{D}_{22}^H$$
$$+ R\overline{C_{11}^H Z_1(k)D_{11}^H}S + R\overline{\bar{C}_{12}^H \overline{Z_1(k)}\bar{D}_{12}^H}S$$
$$+ R\overline{C_{21}^H Z_2(k)D_{21}^H}S + R\overline{\bar{C}_{22}^H \overline{Z_2(k)}\bar{D}_{22}^H}S]$$
$$(7)$$

Taking the norm of both sides of (6) and (7) and applying the following facts for two square complex matrices $tr(AB) = tr(BA)$, $\| A + B \| \leq \| A + B \|$, $\| RAS \| = \| A \| = \| \bar{A} \|$, we obtain

$$\|\xi_1(k+1)\|^2 = \mathrm{Re}(tr(\xi_1^H(k+1)\xi_1(k+1)))$$
$$= \mathrm{Re}(tr(\xi_1^H(k)\xi_1(k))) - \mu\mathrm{Re}(tr([A_{11}^H Z_1(k)B_{11}^H$$
$$+ \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H + A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H$$
$$+ R\overline{A_{11}^H Z_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S$$
$$+ R\overline{A_{21}^H Z_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S]^H \xi_1(k)))$$
$$+ \frac{\mu^2}{4}\|A_{11}^H Z_1(k)B_{11}^H + \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H$$
$$+ A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H$$
$$+ R\overline{A_{11}^H Z_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S$$
$$+ R\overline{A_{21}^H Z_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S\|^2 \qquad (8)$$

Using the characteristics of a matrix's trace, one has

$$\mathrm{Re}(tr([A_{11}^H Z_1(k)B_{11}^H + \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H + A_{21}^H Z_2(k)B_{21}^H$$
$$+ \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H + R\overline{A_{11}^H Z_1(k)B_{11}^H}S$$
$$+ R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S + R\overline{A_{21}^H Z_2(k)B_{21}^H}S$$
$$+ R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S]^H \xi_1(k))),$$
$$= \mathrm{Re}(tr(B_{11}Z_1^H(k)A_{11}\xi_1(k) + \bar{B}_{12}\overline{Z_1(k)}^H \bar{A}_{12}\xi_1(k)$$
$$+ B_{21}Z_2^H(k)A_{21}\xi_1(k) + \bar{B}_{22}\overline{Z_2(k)}^H \bar{A}_{22}\xi_1(k)$$
$$+ S\overline{B_{11}Z_1^H(k)A_{11}}R\xi_1(k) + S\overline{\bar{B}_{12}\overline{Z_1(k)}^H \bar{A}_{12}}R\xi_1(k)$$
$$+ S\overline{B_{21}Z_2^H(k)A_{21}}R\xi_1(k)$$
$$+ S\overline{\bar{B}_{22}\overline{Z_2(k)}^H \bar{A}_{22}}R\xi_1(k))),$$
$$= \mathrm{Re}(tr(Z_1^H(k)A_{11}\xi_1(k)B_{11} + \overline{Z_1(k)}^H \bar{A}_{12}\xi_1(k)\bar{B}_{12}$$
$$+ Z_2^H(k)A_{21}\xi_1(k)B_{21} + \overline{Z_2(k)}^H \bar{A}_{22}\xi_1(k)\bar{B}_{22}$$
$$+ \overline{Z_1^H(k)}\overline{A_{11}}R\xi_1(k)S\overline{B_{11}} + Z_1^H(k)A_{12}R\xi_1(k)SB_{12}$$
$$+ \overline{Z_2^H(k)}\overline{A_{21}}R\xi_1(k)S\overline{B_{21}}$$
$$+ Z_2^H(k)A_{22}R\xi_1(k)SB_{22})),$$
$$= 2\mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12})$$
$$+ Z_2^H(k)(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22})))$$

Substituting from the preceding relation into (8), gives

$$\|\xi_1(k+1)\|^2$$
$$= \|\xi_1(k)\|^2 - 2\mu\mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11}$$
$$+ A_{12}\overline{\xi_1(k)}B_{12}) + Z_2^H(k)(A_{21}\xi_1(k)B_{21}$$
$$+ A_{22}\overline{\xi_1(k)}B_{22}))) + \frac{\mu^2}{4}\|A_{11}^H Z_1(k)B_{11}^H$$
$$+ \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H + A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H$$
$$+ R\overline{A_{11}^H Z_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S$$
$$+ R\overline{A_{21}^H Z_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S\|^2,$$
$$\leq \|\xi_1(k)\|^2$$
$$- 2\mu\mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12})$$
$$+ Z_2^H(k)(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22})))$$
$$+ \frac{\mu^2}{4}(\|A_{11}^H Z_1(k)B_{11}^H + \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H$$
$$+ A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H\|$$
$$+ \| R\overline{A_{11}^H Z_1(k)B_{11}^H}S + R\overline{\bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H}S$$
$$+ R\overline{A_{21}^H Z_2(k)B_{21}^H}S + R\overline{\bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H}S\|)^2,$$
$$\leq \|\xi_1(k)\|^2$$
$$- 2\mu\mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12})$$
$$+ Z_2^H(k)(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22})))$$
$$+ \mu^2\|A_{11}^H Z_1(k)B_{11}^H + \bar{A}_{12}^H \overline{Z_1(k)}\bar{B}_{12}^H$$
$$+ A_{21}^H Z_2(k)B_{21}^H + \bar{A}_{22}^H \overline{Z_2(k)}\bar{B}_{22}^H\|,$$

$$\|\xi_1(k+1)\|^2 \leq \|\xi_1(k)\|^2$$
$$- 2\mu\mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12})$$
$$+ Z_2^H(k)(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22})))$$
$$+ \mu^2(\|A_{11}\|^2 \|B_{11}\|^2 + \|A_{12}\|^2 \|B_{12}\|^2$$
$$+ \|A_{21}\|^2 \|B_{21}\|^2 + \|A_{22}\|^2 \|B_{22}\|^2)$$

$$\times (||Z_1(k)||^2 + ||Z_2(k)||^2). \qquad (9)$$

Similarly to the preceding, we also write

$$||\xi_2(k+1)||^2 \leq ||\xi_2(k)||^2$$
$$- 2\mu \mathrm{Re}(tr(Z_1^H(k)(C_{11}\xi_2(k)D_{11} + C_{12}\overline{\xi_2(k)}D_{12})$$
$$+ Z_2^H(k)(C_{21}\xi_2(k)D_{21} + C_{22}\overline{\xi_2(k)}D_{22})))$$
$$+ \mu^2(||C_{11}||^2||D_{11}||^2 + ||C_{12}||^2||D_{12}||^2$$
$$+ ||C_{21}||^2||D_{21}||^2 + ||C_{22}||^2||D_{22}||^2)$$
$$\times (||Z_1(k)||^2 + ||Z_2(k)||^2). \qquad (10)$$

From (9) and (10)

$$||\xi_1(k+1)||^2 + ||\xi_2(k+1)||^2 \leq ||\xi_1(k)||^2 + ||\xi_2(k)||^2$$
$$- 2\mu \mathrm{Re}(tr(Z_1^H(k)(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12}$$
$$+ C_{11}\xi_2(k)D_{11} + C_{12}\overline{\xi_2(k)}D_{12})$$
$$+ Z_2^H(k)(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22}$$
$$+ C_{21}\xi_2(k)D_{21} + C_{22}\overline{\xi_2(k)}D_{22})))$$
$$+ \mu^2(||A_{11}||^2||B_{11}||^2 + ||A_{12}||^2||B_{12}||^2$$
$$+ ||A_{21}||^2||B_{21}||^2 + ||A_{22}||^2||B_{22}||^2$$
$$+ ||C_{11}||^2||D_{11}||^2 + ||C_{12}||^2||D_{12}||^2$$
$$+ ||C_{21}||^2||D_{21}||^2 + ||C_{22}||^2||D_{22}||^2)$$
$$\times (||Z_1(k)||^2 + ||Z_2(k)||^2).$$

Defining the nonnegative definite function $\eta(k)$ as follows:

$$\eta(k) = ||\xi_1(k)||^2 + ||\xi_2(k)||^2$$

From the previous results, this function may be calculated as follows:

$$\eta(k+1) = ||\xi_1(k+1)||^2 + ||\xi_2(k+1)||^2$$
$$\leq \eta(k) - 2\mu \mathrm{Re}(tr(Z_1^H(k)Z_1(k) + Z_2^H(k)Z_2(k)))$$
$$+ \mu^2(H)(Z_1(k)^2 + Z_2(k)^2) \qquad (11)$$

where

$$H = ||A_{11}||^2||B_{11}||^2 + ||A_{12}||^2||B_{12}||^2$$
$$+ ||A_{21}||^2||B_{21}||^2 + ||A_{22}||^2||B_{22}||^2$$
$$+ ||C_{11}||^2||D_{11}||^2 + ||C_{12}||^2||D_{12}||^2$$
$$+ ||C_{21}||^2||D_{21}||^2 + ||C_{22}||^2||D_{22}||^2$$

$$\eta(k+1) \leq \eta(k) - 2\mu(||Z_1(k)||^2 + ||Z_2(k)||^2)$$
$$+ \mu^2(H)(||Z_1(k)||^2 + ||Z_2(k)||^2),$$

$$\eta(k+1) \leq \eta(k) - 2\mu\left(1 - \frac{\mu}{2}H\right)(||Z_1(k)||^2 + ||Z_2(k)||^2),$$

$$\eta(k+1) \leq \eta(1) - 2\mu\left(1 - \frac{\mu}{2}H\right)$$

$$\times \left(\sum_{m=1}^{k}||Z_1(m)||^2 + \sum_{m=1}^{k}||Z_2(m)||^2\right).$$

If the convergence factor μ is chosen to satisfy (3), then one has

$$\sum_{m=1}^{\infty}||Z_1(m)||^2 + \sum_{m=1}^{\infty}||Z_2(m)||^2 < \infty.$$

Since the matrix equation (1) has a unique solution pair. It follows from the definition (4) and (5) of $Z_i(k)$ that

$$\lim_{i\to\infty}\xi_1(i) = 0 \quad \text{and} \quad \lim_{i\to\infty}\xi_2(i) = 0.$$

Or

$$\lim_{i\to\infty}V(i) = V^* \quad \text{and} \quad \lim_{i\to\infty}W(i) = W^*.$$

This completes the proof of the theorem.

4. Numerical examples

Two numerical examples are given in this section to test the effectiveness of the algorithms I.

Example 4.1: Consider the coupled Sylvester matrix equations with conjugate of two unknowns given by (1) with the following matrices:

$$A_{11} = \begin{bmatrix} -1+i & i & -3i \\ 1-2i & i & 0 \\ 2i & 1+i & 4 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 1-2i & i & 1+3i \\ 1+i & 3i & 2+2i \\ i & 1+2i & 2i \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -3+2i & 0 & 2+3i \\ 1 & 2i & 1+2i \\ 3-i & 1 & -i \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & -1+2i & 1+2i \\ i & 3i & 2i \\ 4 & -3i & 0 \end{bmatrix},$$

$$C_{11} = \begin{bmatrix} i & 3+i & 0 \\ 1+2i & 2-i & 3+i \\ 5 & -1-i & -3i \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 4-i & 1-i & 2i \\ 0.5 & -2+i & 1+2i \\ -2-2i & 3 & -2+i \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0 & 1+i \\ 3i & 2+i \\ 1+2i & 3i \end{bmatrix}, D_{21} = \begin{bmatrix} 2i & 3-i \\ 1+2i & i \\ 2i & 3 \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} 1+i & 2+2i \\ 1-i & i \\ 2-i & 0 \end{bmatrix}, B_{22} = \begin{bmatrix} -1 & 3+i \\ -2+i & 0 \\ 1+3i & 2-i \end{bmatrix},$$
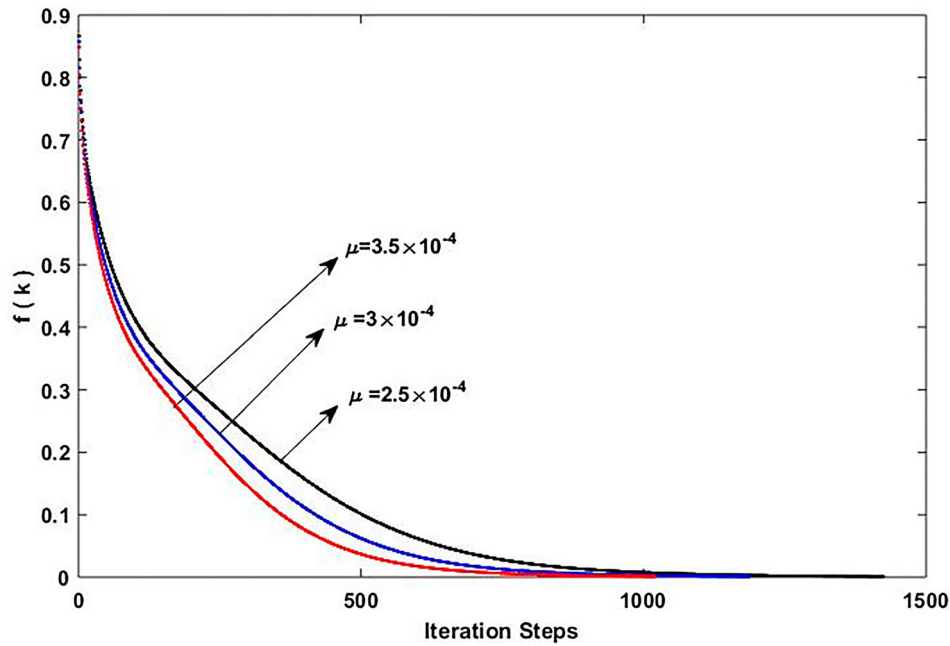
Figure 1. The convergence performance of algorithm I Example 4.1.

$$C_{12} = \begin{bmatrix} i & 2 & 1+i \\ 1+2i & 0 & 2i \\ -i & 2+2i & 1+i \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} i & 2 \\ 1+2i & 0 \\ 0 & 2+2i \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0 & -2+i \\ -1+3i & -1-i \\ 3-i & 1+2i \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 0 & -2+i \\ -1+3i & -1-i \\ 2 & -2+3i \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} -1+i & 3-i & 0 \\ 1+i & 2-i & i \\ 1+i & -2-i & 1+3i \end{bmatrix},$$

$$D_{22} = \begin{bmatrix} i & 1-i \\ -1+3i & 1+i \\ 1+i & -i \end{bmatrix},$$

$$E_1 = \begin{bmatrix} -61-83i & -120+83i \\ -64-36i & -44+95i \\ 54+74i & -124+43i \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 60+21i & 79+13i \\ 111.5-96i & -33.5-89.5i \\ -12+135i & 32-56i \end{bmatrix}.$$

Let R, S as follows

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This coupled Sylvester matrix equation (1) has a unique (R, S) – conjugate solution of the following form

$$V = \begin{bmatrix} -2+2i & 0 & 1+i \\ 4 & 8i & 2i \\ -2-2i & 0 & -1+i \end{bmatrix},$$

$$W = \begin{bmatrix} -3+5i & 1 & 1+2i \\ 4 & 8i & 2i \\ -3-5i & -1 & -1+2i \end{bmatrix}.$$

Choose arbitrary initial (R, S) – conjugate matrices

$$V_1 = \begin{bmatrix} i & 0 & 1+i \\ 1 & 0 & 0 \\ -i & 0 & -1+i \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 1+2i & -1 & 2+i \\ 0 & 0 & 2i \\ 1-2i & 1 & -2+i \end{bmatrix}.$$

As indicated by theorem 3.2.1, the algorithm I is convergent for $0 < \mu < 3.5 \times 10^{-4}$. We can see in Figure 1 that for $\mu = 3.5 \times 10^{-4}, \mu = 3 \times 10^{-4}$ and $\mu = 2.5 \times 10^{-4}$, then the iteration stops at $k = 1021, k = 1189$ and $k = 1425$, respectively.

As k increase, the relative error f decreases and eventually disappears, and algorithm I is efficient. Figure 1 depicts the effect of adjusting the convergence factor μ. We can see that the larger the convergence factor μ, the faster the rate of convergence.

There are two approaches for quantifying approximation errors: the residual error and the relative error. The residual error can be misleading as a measure of precision; however the relative error is more useful because the relative error considers the size of the value.

Table 1. Relative error and the residual error for the convergence factor $\mu = 3.5 \times 10^{-4}$.

Number of iterations n	The relative error	The residual error	The elapsed CPU time
250	0.1924	18.3583	0.510583 sec
500	0.0369	3.1863	0.952949 sec
750	0.0061	0.5268	1.380692 sec
1021	0.0010	0.0988	1.867556 sec

Define the residual error as

$$R(k) = \begin{bmatrix} E_1 - g_1(V(k), W(k)) & 0_{m \times r} \\ 0_{m \times r} & E_2 - g_2(V(k), W(k)) \end{bmatrix}$$

and the relative error as $f(k) = \sqrt{\dfrac{\|V(k) - V\|^2 + \|W(k) - W\|^2}{\|V\|^2 + \|W\|^2}}$.

In Table 1, we compare relative error, residual error, and the elapsed CPU time for the convergence factor $\mu = 3.5 \times 10^{-4}$.

Example 4.2: Consider the coupled Sylvester complex matrix equations with conjugate of two unknowns

$$\left. \begin{aligned} A_{11} V B_{11} + C_{12} \bar{W} D_{12} = E_1 \\ A_{21} V B_{21} + C_{22} \bar{W} D_{22} = E_2 \end{aligned} \right\} \tag{12}$$

With

$$A_{11} = \begin{bmatrix} i & 2 & -1 & 1+i \\ 1 & 0 & 2-i & 2i \\ i & 1 & -i & -1+i \\ -i & -5i & 0 & 2+i \\ 2i & 1-i & -i & -2 \\ -1 & 1 & 1+i & 3 \end{bmatrix},$$

$$C_{12} = \begin{bmatrix} 4+i & -i & -3i & 2+i \\ 1+i & i & 1 & 3-i \\ 0 & 0 & 2-i & 2i \\ 2i & 1+2i & 3-i & 0 \\ 1+i & -1-i & 2i & -i \\ 2+i & 0 & 2 & i \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 2 & i & 0 & 1-2i \\ 0 & 2 & i & 1-i \\ 3+i & 2i & 1 & 2i \\ -i & 1-i & 2-i & 1+i \\ -2i & 2-i & 2i & 3i \\ 1+3i & 0 & 0 & i \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} 1 & 1+3i & 0 & 1+2i \\ -1 & 1-i & i & 2-i \\ 1 & 2i & -1 & i \\ 2-i & 0 & 5 & -3i \\ -3i & i & -i & 0 \\ i & 2 & 1+i & i \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} i & 1+i & i & 2-i & 2i \\ -3i & 1-i & -5i & -i & 1+2i \\ -i & 0 & 2-i & i & 2+3i \\ 1+2i & -i & 0 & 1-i & 4+i \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 5 & 3-i & 3+i & -3i & 2-i \\ 2 & 2i & 2+i & -i & 4+i \\ -1-i & 2+3i & -1 & 3+i & 1+i \\ 0 & 1+2i & 0 & 1+2i & 2i \end{bmatrix},$$

$$D_{22} = \begin{bmatrix} -2-3i & 2i & -i & 1+2i & 1+i \\ -2-i & 2 & -3i & -i & 0 \\ -1-i & -2i & 2i & 3-i & 2i \\ -1+3i & -2+i & -4i & 2+i & -3i \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 3+i & 1+2i & 4i & -2 & 3-3i \\ -i & 3-i & 0 & 2-2i & 2-i \\ 2i & 0 & 5i & 1 & 2i \\ 3+2i & -i & 2-i & 1+3i & 5i \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 3+i & 1+2i & 4i & -2 & 3-3i \\ -i & 3-i & 0 & 2-2i & 2-i \\ 2i & 0 & 5i & 1 & 2i \\ 3+2i & -i & 2-i & 1+3i & 5i \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 233+157i & -4+150i & 147+105i \\ 41-25i & 18+42i & 115-11i \\ 32+119i & -50+28i & -10+61i \\ -168+83i & -61+i & -75-52i \\ -17-81i & -9-i & 45-54i \\ 9+139i & -39+51i & 58+52i \end{bmatrix}$$

$$\begin{bmatrix} -15+112i & 143+169i \\ 19-7i & 22+8i \\ -51+16i & 17+104i \\ -71-14i & -153+142i \\ 22-21i & 5-13i \\ -35+28i & -77+149i \end{bmatrix},$$

$$E_2 = \begin{bmatrix} -42-44i & -54-53i & 102+48i \\ -36+75i & -39+26i & -47+4i \\ -44-87i & -3-98i & 31-6i \\ 36-i & -128+23i & 164-53i \\ -59+2i & -47+28i & -28-123i \\ -4-38i & -40-105i & 74+32i \end{bmatrix}$$

$$\begin{bmatrix} -8-5i & 40+41i \\ 5-35i & -20+27i \\ -21-19i & -26-11i \\ -174+24i & 52+7i \\ -55+37i & -58-31i \\ -6+3i & 48+5i \end{bmatrix}$$

Let R, S as follows

$$R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
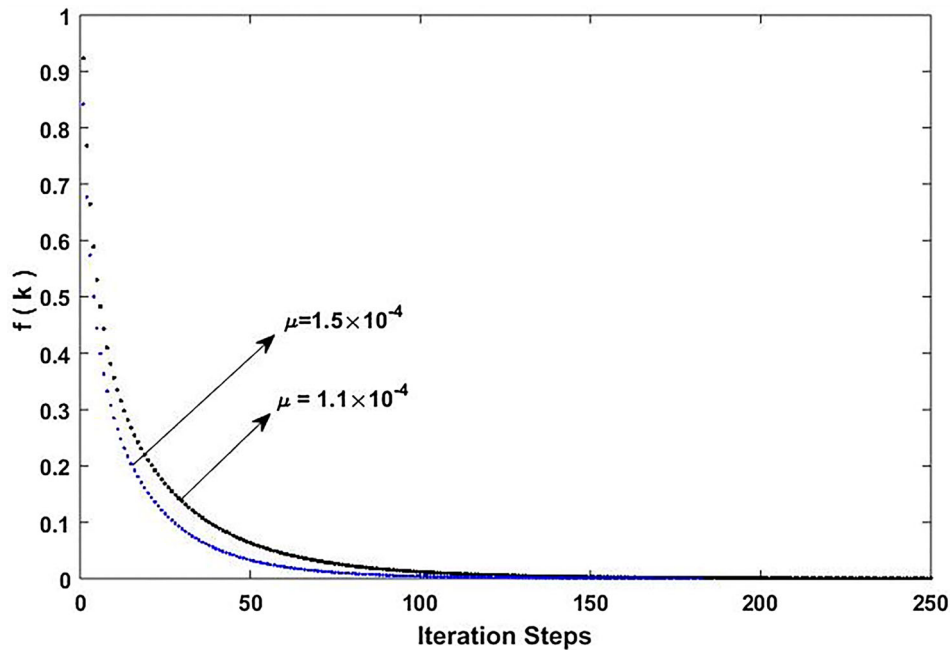
Figure 2. The convergence performance of algorithm I Example 4.2.

This coupled Sylvester matrix equation (12) has a unique (R, S) – conjugate solution of the following form.

$$V = \begin{bmatrix} -1-i & 0 & -2i & -2 \\ -2 & 1+3i & -1+i & 0 \\ 0 & 1+i & -1+3i & 2 \\ -2 & 2i & 0 & -1+i \end{bmatrix},$$

$$W = \begin{bmatrix} 1-3i & -5i & 4i & 2-3i \\ -4i & 1+2i & 1 & 3-5i \\ -3-5i & -1 & -1+2i & -4i \\ 2+3i & -4i & 5i & 1+3i \end{bmatrix}.$$

Choose arbitrary initial (R, S) – conjugate matrices

$$V_1 = \begin{bmatrix} 0 & -2i & -i & 1 \\ i & 1 & 0 & -i \\ -i & 0 & -1 & i \\ 1 & i & 2i & 0 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} i & 1-i & -3i & 2+i \\ 2+2i & 2+i & -1 & -1-2i \\ 1-2i & 1 & -2+i & -2+2i \\ 2-i & 3i & 1+i & -i \end{bmatrix}.$$

As indicated by theorem 3.2.1, the algorithm I is convergent for $0 < \mu < 1.5 \times 10^{-4}$. We can see in Figure 2 that for $\mu = 1.1 \times 10^{-4}$ and $\mu = 2.5 \times 10^{-4}$, then the iteration stops at $k = 184$ and $k = 251$, respectively.

It can be observed from Figure 2. As k increase, the relative error f decreases and eventually disappears, and algorithm I is effective. We can observe that the larger the convergence factor μ, the faster the convergence rate.

5. Conclusions

In this paper, we have constructed an effective algorithm to find (R, S) – conjugate solutions to coupled Sylvester complex matrix equations with conjugate of two unknowns. When these two matrix equations are consistent, for any initial arbitrary (R, S) – conjugate matrices V_1, W_1 the solutions can be obtained by utilizing this iterative algorithm. Sufficient conditions are provided to ensure the proposed algorithm's convergence. We test the proposed algorithm utilizing MATLAB and the results of numerical experiments support our algorithm.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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