DOI: 10.31534/engmod.2023.1.ri.01v
Original scientific paper
Received: 23.02.2022.

A Modified Approach to the Mathematical Model of Crack with Pre-destruction Zones

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SUMMARY

Generalized Griffith's criterion and models with pre-destruction zones are considered in this paper. Unlike those models that used linear dependences, the authors proposed the destruction process to be represented by differential equations. The positive effect of such representation is the possibility to formulate boundary conditions using the corresponding constant in the differential equation solution. The result is that the critical load values responsible for the occurrence and propagation of quasi-brittle cracks in materials are obtained. It is stated that the maximum load of crack propagation completely or essentially depends on its initial length. These generalizations estimate the influence of stress caused by hydrogen close to crack-like defects. In the case of defect-free material, the established formula is used to determine the critical forces necessary for the occurrence of cracks with a definite length. Numerical examples for some types of materials are given to illustrate the theoretical estimates.

KEYWORDS: *cracked body models; potential energy; surface energy; deformation energy; fracture; critical loads.*

1. INTRODUCTION

The present-day fracture mechanics as an independent science on the scientific scene is closely connected with the publication of fundamental works [1] by the famous scientist A. Griffith. Various analytical scenarios of these works are demonstrated in detail in [2]. For the first time, the author proposed the model of an elastic body and assumed the presence of a discontinuity in the form of a through fracture whose edges carry the surface energy. The surface energy here is understood as the formation of new surfaces caused by the origination and propagation of fractures. In the case of the plane elasticity problem [3] within the framework of such a model, the author of these works has calculated the potential energy P of an elastic isotropic body with the crack length of $2l$ under the evenly distributed stretching forces of intensity p perpendicular to the rectilinear contour of such cavity and sufficiently distant from the crack surfaces. The general representation of such energy is as follows:

$$P = P_0 - W(p, l) + U(l), \tag{1}$$

where P_0 is the potential energy of a deformed body without cracks, $W(p, l) = \pi^2 p^2 c l^2$ is the energy of elastic deformations caused by the opening of a $2l$ long crack under the force intensity p, furthermore, $c = (1 - \mu^2)/\pi E$ for the plane deformation, $c = 1/\pi E$ under the plane stress, μ is the Poisson's ratio, E is the material Young's modulus; $U(l) = 4\gamma l$, γ is the density of material effective surface energy. Figure 1 illustrates the scheme of Griffith crack in the coordinate system Oxy.

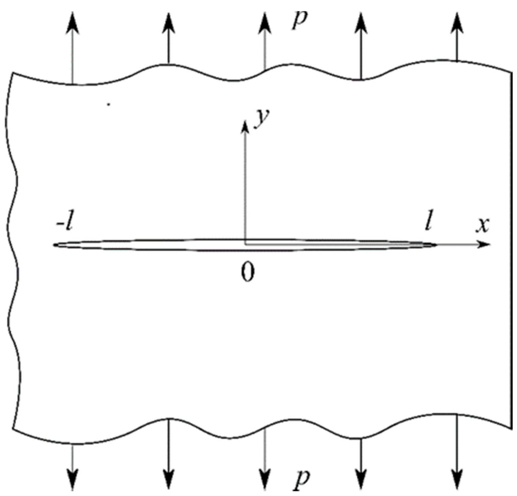

Fig. 1 *Griffith crack scheme*

Differentiating the energy Eq. (1) concerning the variable l and equating the obtained derivative to zero, Griffith postulated the value p_* of a critical load at which the limit state of a body with a $2l$ long crack occurs, and it becomes possible for the crack to start and propagate spontaneously. In paper [1], the following formulae for the critical (destructive) value p_* is established:

$$p_* = \sqrt{\frac{2E\gamma}{\pi(1-\mu^2)l}} \quad and \quad p_* = \sqrt{\frac{2E\gamma}{\pi l}} \tag{2}$$

for the plane deformation and plane stress state, respectively.

However, it is not difficult to note that at the above critical load p_* some arbitrariness and inaccuracy are presumed. Via the differentiation of energy, specified by the Eq. (1), with respect to the variable l, we establish not a critical p_* but a critical value of l_* at the respective p. Concurrently, differentiating Eq. (1) with respect to the parameter p, one can calculate that $\partial P/\partial p = \partial W/\partial p = 2\pi^2 p c l^2$ and thus receive no critical load values. The specified fact suggests an idea of the necessity to refine the algorithm used to define the critical parameter p. Therefore it is logical to assume that the critical value of the intensity p of the external load necessary for cracks origination functionally depends on the initial half-length l of the material defect. Since an explicit dependence of these parameters is still unknown, let us assume that the load p is an implicit function of the crack half-length, i.e.:

$$5p = p(l). \tag{3}$$

The above assumption comes naturally since the destructive load for minor fractures should be greater than for large fractures [4]. Here one can notice the spontaneity of destruction in respect of the load scheme discussed above. At the same time, in contrast to Griffith's theory, this assumption leads to a differential equation, the solution of which generalizes relations obtained through the application of this theory.

Differential equations (ODEs and PDEs as well) are a modern instrument in the mathematical modelling of propagation of the different types of material fractures. For example, ordinary differential equations describing the transfer intensity of tangent shearing stress and normal stress on the crack edge are considered by Verveiko et al. [5]. Due to the obtained exact solutions in the case of unstressed material classical criteria of brittle destruction were specified. Sendova et al. studied the analysis of differential equations arising under a new approach to model brittle destruction based on mechanics of continuous media being extended to nano-dimensions [6]. There are analysed transient characteristics of linear elastic solid body weakened by a nano-dimensional crack on condition that solid body is under impact loading [7]. The problem is formulated as a non-classical mixed initial-boundary problem. The corresponding boundary problem is reduced to the singular differential equation that can be numerically solved. Lalegname et al. studied crack propagation in the boundary linear elastic body under shearing waves influence for the simplified 2D-model [8]. This model is described by two associated differential equations, namely, a two-dimensional scalar wave equation in the boundary domain and an ordinary differential equation obtained from the energy balance law. There is an established improved formula for the ordinary differential equation of crack tip movement with stress- intensity factor. Yankelevsky et al. obtained by modelling crack propagation via an ordinary second-order differential equation [9]. There are estimated parameters of crack width which are the main effectiveness criterion in ferroconcrete constructions and also are essential for bridge safety exploitation. A differential equation model is used to select directly experiment data of a-N fatigue crack propagation [10]. Zheng proposed the method of estimation for the parameters of Weibull distribution via differential modelling [11].

It makes sense to emphasize additionally the application of differential equations modelling to the influence of hydrogen-induced stress estimation. In particular, Ivanyts'kyi et al. established the exact analytical solution in the closed form of the problem of hydrogen diffusion in the process zone near the crack tip [12]. A new criterion for hydrogen-induced cracking that includes both the embrittlement effect and the loading effect of hydrogen was obtained theoretically [13]. Stashchuk and Dorosh considered the generalized mathematical model to estimate the influence of hydrogen-induced stress generated close to crack-like defects [14].

The paper aims to generalize Griffith criterion and the δ_k criteria [15] to model the material destruction. The generalization is made by formulating and solving the corresponding differential equations for the specified models of deformable solid bodies. As a result, values of critical loads responsible for the origination and quasi-brittle crack propagation in materials are found.

2. DIFFERENTIAL EQUATIONS OF CRACKED BODY MODEL

Theorem 1. The critical load for the start of a micro-crack of length in the framework of Griffith's theory to length $2l$ modelling by a differential equation is determined by the formula:

$$p(l) = \sqrt{\frac{4\gamma}{\pi^2 cl}\left(1 - \frac{l_0}{l}\right) + p_*^2 \frac{l_0^2}{l^2}}.$$

Proof. In view of Eq. (3), Eq. (1) is rewritten in the following form:

$$P = P_0 - \pi^2 p^2(l) cl^2 + 4\gamma l. \tag{4}$$

Let us find a derivative:

$$\frac{dP}{dl} = 4\gamma - 2\pi^2 clp(l)[lp'(l) + p(l)] \tag{5}$$

and use a necessary extreme condition:

$$\frac{dP}{dl} = 0. \tag{6}$$

Here from one can get a differential equation to define the critical value p, which is necessary to shift a crack with a half-length l:

$$p\frac{dp}{dl} + \frac{p^2}{l} = \frac{2\gamma}{\pi^2 cl^2}. \tag{7}$$

Let us call the last equation as a differential equation within Griffith's model for the description of the limit state of an elastic body with a sharp-pointed crack. The general solution of the above equation is:

$$p^2(l) = \left(\frac{4\gamma l}{\pi^2 c} + C_*\right) l^{-2}. \tag{8}$$

Hence, for $2l$ long crack the corresponding intensive force is:

$$p(l) = \sqrt{\frac{4\gamma}{\pi^2 cl} + \frac{C_*}{l^2}}. \tag{9}$$

Equation (9) shows that the load can be defined up to a constant C_*, thus it is a non-unique one. It is logical to assume that the critical value p for a crack shift depends on its initial length. Therefore, modelling of crack initiation in materials via differential equations can set up a condition that for each initial value l_0 the specified value p_* is relevant, i.e.:

$$p(l_0) = p_*, \tag{10}$$

where p_* defines the value of p when the limit state of a body having the crack of a half-length l_0 occurs. Due to condition defined by Eq. (10):

$$C_* = p_*^2 l_0^2 - \frac{4\gamma}{\pi^2 c} l_0. \tag{11}$$

Thus Eq. (9) changes as:

$$p(l) = \sqrt{\frac{4\gamma}{\pi^2 cl}\left(1 - \frac{l_0}{l}\right) + p_*^2 \frac{l_0^2}{l^2}}. \tag{12}$$

Theorem 1 is proved.

As one can see, the latter formula contains the initial half-length l_0 of a crack in a solid body. Griffith's Eq. (2) doesn't comprise this parameter, though the author projected in his reasoning the presence of fractures before the destruction. Let's also note that from Eq. (12) with zero initial half-length ($l_0 = 0$), one can get the relation analogous to Eq. (2). In this case, the numerical values increase $\sqrt{2}$-fold. Consequently, the question about the physical meaning of the obtained relation arises. That means: if $l_0 = 0$ (no initial crack is present), Eq. (12) in this specific case allows defining the critical value p_*, necessary for the origination of a fracture of half-length l in a continuous (with no defects) solid body.

Consequence 1. The critical nucleation stress of a half-length crack l is:

$$p(l) = 2\sqrt{\frac{\gamma}{\pi^2 c l}}.$$

3. DIFFERENTIAL EQUATION OF A CRACKED BODY MODEL WITH PRE-FRACTURE ZONES

In this section, a more effective generalized differential model of the crack theory would be considered.

Theorem 2. The critical load for the start of a microcrack of length $2l_0$ in the framework of the theory of cracks with zones of pre-destruction to length $2l_0$ modelling by a differential equation is determined by the formula:

$$p(l) = \frac{2\sigma_0}{\pi} cos^{-1}\left[exp\left(\frac{\gamma l_0}{2\sigma_0^2 c l}\left(\frac{1}{l} - \frac{1}{l_0}\right)\right)\right].$$

Proof. Firstly, let us write the potential energy for a cracked body model, which stipulates the presence of pre-fracture zones in material defect apices, i.e., for the so-called crack model with pre-destruction zones. Mathematical calculations regarding the crack model with pre-destruction zones are based on the problem of elasticity theory considering a crack with weakened interatomic bounds in its apices. It is supposed that an infinite plate has a $2l_0$ long crack and is loaded in the infinite remote points with the intensive force p directed perpendicularly to the crack edges. Under these forces and within the limit state at the finite segments close to the crack edges the constant stress σ_0, caused by the interatomic (intermolecular) resistance to material strength, is allowed. Coordinate system Oxy is associated with an elastic plate and a crack so that axis Ox is aligned with the crack surface (see Figure 2).

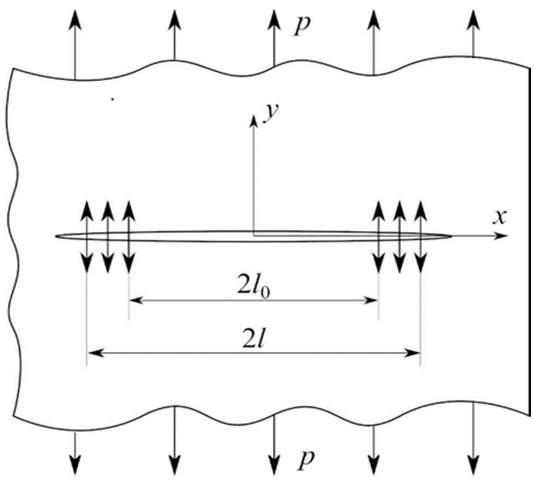

Fig. 2 *Pre-destruction zones*

Then the boundary conditions of a respective elasticity theory problem used to establish stress tensor components σ_y, τ_{xy} are the following [15]:

$$\tau_{xy}(x,0) = 0,\ if\ -\infty < \boldsymbol{x} < +\infty, \quad \sigma_y(x,0) = \begin{cases} 0, & if\ |x| < l_0, \\ \sigma_0, & if\quad l_0 < |x| < l, \end{cases} \tag{13}$$

and it is supposed that at infinity points of the stress plane:

$$\sigma_y(x,\infty) = p. \tag{14}$$

Based on calculations [15], the stress is:

$$\sigma_y(x,0) = \frac{1}{\pi\sqrt{x^2-l^2}}\Big\{\pi(p-\sigma_0)\left(x-\sqrt{x^2-l^2}+2\sigma_0 x\sin^{-1}\frac{l_0}{l}+\sigma_0\sqrt{x^2-l^2}\right)\cdot \left[\sin^{-1}\frac{l^2-xl_0}{l(x-l_0)}-\sin^{-1}\frac{l^2+xl_0}{l(x+l_0)}\right]\Big\}, x \geq l. \tag{15}$$

Notice, according to the δ_k stress model, crack apices (i.e., when $x \to 0$) should be limited. The stress limit is provided when the multiplier equals zero in a singular part $1/\sqrt{x^2-l^2}$ of Eq. (3) at points $x = \pm l$, i.e., when:

$$p = \frac{2}{\pi}\sigma_0\cos^{-1}\frac{l_0}{l} \tag{16}$$

or

$$l = l_0\sec\frac{\pi p}{2\sigma_0}. \tag{17}$$

Then, the crack edges shift is:

$$v(x,0) = c\sigma_0\{(x-l_0)\Gamma(l,x,l_0)-(x+l_0)\Gamma(l,x,-l_0)\}, |x| \leq l, \tag{18}$$

where:

$$\Gamma(l,x,\xi) = \ln\frac{l^2-x\xi-\sqrt{(l^2-x^2)(l^2-\xi^2)}}{l^2-x\xi+\sqrt{(l^2-x^2)(l^2-\xi^2)}}. \tag{19}$$

By Eqs. (15) and (18) the energy W of elastic deformations caused by crack opening originated due to the intensity stress p perpendicular to its plane is calculated. Using the results presented in [2], [15] it can be shown that:

$$W = -\int_{-l}^{l}\sigma_y(x,0)v(x,0)dx = p\int_{-l}^{l}v(x)dx - \sigma_0\left(\int_{-l}^{-l_0}v(x)dx+\int_{l_0}^{l}v(x)dx\right). \tag{20}$$

Calculating integrals on the right-hand side of Eq. (20) following is obtained:

$$W = 2\pi c l_0\sigma_0\sqrt{l^2-l_0^2}\left(p-\frac{2}{\pi}\sigma_0\cos^{-1}\frac{l_0}{l}\right)-2\sigma_0^2 l^2 c[\Gamma(l,-l_0,l_0)+\Gamma(l,l_0,-l_0)], \tag{21}$$

taking into account, Eqs. (16) and (17):

$$W = -8\sigma_0^2 l_0^2 c\ln\frac{l_0}{l}. \tag{22}$$

Using Eq. (17), one can obtain the following formula:

$$W = -8\sigma_o^2 l_0^2 c\ln\left(\cos\frac{\pi p}{2\sigma_0}\right). \tag{23}$$

As Griffith's theory suggests, the potential energy P of the body with a crack, modelled by weakened pre-fracture zones at its ends, can be received. Substitution of Eqs. (21) and (23) into Eq. (1) gives the final analytical formula of the potential energy of the body weakened by a real crack taking into account the pre-fracture zones in its apices:

$$P = P_0 + 4\gamma l_0 + 8\sigma_0^2 l_0^2 c\, ln\left(cos\frac{\pi p}{2\sigma_0}\right). \tag{24}$$

The following considerations support the validity of the Eq. (23). If the crack growth conditions proposed by Griffith are applied:

$$\left.\frac{\partial P}{\partial l_0}\right|_{p=p_*} = 0, \quad \frac{\partial^2 P}{\partial l_0^2} < 0, \tag{25}$$

then after differentiation of the Eq. (24) and its substitution into the first of Eqs. (25), we obtain the following relation:

$$\gamma = -4\sigma_0^2 l_0 c\, ln\left(cos\frac{\pi p_*}{2\sigma_0}\right). \tag{26}$$

Hence:

$$p_* = \frac{2}{\pi}\sigma_0\, cos^{-1}\left[exp\left(-\frac{\gamma}{4\sigma_0^2 l_0 c}\right)\right]. \tag{27}$$

In Eqs. (24) and (25) p_* is the intensity of critical load. The second condition in Eq. (25) holds automatically. If in the Eq. (27) $\gamma = \delta_k\sigma_0/2$ [15], where δ_k is a critical distance between the non-interacting crack edges, we automatically obtain the known formula:

$$p_* = \frac{2}{\pi}\sigma_0\, cos^{-1}\left[exp\left(-\frac{\delta_k}{8 l_0 \sigma_0 c}\right)\right]. \tag{28}$$

Equation (28) confirms the reliability of writing potential energy in the form of Eq. (24). However, applying the condition of Griffith Eq. (6) and taking into account dependence Eq. (3), the differential equation is obtained:

$$2\sigma_0^2 c l^2 \frac{dz}{dl} + 4\sigma_0^2 l c z + \gamma = 0, \tag{29}$$

where $z = ln\left(cos\frac{\pi p}{2\sigma_0}\right) = 0$. The Eq. (29) corresponds to δ_k-model of the body with a l long crack and weak zones on its stretch up to $l > l_0$. The general solution of this differential equation is:

$$p(l) = \frac{2\sigma_0}{\pi} cos^{-1}\left[exp\left(\frac{C_*}{l^2} - \frac{\gamma}{2\sigma_0^2 c l}\right)\right]. \tag{30}$$

To define C_*, let's impose a condition: the rupture load at the end of an initial crack of half-length l_0 equals the material theoretical strength, i.e.:

$$p(l_0) = \sigma_0. \tag{31}$$

This condition is satisfied if:

$$\frac{C_*}{{l_0}^2} - \frac{\gamma}{2\sigma_0^2 c l_0} = 0. \tag{32}$$

Hence:

$$C_* = \frac{\gamma l_0}{2\sigma_0^2 c}. \tag{33}$$

So the limited effort necessary for a crack to grow is:

$$p(l) = \frac{2\sigma_0}{\pi} cos^{-1}\left[exp\left(\frac{\gamma l_0}{2\sigma_0^2 c l}\left(\frac{1}{l} - \frac{1}{l_0}\right)\right)\right], \tag{34}$$

where l_0 is the initial half-length of a crack in the material. Equation (34) differs from the analytic Eq. (27) by the presence of a half-length l_0 of the initial section. Theorem 2 is proved.

Consequence 2. Assuming that the initial crack half-length $l_0 = 0$ (defect-free body), the value:

$$p(l) = \frac{2\sigma_0}{\pi} cos^{-1}\left[exp\left(-\frac{\gamma}{2\sigma_0^2 cl}\right)\right] \quad \textbf{(35)}$$

can be considered as corresponding to the origination of a $2l$ long fracture in a solid deformed body.

It should be noted that if the Eq. (34) is expanded into a series, then, keeping the terms of the first-order smallness, Eq. (12) is obtained. The critical value of the half-length l_0 has already been calculated, furthermore, initial data were taken as deformable body output parameters [16]. We obtain similar results with small deviations considering the origin of a fracture on the planar rigid inclusion continuum.

4. APPLICATION OF THE OBTAINED RESULTS. DETERMINATION OF THE CRITICAL LOAD OF THE TRANSITION FROM MICRO- TO MACRO-CRACKS

To determine the critical load of crack formation, it is necessary to know the critical length of the microcrack during the transition to the macrocrack. Using the results presented in [2, 4] it is possible to establish the minimum size of a microcrack at its transition to a macrocrack. It was considered that a microcrack becomes a macrocrack if the energy of deformation of the body with the crack changes its shape, i.e. suffers a catastrophe [4]. It is established that the critical half-length of such a microcrack is equal to:

$$l_* = \frac{\gamma}{8(2-\sqrt{3})c\sigma_0^2}. \quad \textbf{(36)}$$

Due to Consequence 1 and Griffith's theory, the critical load required to generate a microcrack of this length is:

$$p_{*1} = \frac{4(\sqrt{3}-1)}{\pi}\sigma_0. \quad \textbf{(37)}$$

The critical load in the framework of the theory of cracks with pre-fracture zones is as follows:

$$p_{*2} = \frac{2\sigma_0}{\pi} cos^{-1}\left[exp\left(\frac{4(\sqrt{3}-2)E}{\sigma_0}\right)\right]. \quad \textbf{(38)}$$

The numerical values of the half-length of the crack, as well as the critical load found within these theories (Eqs. (37) and (38)), are presented in Table 1.

Table 1 *Crack half-length values and critical loads for some types of materials*

Material	μ	$E, \frac{N}{m^2}$	$\gamma, \frac{J}{m^2}$	l_*, m	$p_{*1}, \frac{N}{m^2}$	$p_{*2}, \frac{N}{m^2}$
Plexiglass	0.2	$2.45 \cdot 10^9$	$1.5 \cdot 10^2$	$0.934 \cdot 10^{-3}$	$2.28 \cdot 10^7$	$2.45 \cdot 10^7$
Silicate glass	0.24	$6.7 \cdot 10^{10}$	$2.1 \cdot 10^2$	$0.048 \cdot 10^{-3}$	$62.45 \cdot 10^7$	$67 \cdot 10^7$
Carbon steel U8	0.28	$2.06 \cdot 10^{11}$	$7.5 \cdot 10^2$	$0.057 \cdot 10^{-3}$	$192 \cdot 10^7$	$206 \cdot 10^7$

For given values l_*, p_{*1}, p_{*2} calculated values of surface energy γ are stated in the paper [15], where $\sigma_0 = 0{,}01E$ are intermolecular forces. Mechanical characteristics of materials, namely Poisson's ratio μ and Young's modulus E, are taken from the handbook [17]. The difference in critical loads according to the considered theories is the most essential for steel.

5. CONCLUSIONS

By writing the corresponding differential equations, the criteria of Griffith were supplemented in relation to material destruction. In the formula corresponding to this theory, the initial fracture length was introduced, which Griffith did not use in his theoretical analysis.

It was pointed out that the inaccuracy, seen as incorrect from the mathematical analysis point of view, was allowed in Griffith analytical studies to determine the critical load. Determining the value of the length of a crack suspected of a potential energy extreme should be critical.

It was established that the maximum load of the crack propagation in the material depends entirely or significantly on the initial crack length.

The working formula for critical loads required for the movement of the crack of a specific length, i.e., the weakening of the material, is obtained.

It was shown that Griffith criterion is more significant for the formation of a $2l$ long fracture than it is responsible for the growth conditioned by its size.

It was proved that the well-known crack model with pre-destruction zones, which generalizes Griffith criterion, is related to energy criteria.

The obtained analytical relations are used to determine critical values of acting mechanical stress necessary for the destruction of the deformed material with a predetermined fixed crack within the known crack model with pre-destruction zones, which becomes its generalization. In this case, for a defect-free material, a working formula was established for determining the critical forces required for the occurrence of a fracture of a certain length.

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