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Parabola-Inscribed Poncelet Polygons Derived from the Bicentric Family

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ABSTRACT

We study loci and properties of a Parabola-inscribed family of Poncelet polygons whose caustic is a focus-centered circle. This family is the polar image of a special case of the bicentric family with respect to its circumcircle. We describe closure conditions, curious loci, and new conserved quantities.

Key words: Poncelet, closure, porism, parabola, bicentric, conservation, invariants

MSC2020: 37M05, 00A72, 51N20, 37-04

Ponceletovi poligoni upisani paraboli i dobiveni iz bicentričkih familija

SAŽETAK

Proučavamo geometrijska mjesta i svojstva familija Ponceletovih poligona upisanih paraboli koji omataju kružnicu sa središtem u fokusu parabole. Ova familija je polarna slika specijalnog slučaja bicentrične familije s obzirom na svoju opisanu kružnicu. Opisujemo uvjete zatvaranja, geometrijska mjesta, i nove invarijante.

Ključne riječi: Poncelet, zatvaranje, porizam, parabola, bicentričan, očuvanost, invarijante

1 Introduction

This is a continuation of our investigation of Euclidean phenomena of Poncelet families [11, 12, 14, 20]. Recall Poncelet's porism: specially-chosen pairs of conics $\mathcal{C},\mathcal{C}'$ admit a one-parameter family of polygons inscribed in $\mathcal{C}$ while simultaneously circumscribed about $\mathcal{C}'$ [5, 7, 8].

Here we consider a certain family such that $\mathcal{C}$ is a parabola $\mathcal{P}$ while $\mathcal{C}'$ is a circle centered on the focus of $\mathcal{P}$. As shown in Figure 1, this is simply the polar image of the *bicentric family* (interscribed between two circles) with respect to its circumcircle, see Appendices A and B for construction details. We derive closure conditions for this new family for $N = 3,4,5,6$ cases (N is the number of sides) and describe some of its properties and loci of associated points. Also considered is its polar image with respect to $\mathcal{P}$.

Main results

- The loci of vertex, perimeter, and area centroids are parabolas. Recall that in general, the locus of the perimeter centroid is not a conic [22].
- The loci of vertex and area centroids of polar polygons are straight lines, whereas that of the perimeter centroid is a non-conic.
- In the $N = 3$ case, the locus of the orthocenter is a straight line as are those of many triangle centers of the polar family. The Euler line of the polar family always passes through the parabola's focus.
- Several centers of the $N = 3$ polar family are stationary and/or sweep circles. In the latter case, they all belong to a single parabolic pencil.
- We prove that the quantity $\sum \sin\theta_i/2$ is conserved, where θ_i are the interior angles of parabola-inscribed polygons. In fact, this quantity is conserved by any conic-inscribed polar image of the bicentric family.

Most of the above properties were first noticed via simulation [25], and later proved with a computer-algebra system (CAS) [17], using the explicit parametrizations given in Appendix A. For brevity, we omit any CAS-based proofs.

Related work

We can roughly divide it into three groups: (i) the study of point loci over certain triangle families [18, 19, 27], (ii) proving that loci of certain Poncelet triangle families are of a given curve type [9, 13, 21, 23], and (iii) proving properties and invariants over $N \geq 3$ Poncelet families [2, 4, 6, 22]. Also related is the Steiner-Soddy Poncelet family which are the polar image of the so-called Brocard porism with respect to the circumcircle [10].

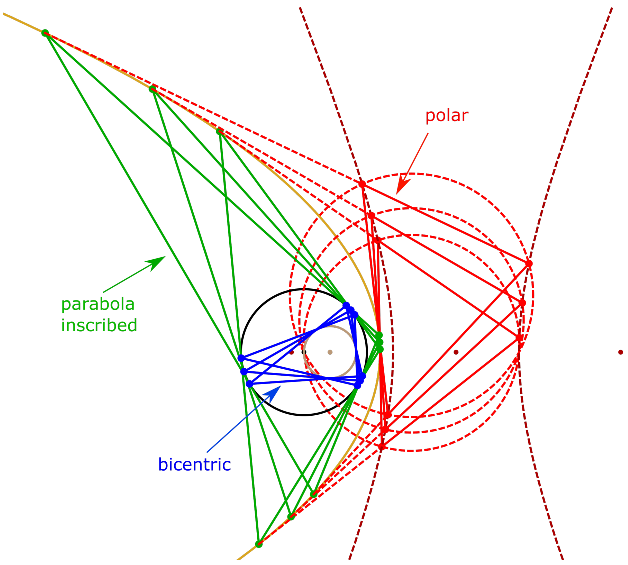

Figure 1: *Several configurations of the parabola-inscribed Poncelet family (green), obtainable as the polar image of the bicentric family (blue) with respect to the outer circle (black), provided the bicentric incircle passes through the circumcenter, see Apendix B. Also shown is the polar family (red) of the parabola-inscribed one with respect to the parabola itself. This family is inscribed in a hyperbola (dark dashed red). So you can think of this trio (blue, green, red) as successive polar images with respect to the outer conic of each preceding family.*

Article organization

In Sections 2 and 3 we examine parabola-inscribed Poncelet triangles (as well as its polar polygons with respect to the parabola), deriving closure conditions and expressions for many of its their triangle center loci. In Section 4 we derive geometric closure conditions for $N = 4,5,6$ families, respectively, detecting the abovementioned pattern for the loci of their centroids (as well as in the polar family), followed by conjectured generalizations in Section 7. In Section 8 we describe a new quantity conserved by the parabola-inscribed family (and variations thereof).

In Appendix A we provide explicit parametrizations for the vertices of both the $N = 3$ and $N = 4$ families, as well as their respective polar families. In Appendix B we explore the relation of parabola-inscribed families with the traditional bicentric family.

2 Loci of parabola-inscribed triangles

Referring to Figure 2, consider a Poncelet family T of triangles inscribed in a parabola $\mathcal{P}$, and circumscribed about a focus-centered circle. Let $F = [-f, 0]$ and $V = [0, 0]$ denote focus and vertex, respectively, where f is the focal distance. Consider a circle $\mathcal{C}$ centered at F with radius r.

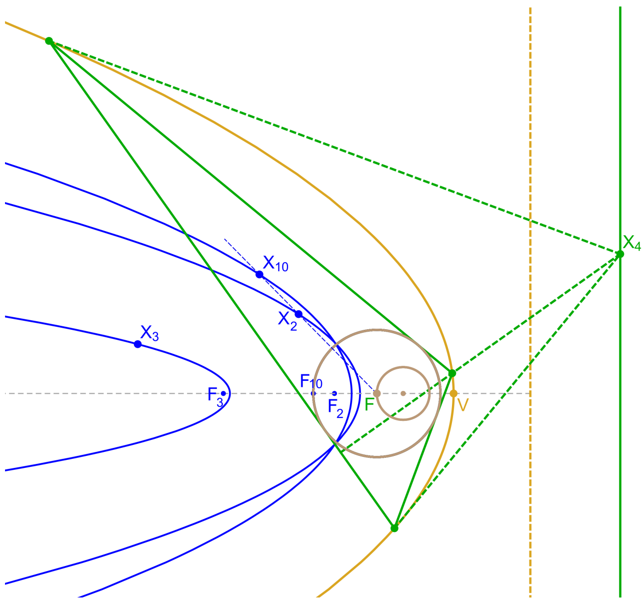

Figure 2: *A Poncelet triangle (green) is shown inscribed in a parabola $\mathcal{P}$ (gold), circumscribed about a focus-centered circle $\mathcal{C}'$ (brown). Over the family, X_4 sweeps a line (solid green) parallel to the directrix (dashed gold). The loci of barycenter X_2, circumcenter X_3, and Spieker center X_{10} are coaxial parabolas (blue); their foci are labeled F_2, F_3, and F_{10}, respectively. Notice the latter is on an intersection of $\mathcal{C}'$ and the axis of the $\mathcal{P}$ (dashed gray). Since the family circumscribes a circle centered on F, F, X_2, X_{10} are collinear (dashed blue) and $X_{10} = F + (3/2)(X_2 - F)$, see Remark 1.*

Proposition 1 *$\mathcal{P}$ and $\mathcal{C}$ will admit a Poncelet family of triangles if, and only if, $r/f = 2(\sqrt{2} - 1)$.*

Proof. Consider the Poncelet triangle with two parallel sides shown in Figure 3, inscribed in the parabola $y = x^2/(4f)$, where f is the focal length. At $x = r$ the parabola must be at $y = f - r$, i.e., $f - r = r^2/(4f)$, and the result follows. $\square$

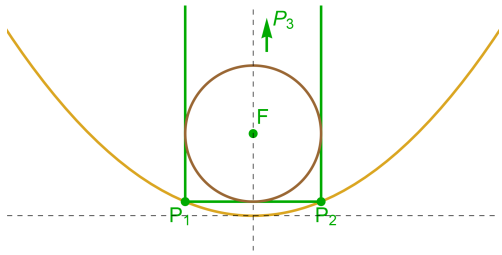

Figure 3: *Construction used to derive r/f in Proposition 1. Side P_1P_2 of the Poncelet triangle is perpendicular to the axis while the other two sides are parallel to it, i.e., vertex P_3 lies on the line at infinity.*

2.1 Straight-line orthocenter locus

let $\mathcal{T}$ denote our parabola-inscribed triangle family and $\mathcal{D}$ the directrix of $\mathcal{P}$. Henceforth we shall adopt Kimberling's notation X_k to refer to triangle centers [16].

Proposition 2 *Over $\mathcal{T}$, the locus of the orthocenter X_4 is the line parallel to $\mathcal{D}$ given by $x = (5-2\sqrt{2})f$, with $y \in \mathbb{R}\backslash\{\pm 2(\sqrt{2}-1)f\}$.*

The proof below was kindly contributed by Alexey Zaslavsky [26]:

Proof. Let $\mathcal{C}$ be the unit circle in the complex plane and A, B, C the touching points with the sides of the parabola-inscribed triangles. The polar transformation with center F maps the parabola to a circle with center I passing through F and touching AB, BC, CA. Using Euler's formula $|FI|^2 = r^2 = R(R-2r)$ [24], with $R = 1$ its radius, and $r = \sqrt{2}-1$. Consider the line FI as the real axis. Since I is self-conjugated with respect to ABC, we have $a+b+c = 2\sqrt{2}-2+(3-2\sqrt{2})abc$, $ab+bc+ca = 3-\sqrt{2}+(2\sqrt{2}-2)abc$. The polar images of the altitudes of the original triangle are the common points of BC, CA, AB with the lines passing through F, and perpendicular to FA, FB, FC respectively. We have to calculate the common point of the line passing through these three points and the real axis. The coordinate functions of this point are symmetric functions in a, b, c, so we can express them as elementary symmetric functions on said variables, and verify that they are constant. The restriction on y coordinates are poles in the parametric equation that describes the locus. □

Note that in [10, Section 4] a more general result was proved: the locus of the orthocenter X_4 of any Poncelet triangle family inscribed in a parabola $\mathcal{P}$ whose caustic is a circle centered on the axis of $\mathcal{P}$, is a straight line parallel to the directrix of $\mathcal{P}$.

In Appendix B we describe how the parabola-inscribed family is the polar image of the bicentric family with respect to its circumcircle. Referring to Figure 4, Proposition 2 is actually a special case of:

Proposition 3 *The locus of X_4 of an family which is the polar image of $N = 3$ bicentrics with respect to its outer circle is an ellipse, straight line, or hyperbola if the circumcenter of the bicentric triangle lies in the interior, on top, or outside its incircle.*

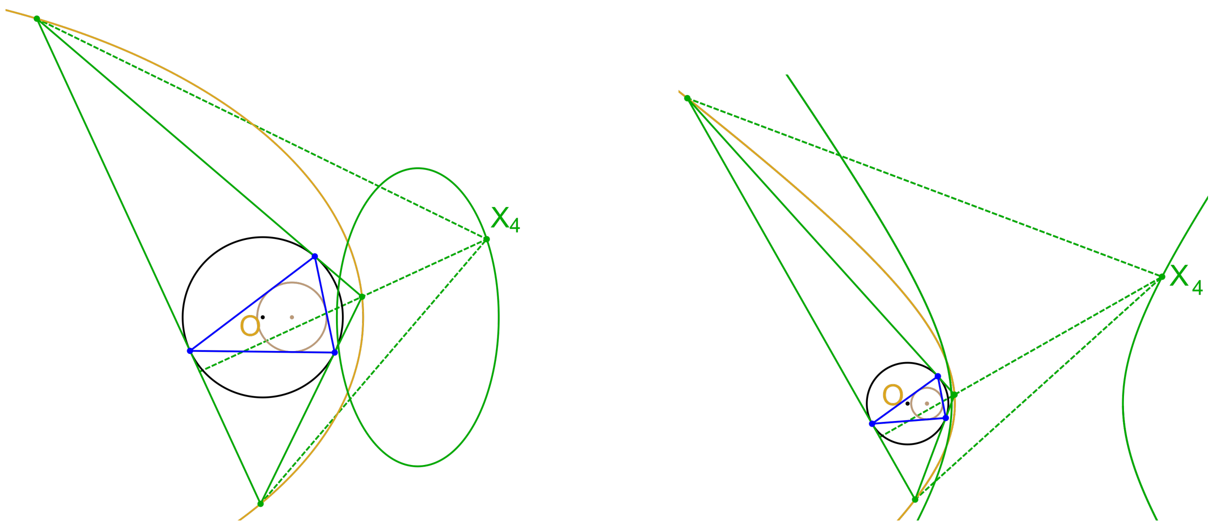

Figure 4: *Consider perturbing to the bicentric family (blue) such that the circumcenter O is interior (resp. exterior) to the incircle, as shown on the left (resp. right). The tangential family (green) becomes ellipse- (resp. hyperbola-) inscribed (gold curve). In the former (resp. latter) case, the locus of the orthocenter X_4 is an ellipse (resp. hyperbola).*

2.2 Three parabolic loci

Referring to Figure 2, we show below that over $\mathcal{T}$, the loci of the barycenter, circumcenter, and Spieker centers are all parabolas. The first and last correspond to the vertex and perimeter centroids of a triangle. This is curious since, in general, the locus of the perimeter centroid of a Poncelet family is not a conic [22].

Proposition 4 *Over $\mathcal{T}$, the locus of the barycenter X_2 is a parabola coaxial with $\mathcal{P}$, with focus $F_2 = [-f/3, 0]$, and vertex $V_2 = [2f(1-2\sqrt{2})/3, 0]$.*

Proposition 5 *Over $\mathcal{T}$, the locus of the circumcenter X_3 of T is a parabola coaxial with $\mathcal{P}$, with focus $F_3 = [-f(2\sqrt{2}-3)/2, 0]$, and vertex $V_3 = [-f(2\sqrt{2}+3)/2, 0]$.*

Proposition 6 *Over $\mathcal{T}$, the locus of the Spieker center X_{10} is a parabola coaxial with $\mathcal{P}$, with focus $F_{10} = [(1-2\sqrt{2})f, 0]$ and vertex $V_{10} = [f(3/2-2\sqrt{2}), 0]$. In particular, $F_{10} = [-f-r, 0]$, i.e., it lies on the left extreme of $\mathcal{C}$.*

Note that X_{10} is the perimeter centroid of a triangle, while X_2 doubles up as both the vertex and area centroid. A. Akopyan has reminded us of the following general fact:

Remark 1 *If a polygon circumscribes a circle (let its center be O), then C_1, C_2, O are collinear and $(C_1 - O) = (3/2)(C_2 - O)$.*

Therefore:

Corollary 1 *Over $\mathcal{T}$, X_{10} is collinear with X_2 and $X_{10} = F + (3/2)(X_2 - F)$.*

3 The polar $N = 3$ family

Referring to Figure 5, let T' denote the *polar* triangle of a triangle T in $\mathcal{T}$, i.e., whose sidelines are the polars of T with respect to $\mathcal{P}$. Since T is inscribed in $\mathcal{P}$ these are simply the tangents.

Recall some known properties of the polar triangle with respect to any parabola [3]: (i) the circumcircle of T' passes through the focus F; (ii) the orthocenter of T' is on the directrix; (iii) its area is half that of the reference triangle.

Proposition 7 *The T' family is Ponceletian. It is circumscribed about $\mathcal{P}$ and is inscribed in a hyperbola $\mathcal{H}$ with center $[f, 0]$. Its axes are the axis and directrix of $\mathcal{P}$. Its implicit equation reads*

$$\mathcal{H}: \left(\sqrt{2}+\frac{3}{2}\right)(x-f)^2 - \frac{y^2}{2} - 2f^2 = 0.$$

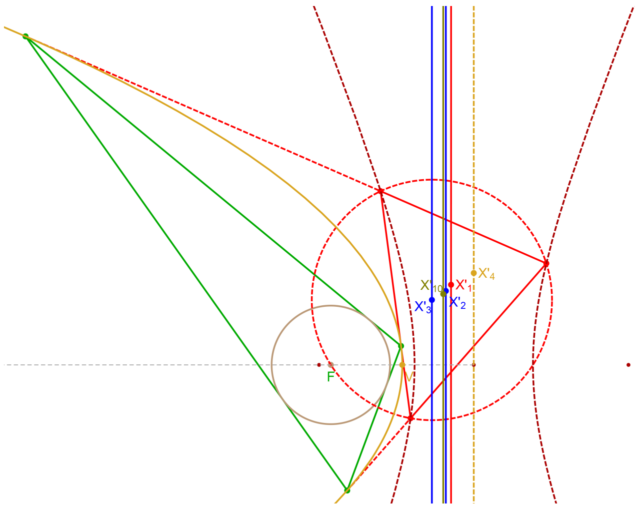

Figure 5: *The polar triangle T' (red) with respect to the parabola $\mathcal{P}$ (gold) to which our Poncelet family (green) is inscribed. It is Ponceletian as it is inscribed in a hyperbola (dashed dark red). Well-know properties include (i) the circumcircle (dashed red) passes through the focus F, and (ii) the orthocenter X_4' lies (and therefore sweeps) the directrix of $\mathcal{P}$ [3]. Also shown are the visually-straight, though quartic loci of the polar incenter X_1' and Spieker center X_{10}' (red and olive, respectively). The loci of X_k', $k = 2, 3, 4$ are straight lines parallel to the directrix (red, red, and dashed gold), the latter the directrix itself.*

3.1 Straight and nearly-straight loci

An enduring conjecture has been that the locus of the incenter X_1 of a Poncelet triangle family can only be a conic if the pair is confocal [15].

As shown in Figure 5, over the polars, the locus of the incenter is, to the naked eye, a straight line. However, upon an algebraic investigation:

Proposition 8 *The locus of the incenter X_1' of T' is one of four branches of the following quartic:*

$$X_1' : (-5\sqrt{2}-6)x^2y^2 + (4\sqrt{2}+2)f^2x^2 + (10\sqrt{2}+12)fxy^2 + (8\sqrt{2}+4)f^3x + (3\sqrt{2}-16)f^2y^2 - 14f^4 = 0.$$

Specifically, the branch

$$X_1' = \left[\frac{\sqrt{2}y^2+2+2y^2-\sqrt{-4y^2+4y^4+8\sqrt{2}+8\sqrt{2}y^2-2\sqrt{2}y^4}}{y^2(\sqrt{2}+2)-2}, y\right],$$

where $y \neq \pm\sqrt{2-\sqrt{2}}$.

The locus of X_1' is bounded by two lines parallel to the directrix and approximately $f/850$ apart, see Figure 6.

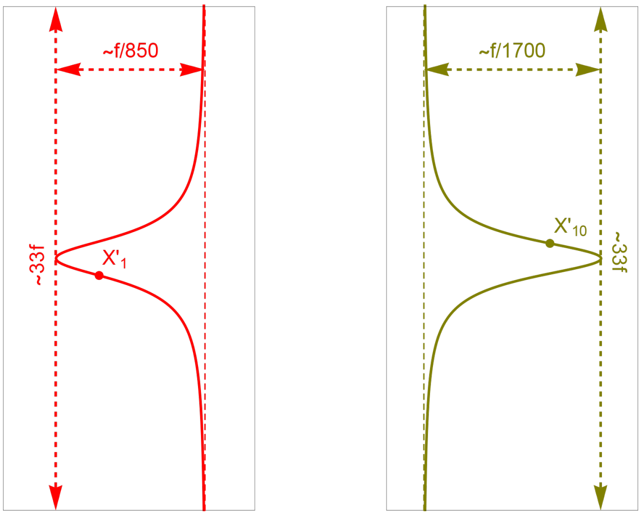

Figure 6: ***Left:*** *The locus of the polar incenter X_1' is the branch of a quartic which visually is a straight line. It fits within to lines parallel to the directrix and at a distance of $f/850$. In the figure the curve is shown at aspect ratio of 28,000.* ***Right:*** *The locus of the polar Spieker center X_{10}' (perimeter centroid) is an algebraic curve of degree at least four, bounded by two vertical lines separated by $f/1700$. The aspect ratio of the figure is 56,000.*

Still referring to Figure 5:

Proposition 9 *The locus of the barycenter X_2' of T' is a line parallel to $\mathcal{D}$ and parametrized by*

$$X_2' = \frac{1}{3}\left[\left(2\sqrt{2}-1\right)f, \frac{(4-8\sqrt{2})f^2y+y^3}{(8\sqrt{2}-12)f^2+y^2}\right].$$

Proposition 10 *The locus of the circumcenter X_3' of T' is a line parallel to $\mathcal{D}$ and parametrized by*

$$X_3' = \left[(\sqrt{2}-1)f, \frac{(3\sqrt{2}+2)(2\sqrt{2}y^2-28f^2+y^2)y}{14(y\sqrt{2}-2f+y)(y\sqrt{2}+2f+y)}\right].$$

Referring to Figure 7, the above expressions for X_2' and X_3' yield:

Corollary 2 *The (varying) Euler line $X_2'X_3'$ of the polar family passes through the focus $F=[-f,0]$ of $\mathcal{P}$.*

Still referring to Figure 7, the next 4 propositions were obtains from experimental evidence and verification by CAS:

Proposition 11 *The locus of the symmedian point X_6' of T' is a line parallel to $\mathcal{D}$ and parametrized by*

$$X_6' = \left[(5-3\sqrt{2})f, \frac{(3\sqrt{2}+4)(2\sqrt{2}y^2-28f^2+y^2)y}{14(y\sqrt{2}-2f+y)(y\sqrt{2}+2f+y)}\right].$$

Proposition 12 *The locus of X_{10}' of the polar family is an algebraic curve of degree four given by*

$$\begin{aligned} X_{10}' : \;& 4\left(11\sqrt{2}+16\right)x^4 - 4\left(3\sqrt{2}+5\right)x^2y^2 - 4\left(37\sqrt{2}+50\right)fx^3 \\ &+ 8\left(2\sqrt{2}+1\right)fxy^2 + 21\left(5\sqrt{2}+8\right)f^2x^2 - 4\left(9\sqrt{2}+8\right)f^3x \\ &- \left(\sqrt{2}+4\right)f^2y^2 + 7f^4 = 0 \end{aligned}$$

This locus is tightly bound by the following two lines parallel to the directrix:

$$x = \left(\sqrt{2}-1+\frac{\sqrt{10-7\sqrt{2}}}{2}\right)f \text{ and } x = \left(\sqrt{2}-2^{-1/4}\right)f.$$

The distance between these lines is approx. $f/1700$.

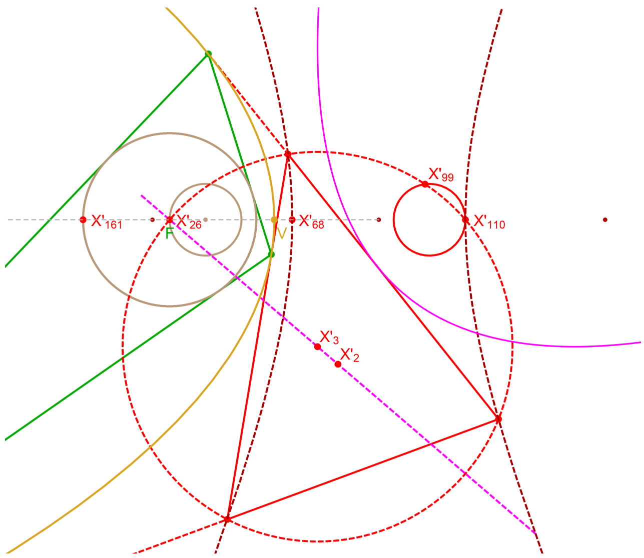

Figure 7: *Over the polar family (red), the Euler line (dashed magenta) will always pass through the focus F of the parabola-inscribed family (green). X_{26}' (resp. X_{68}' and X_{110}') remain stationary at the focus F (resp. the two vertices of the hyperbola to which the polar family is inscribed). Experimentally, X_{161} is stationary at the intersection of the caustic with the parabola axis farthest from the latter's vertex. Also shown is the Kiepert inparabola (magenta), whose focus is X_{110} and directrix is the Euler line. Thus the polar family simultaneously inscribes the original parabola (gold) and the Kiepert (magenta). Finally, the figure depicts the circular locus of Steiner point X_{99}' of the polar family.*

3.2 Stationary points

The circumcenter of the tangential triangle appears as X_{26} on [16].

Proposition 13 *Point X'_{26} of T' is stationary at the focus F of $\mathcal{P}$.*

Note X_{26} does not lie in general on the circumcircle of a reference triangle. In our case it does since, as mentioned above, the circumcircle of the polar contains the focus.

The Kiepert parabola of a triangle is an inscribed parabola whose focus is labeled X_{110} on [16]. Its directrix is the Euler line [24]. Referring to Figure 7:

Proposition 14 *The focus X'_{110} of the Kiepert parabola (resp. the Prasolov point X'_{68}) of the polar family is stationary at the vertex of $\mathcal{H}$ farthest (resp. closest) to the focus of $\mathcal{P}$. Furthermore, X'_{161} is stationary at the intersection of the incircle with the parabola axis farthest from the parabola vertex, i.e., at $[(1-2\sqrt{2})f,0]$.*

Observation 1 *Over the polar family, the vertex of its Kiepert parabola sweeps a circle.*

3.3 Linear loci galore

Referring to Figure 8:

Observation 2 *Over the first 1000 triangle centers in [16], the following triangle centers of T' sweep linear loci parallel to $\mathcal{D}$: $X'_k, k=$2, 3, 4, 5, 6, 20, 22, 23, 24, 25, 49, 51, 52, 54, 64, 66, 67, 69, 74, 113, 125, 140, 141, 143, 146, 154, 155, 156, 159, 182, 184, 185, 186, 193, 195, 206, 235, 265, 323, 343, 368, 370, 373, 376, 378, 381, 382, 389, 394, 399, 403, 427, 428, 468, 546, 547, 548, 549, 550, 567, 568, 569, 575, 576, 578, 597, 599, 631, 632, 858, 895, 973, 974.*

3.4 A pencil of circular loci

Referring to Figure 7:

Proposition 15 *The locus of the Steiner point X'_{99} is a circle whose center O'_{99} lies on the axis of $\mathcal{P}$ of radius R'_{99} such that at its right endpoint it touches X'_{110}. Explicitly,*

$$O'_{99} = \left[(6\sqrt{2}-7)f, 0\right], \quad R'_{99} = 2f\sqrt{17-12\sqrt{2}}\,.$$

Referring to Figure 9:

Observation 3 *Over the first 1000 triangle centers in [16], the following triangle centers of T' sweep circular loci with centers on the axis of $\mathcal{P}$ and passing through X'_{110}: $X'_k, k=$99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935.*

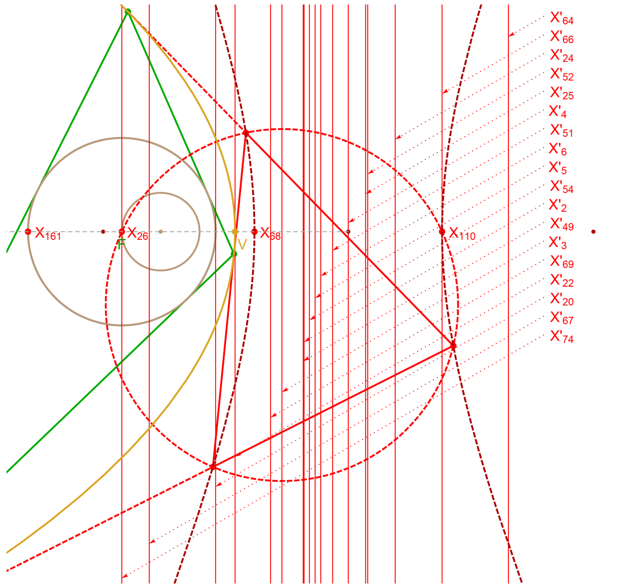

Figure 8: *Many triangle centers of the polar family sweep lines parallel to the directrix. The following are shown: X_k, $k=$2, 3, 4, 5, 6, 20, 22, 24, 25, 49, 51, 52, 54, 64, 66, 67, 69, 74.*

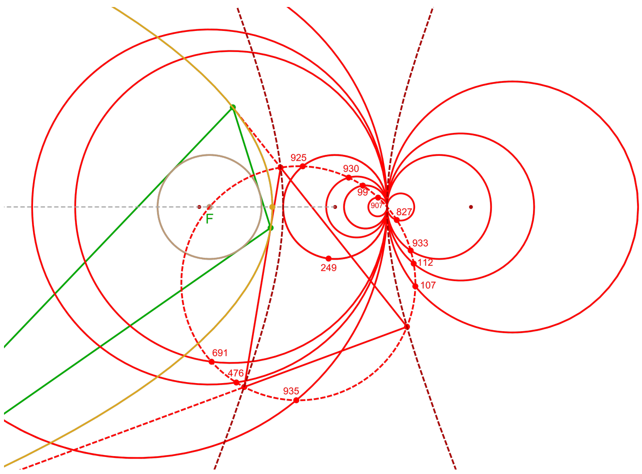

Figure 9: *Over the polar family we find that if a certain triangle center sweeps a circular locus, said locus will be an element of a parabolic pencil with X_{110} as their common point (not labeled). In the figure the circular loci of X_k, $k=$99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935 are shown. Notice all lie on the dynamically-moving circumcircle (dashed red) except for X_{249}.*

This gives credence to:

Conjecture 1 *If the locus of X'_k is a circle with nonzero radius, it is in the parabolic pencil with X_{110} as a common point.*

4 Parabola-inscribed quadrilaterals

Referring to Figure 10, consider a Poncelet family $\mathcal{Q}$ of quadrilaterals inscribed in a parabola $\mathcal{P}$, and circumscribed about a focus-centered circle $\mathcal{C}$ of radius r. As before, let f denote the parabola's focal distance, and $V = [0,0]$, $F = [-f,0]$, its vertex and focus, respectively.

Proposition 16 *$\mathcal{P}$ and $\mathcal{C}$ will admit a Poncelet family of convex quadrilaterals if, and only if, $r/f = 2\sqrt{\sqrt{5}-2}$.*

Proof. Referring to Figure 11, consider the symmetric Poncelet quadrilateral $P_i = [x_i, y_i]$, $i = 1,\ldots,4$, inscribed in the parabola $y = x^2/(4f)$, i.e., $x = 2\sqrt{fy}$. Clearly, $y_1 = f - r$, and $y_2 = f + r$. Requiring that P_1P_2 be tangent to $\mathcal{C}$ yields the quartic $r^2 + 4f\sqrt{f^2 - r^2} = 0$. The claim is the one positive root of this quartic. □

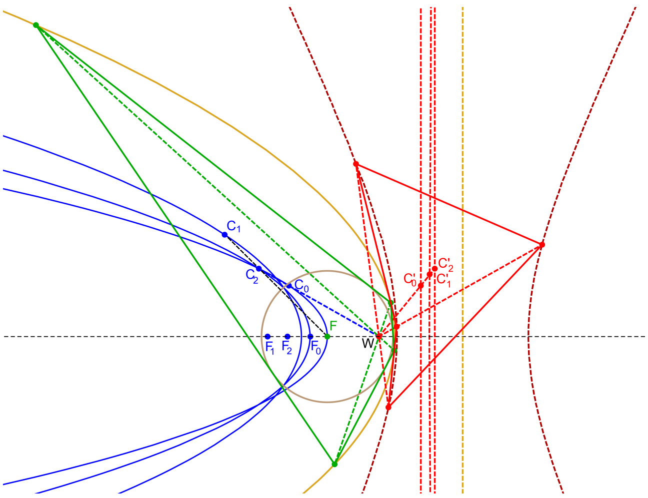

Figure 10: *A Poncelet quadrilateral (green) is shown inscribed in a parabola $\mathcal{P}$ (gold) and circumscribed about a focus-centered circle (brown). Over the family, (i) the intersection W of its diagonals (dashed green) is stationary; (ii) the loci of vertex C_0, perimeter C_1, and area C_2, centroids sweep 3 distinct parabolas (blue) coaxial with $\mathcal{P}$ with foci on F_0, F_1 and F_2. Notice the vertex of C_0 is F and that of C_1 is F_0. (iv) As predicted by Remark 1, C_1 is collinear with F and C_2 (dashed black); (v) C_0, C_2, W are collinear (dashed blue). Also shown is the polar quadrilateral Q' (red) with respect to $\mathcal{P}$, inscribed in a hyperbola (dashed, red) centered at $[f,0]$. One observes that: (a) its diagonals (dashed red) also intersect at W; (b) the loci of its vertex C_0' and area C_2' centroids are lines (dashed orange) perpendicular to the axis of $\mathcal{P}$, (c) C_0', C_2', W are collinear (dashed red); (d) the locus of the polar perimeter centroid C_1' is algebraic and of degree 10.*

Note: more generally, Cayley's conditions may be used to include the non-convex case, see [8].

The next 3 propositions, first identified experimentally, were then confirmed via CAS.

Proposition 17 *Over $\mathcal{Q}$, the two diagonals P_1P_3 and P_2P_4 intersect at a stationary point $W = [(2-\sqrt{5})f, 0]$.*

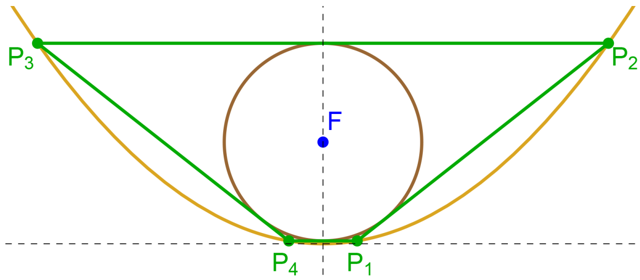

Figure 11: *Construction used to derive r/f in for parabola-inscribed convex quadrilaterals in Proposition 16.*

4.1 The three centroids

Referring to Figure 10, let C_0, C_1, and C_2 denote the vertex, perimeter, and area centroids of the quadrilaterals in $\mathcal{Q}$, respectively.

Proposition 18 *Over the family, C_0, C_2, and W are collinear.*

Proposition 19 *Over the Poncelet family, the loci of C_0, C_2 are parabolas coaxial with $\mathcal{P}$, whose foci and vertices locations are listed in Table 1.*

From Remark 1:

Corollary 3 *The locus of C_1 is a $3/2$-scaled version of the locus of C_2 with F as the homothety center.*

centroid (N=4)	focal dist.	vertex x/f	vtx. x/f (num)
C_0	$f/4$	-1	-1
C_1	$f/2$	$(\sqrt{5}-5)/2$	-1.381966
C_2	$f/3$	$\sqrt{5}/3-2$	-1.25464

Table 1: *Location of centroids C_0, C_1, C_2 in the convex $N = 4$ family.*

4.2 The polar quadrilateral

Referring to Figure 10, consider the polar quadrilateral whose sides are the tangents to $\mathcal{P}$ at the vertices of the original family. Let P_i', $i = 1,\ldots,4$ denote its vertices and C_0', C_1', and C_2' denote its vertex, perimeter, and area centroids.

Proposition 20 *The locus of the polar quadrilateral's vertices is the hyperbola $\mathcal{H}$ given by*

$$\mathcal{H}: \frac{(x-f)^2}{4(\sqrt{5}-2)f^2} - \frac{y^2}{4f^2} - 1 = 0.$$

with center at $[f,0]$ and foci $[f(1 \pm 2\sqrt{\sqrt{5}-1}),0]$.

Let W be defined as in Proposition 17. The next two propositions result from visual (and numerical) detection, followed by verification by CAS.

Proposition 21 *The two diagonals of the polar quadrilateral intersect at W.*

Proposition 22 *Over the polar quadrilateral family, C_0', C_2', and W are collinear.*

Proposition 23 *Over Q, the loci of C_0' and C_2' are lines parallel to the parabola's directrix and given by $C_0' : x = (3-\sqrt{5})f/2$, and $C_2' : x = (4-\sqrt{5})f/3$.*

Rather laborious CAS manipulation yields:

Proposition 24 *Over Q, the locus of C_1' is one connected component of an algebraic curve of degree ten, given by the following equation:*

$$\begin{aligned}
C_1' :\ & -\left(1457008\sqrt{5}+3257968\right) f x^7 y^2 + \left(122156\sqrt{5}+273148\right) f^2 x^4 y^4 \\
& + \left(465164\sqrt{5}+1040132\right) f^2 x^6 y^2 - \left(96506\sqrt{5}+215698\right) f^6 x^2 y^2 \\
& - \left(119256\sqrt{5}+266664\right) f^3 x^3 y^4 + \left(505052\sqrt{5}+1129268\right) f^5 x^3 y^2 \\
& + \left(8564\sqrt{5}+19204\right) f^7 x y^2 - \left(881712\sqrt{5}+1971568\right) x^{10} \\
& + \left(43955\sqrt{5}+98289\right) f^4 x^2 y^4 + \left(24568\sqrt{5}+54936\right) f x^5 y^4 \\
& - \left(7250\sqrt{5}+16210\right) f^5 x y^4 - \left(1274930\sqrt{5}+2850838\right) f^4 x^4 y^2 \\
& + \left(1235568\sqrt{5}+2762832\right) f^3 x^5 y^2 + \left(4457696\sqrt{5}+9967712\right) f x^9 \\
& - \left(7787152\sqrt{5}+17412608\right) f^2 x^8 + \left(5470456\sqrt{5}+12232344\right) f^3 x^7 \\
& - \left(1690535+755997\sqrt{5}\right) f^4 x^6 - \left(812098\sqrt{5}+1815898\right) f^5 x^5 \\
& + \left(330322\sqrt{5}+738968\right) f^6 x^4 + \left(1002+448\sqrt{5}\right) f^6 y^4 \\
& - \left(228\sqrt{5}+672\right) f^8 y^2 - \left(7300\sqrt{5}+16956\right) f^7 x^3 \\
& + \left(2750\sqrt{5}+7150\right) f^9 x - \left(16145\sqrt{5}+36103\right) f^8 x^2 \\
& - \left(84196\sqrt{5}+188268\right) x^6 y^4 + \left(544928\sqrt{5}+1218496\right) x^8 y^2 - 726 f^{10} = 0.
\end{aligned}$$

Furthermore, C_1' is bound by the following two lines parallel to the directrix and approximately $f/25$ apart: $x = \left(5+\sqrt{2}-\sqrt{5}\sqrt{2}-\sqrt{5}\right)f/2$, and $x = \left(\sqrt{5}\sqrt{2}-\sqrt{5}-2\sqrt{2}+3\right)f/2$.

5 Parabola-inscribed pentagons

Referring to Figure 12, consider a family of pentagons inscribed in a parabola $\mathcal{P}$ of focal distance f, and circumscribed about a focus-centered circle $\mathcal{C}$ of radius r.

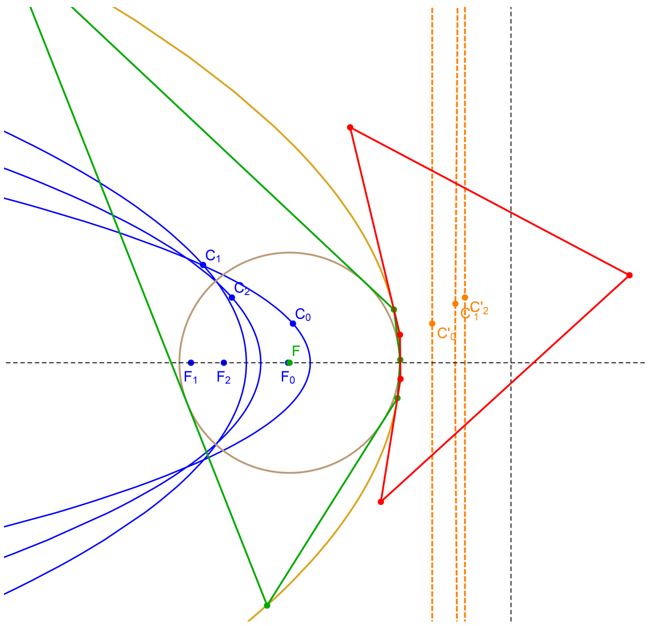

Figure 12: *Parabola-inscribed pentagons (green), and their polar polygon (red). The loci of vertex C_0, perimeter C_1, and area centroids C_1 are parabolas (blue) coaxial with $\mathcal{P}$ (gold). Over the polar family, C_0' and C_2' are straight lines (dashed orange) perpendicular to the directrix (dashed black). Though the locus of the perimeter centroid C_1' is indistinguishable from a straight line, it is an algebraic curve of degree likely much higher than 10 (since that is the degree for C_1' on $N = 4$).*

Proposition 25 *The pair $\mathcal{P},\mathcal{C}$ will admit a Poncelet family of pentagons if, and only if, r/f is the only positive root of the following sextic polynomial ($r/f \approx 0.995219$):*

$$x^6 + 12x^5 - 28x^4 + 32x^3 + 112x^2 - 64x - 64 = 0.$$

Proof. Referring to Figure 13, without loss of generality, let $\mathcal{P}$ be the unit parabola $y = x^2$ with focus $F = [0,1/4]$

and let $\mathcal{C}$ be a circle of radius r centered at F. Consider the Poncelet pentagon P_i, $i = 1, \ldots, 5$ with P_4 at infinity, and P_1P_2 horizontal and tangent to $\mathcal{C}$ at $[0, 1/4 - r]$. Compute the next Poncelet vertex $P_3 = [x_3, y_3]$ as the intersection of a tangent to $\mathcal{C}$ from P_2 with $\mathcal{P}$. By requiring that $x_3 = r$, we obtain the sextic in the claim. □

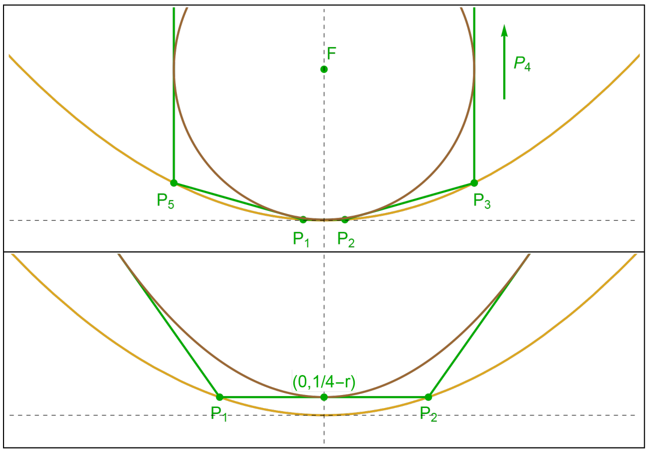

Figure 13: *Construction used to derive r/f in Proposition 25. Top (resp. bottom) shows the complete picture (resp. a detailed view near the vertex)*

Referring to Figure 12:

Conjecture 2 *Over the parabola-inscribed pentagon family, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $\mathcal{P}$.*

Conjecture 3 *Over the family of polar polygons to parabola-inscribed pentagons, the locus of vertex and area vertices are lines perpendicular to the directrix while that of the perimeter centroid is an algebraic curve of degree at least four.*

6 Parabola-inscribed hexagons and summary

6.1 Hexagons and summary

Referring to Figure 14, we can also consider a family of parabola-inscribed hexagons.

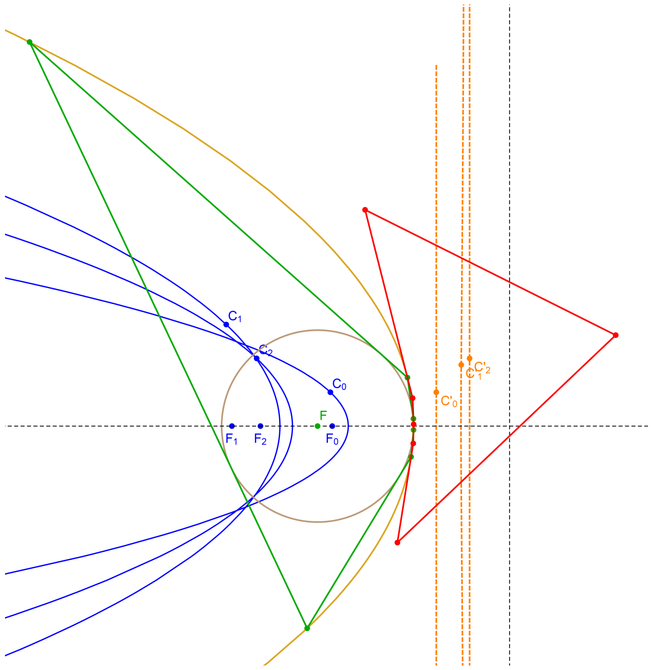

Figure 14: *Hexagons (green) inscribed in a parabola $\mathcal{P}$. As before, the loci of C_0, C_1, and C_2 are parabolas (blue) coaxial $\mathcal{P}$. Over the polar family (red), the loci of C_0', C_2' are lines perpendicular to the axis while that of C_1' is algebraic, and though visually a straight line, its degree is likely much higher than 10 (since that is the degree for C_1' on $N = 4$).*

An analogous construction (based on symmetric configurations) was used to obtain r/f required for convex $N = 6$. A summary of all r/f thus obtained appears in Table 2.

N	r/f	r/f (num.)	Cayley
3	$2(\sqrt{2}-1)$	0.828427	4
4	$2\sqrt{\sqrt{5}-2}$	0.971737	4
5	n/a	0.995219	8
6	n/a	0.999183	8

Table 2: *Table of r/f required for closure of convex N-gons inscribed in a parabola, and circumscribed about a focus-centered circle. Algebraic expressions (2nd column) are only possible for $N = 3, 4$. The last column shows the number of possible solutions for r/f if one were to include cases where circle and parabola intersect (the Poncelet polygon may be self-intersecting and/or non-convex). For Cayley's conditions in the general case, see [8].*

7 Generalizing centroidal loci

Let $\mathcal{R}$ be a Poncelet family of N-gons inscribed to a parabola $\mathcal{P}$, and circumscribed about a focus-centered circle

$\mathcal{C}$. Experimental the evidence for the $N = 3,4,5,6$ cases, we propose the following generalizations (reader contributions are encouraged):

Conjecture 4 *Over $\mathcal{R}$, for any $N \geq 3$, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $\mathcal{P}$.*

Conjecture 5 *Over $\mathcal{R}$ and for any $N \geq 3$, the loci of vertex and area centroids of the polar polygons with respect to $\mathcal{P}$ are straight lines parallel to the directrix of $\mathcal{P}$.*

Let $\mathcal{B}'$ be the conic-inscribed polar image of a generic bicentric family of N-gons with respect to the bicentric circumcircle (see Appendix B).

Recall that the locus of vertex and area centroids C_0, C_2 are conics over any Poncelet family, while that of the perimeter centroid C_1 is not, in general, a conic [22]. A consequence of Remark 1, analogously exploited in [10, Corollary 2], is that:

Corollary 4 *Over $\mathcal{B}'$, the locus of the perimeter centroid is a conic.*

Let $\mathcal{P}'$ be the conic to which $\mathcal{B}'$ is inscribed.

Conjecture 6 *Over the polar polygons of $\mathcal{B}'$ with respect to $\mathcal{P}'$, the locus of the perimeter centroid is never a conic.*

8 A conserved quantity

As in Appendix B, let $\mathcal{B}$ denote a bicentric family of N-gons inscribed to a circle $\mathcal{C} = (O,R)$, and circumscribed about a second, nested circle $\mathcal{C}'$. Let d_i denote the perpendicular distance from the bicentric circumcenter O to side P_iP_{i+1}.

Referring to Figure 15:

Lemma 1 *Over $\mathcal{B}$, the quantity $\sum d_i$ is conserved.*

The argument below was kindly provided by A. Akopyan [1].

Proof. The above statement is equivalent to stating that over $\mathcal{B}$ the sum of unit vectors from a point P in the direction perpendicular to bicentric sides is constant. In turn, the latter is a corollary of the well-known fact that over $\mathcal{B}$, the centroid of the touchpoints of sidelines with $\mathcal{C}'$ is stationary. □

Let θ_i, $i = 1,\ldots,N$, denote the angles interior to a polygon $\mathcal{B}$.

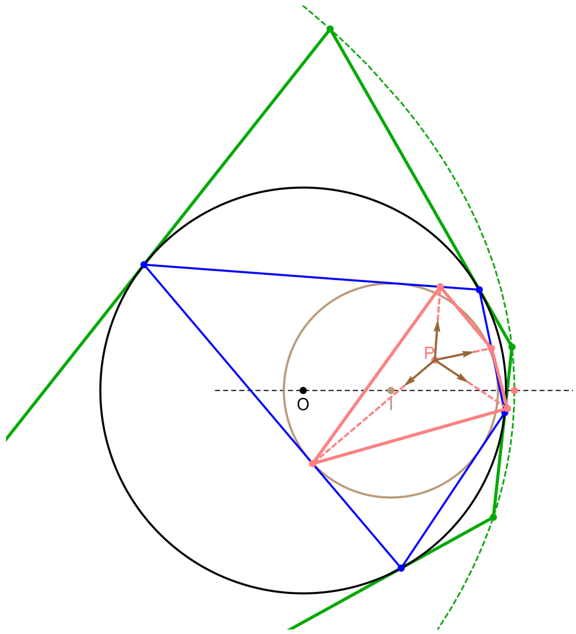

Figure 15: *An $N = 4$ bicentric polygon is shown (blue). Without loss of generality, in the case shown the circumcenter O is interior to the incircle, i.e., the polar family (green) is ellipse-inscribed. Also shown is the pedal polygon (pink) with respect to a point P in the interior of the circumcircle and the unit vectors (brown) along each perpendicular dropped from P onto the sides.*

Proposition 26 *For all N, the porism of polygons polar to $\mathcal{B}$ with respect to its circumcenter conserves $\sum_{i=1}^{N} \sin\theta_i/2 = (1/R)\sum d_i$.*

Proof. The vertices of the tangential polygon are the poles of each side of $\mathcal{B}$ with respect to the circumcircle. Therefore, said vertices are at a distance $D_i = R^2/d_i$ from the O. Since $\sin\theta_i/2 = R/D_i = d_i/R$, per Lemma 1, the claim follows. □

Note that in general, θ_i is the directed angle $P_{i-1}P_iP_{i+1}$. In the case when $r < d$, the tangential polygon will be inscribed in two branches of a hyperbola. There are only two cases: Either (i) all vertices lie on a first proximal branch of the hyperbola, or (ii) all but one vertex P_k will lie on said branch, with P_k lying on the distal branch. In case (i), all θ_i are positive whereas in (ii) all are positive except for θ_k. Furthermore, in this case, the supplement of angles θ_{k-1} and θ_{k+1} need to be used in the sum. So the invariant sum becomes

$$\sin\frac{\theta_1}{2} + \ldots + \sin\frac{\pi - \theta_{k-1}}{2} - \sin\frac{\theta_k}{2}$$
$$+ \sin\frac{\pi - \theta_{k+1}}{2} + \ldots + \sin\frac{\theta_N}{2} =$$
$$\sin\frac{\theta_1}{2} + \ldots + \cos\frac{\theta_{k-1}}{2} - \sin\frac{\theta_k}{2} + \cos\frac{\theta_{k+1}}{2} + \ldots + \sin\frac{\theta_N}{2}.$$

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Appendix A. Vertex parametrizations

A.1. Parabola-inscribed triangles

A 3-periodic orbit $P_i = [x_i, y_i] = [-y_i^2/(4f), y_i]$ is such that

$$y_2 = \frac{2\left(1-\sqrt{2}\right)(4fy_1+\Delta)f}{8f^2\sqrt{2}-12f^2+y_1^2},$$

$$y_3 = \frac{2\left(\sqrt{2}-1\right)f\Delta}{8f^2\sqrt{2}-12f^2+y_1^2},$$

where $\Delta = \sqrt{16(8\sqrt{2}-11)f^4+8f^2y_1^2+y_1^4}$.

A.2. Hyperbola-inscribed polar triangles

A 3-periodic orbit $Q_i = [q_{1,i}, q_{2,i}]$ is such that

$$Q_1 = \left(1+\sqrt{2}\right)\cdot \left[\frac{(4fy_1+\Delta)y_1}{2(2\sqrt{2}+3)y_1^2-8f^2}, \frac{\left(1+\sqrt{2}\right)y_1^3-4\left(1+\sqrt{2}\right)f^2y_1-2\Delta f}{2(2\sqrt{2}+3)y_1^2-8f^2}\right],$$

$$Q_2 = \left(1+\sqrt{2}\right)\cdot \left[\frac{(4fy_1-\Delta)y_1}{2(2\sqrt{2}+3)y_1^2-8f^2}, \frac{\left(1+\sqrt{2}\right)y_1^3-4\left(1+\sqrt{2}\right)f^2y_1+2\Delta f}{2(2\sqrt{2}+3)y_1^2-8f^2}\right],$$

$$Q_3 = \left(1+\sqrt{2}\right)\cdot \left[\frac{\left(5-3\sqrt{2}\right)\left((1+2\sqrt{2})y_1^2-28f^2\right)f}{7((3+2\sqrt{2})y_1^2-4f^2)}, -\frac{8f^2y_1}{(3+2\sqrt{2})y_1^2-4f^2}\right],$$

where $\Delta = \sqrt{y_1^4+8f^2y_1^2+16(8\sqrt{2}-11)f^4}$.

A.3. Parabola-inscribed quadrilaterals

A 4-periodic orbit $P_i = [x_i, y_i] = [-\frac{1}{4f}y_i^2, y_i]$ is such that:

$$y_2 = \frac{\left(2\sqrt{\sqrt{5}-2}\,\Delta_1+4fy_1\left(3-\sqrt{5}\right)\right)f}{4f^2\sqrt{5}-8f^2-y_1^2},$$

$$y_3 = \frac{4\left(2-\sqrt{5}\right)f^2}{y_1},$$

$$y_4 = -\frac{\left(2\sqrt{\sqrt{5}-2}\,\Delta_1+4fy_1\left(\sqrt{5}-3\right)\right)f}{4f^2\sqrt{5}-8f^2-y_1^2},$$

where $\Delta_1 = \sqrt{y_1^4+8f^2y_1^2+16\left(9-4\sqrt{5}\right)f^4}$.

A.4. Hyperbola-inscribed polar quadrilaterals

A 4-periodic orbit $P_i = [p_i, q_i]$ is such that:

$$p_1 = \frac{\sqrt{\sqrt{5}-2}\left(\Delta_1+6fy_1\sqrt{5}\sqrt{\sqrt{5}-2}+14fy_1\sqrt{\sqrt{5}-2}\right)y_1}{4y_1^2+2y_1^2\sqrt{5}-8f^2}$$

$$q_1 = \frac{\left(2\sqrt{\sqrt{5}+2}\,\Delta_1 f+4f^2y_1-yI^3\right)\left(4f^2\sqrt{5}+8f^2+y_1^2\right)}{32f^4-32f^2y_1^2-2y_1^4}$$

$$p_2 = \frac{2\sqrt{\sqrt{5}-2}\left(\Delta_1+2\sqrt{2}(\sqrt{5}-1)y_1\right)\left(\left(\sqrt{5}-2\right)y_1^2+4f^2\right)f^2}{y_1\left(16f^4-16f^2y_1^2-y_1^4\right)}$$

$$q_2 = \sqrt{\sqrt{5}+2}\left(y_1\Delta_1+2\sqrt{\sqrt{5}+2}fy_1^2-8\left(\sqrt{5}-2\right)^{3/2}f^3\right)\cdot \left((\sqrt{5}-2)y_1^2+4f^2\right)f$$

$$p_3 = \frac{-2\sqrt{\sqrt{5}-2}\left(\Delta_1-2\sqrt{2}\sqrt{\sqrt{5}-1}fy_1\right)}{y_1\left(16f^4-16f^2y_1^2-y_1^4\right)}\cdot \left(y_1^2\sqrt{5}+4f^2-2y_1^2\right)f^2$$

$$q_3 = \frac{-\sqrt{\sqrt{5}+2}\left(yI\,\Delta_1-2\sqrt{\sqrt{5}+2}fy_1^2+8\left(\sqrt{5}-2\right)^{3/2}f^3\right)}{y_1\left(16f^4-16f^2y_1^2-y_1^4\right)}\cdot \left((\sqrt{5}-2)y_1^2+4f^2\right)f$$

$$p_4 = \frac{\sqrt{\sqrt{5}-2}\left(\Delta_1-2\sqrt{2}(\sqrt{5}-1)fy_1\right)\left((\sqrt{5}-2)y_1^2+4f^2\right)y_1}{y_1\left(16f^4-16f^2y_1^2-y_1^4\right)}$$

$$q_4 = \frac{\left(-2f\Delta_1\sqrt{\sqrt{5}-2}+4f^2y_1-y_1^3\right)\left(4f^2\sqrt{5}+8f^2+y_1^2\right)}{y_1\left(16f^4-16f^2y_1^2-y_1^4\right)}$$

Appendix B. Relation to the bicentric family

Referring to 16, the bicentric family $\mathcal{B}$ of N-gons is a family of Poncelet N-gons inscribed in a circles $\mathcal{C} = (O, R_b)$, and circumscribed about another circle $\mathcal{C}' = (O', r_b)$. Let $d = |O - O'|$. Relations between d, R, r_b are known for many "low N" and are listed in [24, Poncelet's porism].

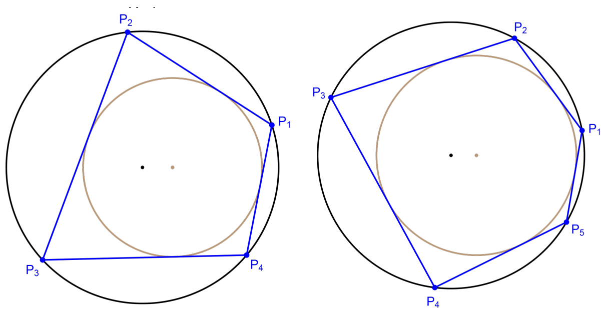

Figure 16: *The bicentric family is a family of Poncelet polygons interscribed between two circles. Shown are the $N = 4$ (left) and $N = 5$ (right) convex cases.*

Definition 1 (Polar polygon) *Given a polygon P, its* polar polygon *P' with respect to a conic $\mathcal{C}$ is bounded by the tangents to $\mathcal{C}$ at the vertices of $\mathcal{P}$.*

Proposition 27 *The polar family $\mathcal{B}'$ of $\mathcal{B}$ with respect to $\mathcal{C}$ is an ellipse, parabola, or hyperbola-inscribed if d is smaller, equal, or greater than R', respectively (O is interior, on the boundary, or exterior to $\mathcal{C}'$, respectively). Furthermore, one of the foci coincides with O'.*

As shown in Figure 17, when the polar family is hyperbola-inscribed, there are two layouts for its vertices: either (i) all lie on the branch of the hyperbola closest to the incenter of the family, or (ii) all but one lie on said branch, while the remaining one lies on the "other" branch.

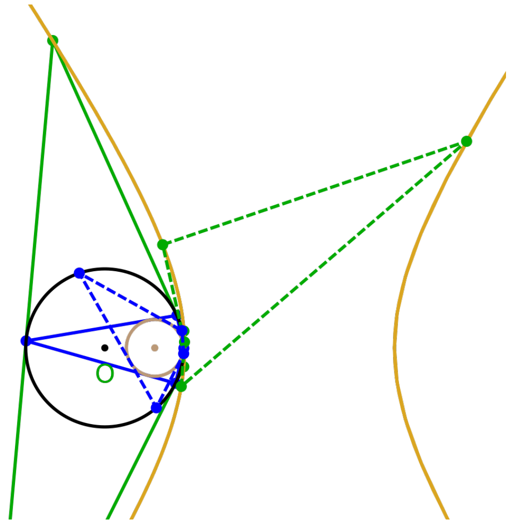

Figure 17: *If the circumcenter O is exterior to the incircle of a bicentric polygon (blue), the polar (i.e., tangential) family will be hyperbola (gold) inscribed. Over the family there are two configurations: (i) solid green: all vertices lie on one branch of the hyperbola; (ii) dashed green: all but one vertex lie on the branch proximal to the incenter, while a lone one lies on the opposite branch.*

Proposition 28 *The parabola $\mathcal{P}$ which is the polar image of $\mathcal{B}$ with $d = r_b$, has focal distance $f = R_b^2/(2r_b)$.*

Proof. Let $O = (0,0)$. Consider a polygon in $\mathcal{B}$ with a vertical side P_1P_2 tangent to the incircle at $(2r_b, 0)$. The vertex V of $\mathcal{P}$ is the pole of said side which can be obtained as the inversion of point $(2r_b, 0)$ with respect to the circumcircle. This yields the result. $\square$

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