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Triads of Conics Associated with a Triangle

Dedicated to Paul Yiu

Triads of Conics Associated with a Triangle

ABSTRACT

We revisit constructions based on triads of conics with foci at pairs of vertices of a reference triangle. We find that their 6 vertices lie on well-known conics, whose type we analyze. We give conditions for these to be circles and/or degenerate. In the latter case, we study the locus of their center.

Key words: triangle, conic, Carnot, Soddy circles

MSC2010: 51M04, 51N20, 51N35, 68T20

Trijade konika pridruženih trokutu

SAŽETAK

Podsjećamo na konstrukcije temeljene na trijadama konika sa žarištima u parovima vrhova referentnog trokuta. Nalazimo da njihovih 6 vrhova leži na dobro poznatim konikama čiji tip analiziramo. Za ove konike dajemo uvjete da budu kružnice i/ili degenerirane konike. U slučaju degeneriranih konika proučavamo geometrijsko mjesto njihovog središta.

Ključne riječi: trokut, konika, Carnot, Soddyjeve kružnice

1 Introduction

Paraphrasing a passage in [13], “new tools of interactive geometry enable the discovery of properties in a way mathematicians in the past could only have dreamed about”. Aided by interactive simulation (mostly Mathematica and GeoGebra), and inspired by a construction by Paul Yiu [17,

Sec. 12.4, p. 148], we tour curious dynamic phenomena manifested by triads of ellipses (or hyperbolas) naturally associated with a triangle. Namely, we attach their foci to a pair of vertices and impose that the conic pass through either (i) the remaining vertex, or (ii), some chosen point P . We call these “V-” or “P”-conics, respectively, see Figure 1.

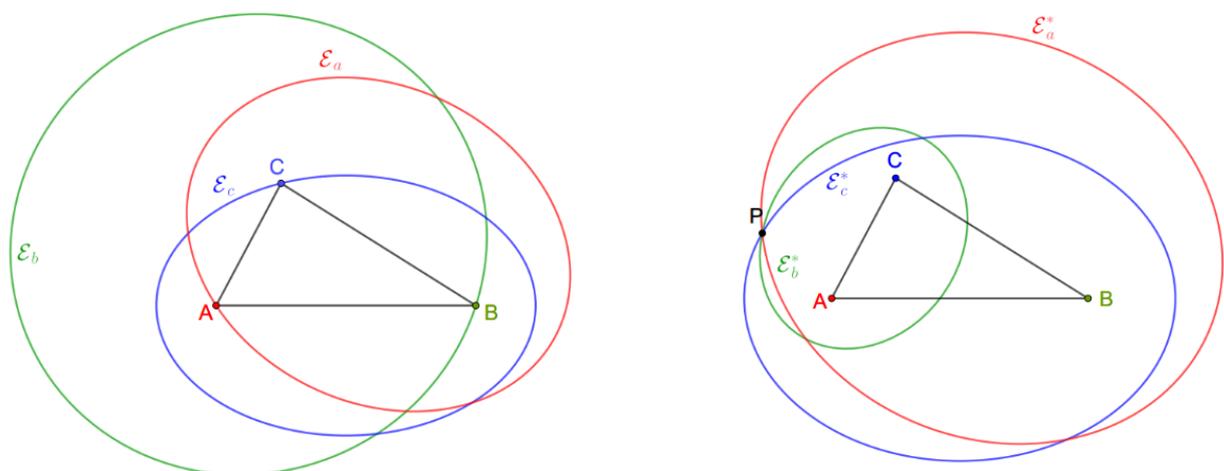


Figure 1: **Left:** A $\triangle ABC$, and a V-triad of ellipses passing through a vertex and with foci on the remaining pair. **Right:** in the P-triad case, ellipses still have foci on pairs of vertices but now pass through a given point P .

Some of our main results include:

- The 6 vertices of V-ellipses always lie on a conic; this conic is degenerate iff the reference triangle is a right triangle.
- The conic passing through the 6 P-ellipse vertices is degenerate iff P lies on the circumcircle.
- The locus of the center of the 6-point conic over the degenerate family is a quartic in the V-ellipse case, and the union of three arcs of ellipses in the P-ellipse case; we derive expressions for them.
- We specify the regions such that various 6-point conics are of a given type (hyperbola, ellipse, parabola, or degenerate).
- We derive conditions such that various 6-point conics are a circle.
- We derive conditions (and loci) under which the co-vertices of conic triad lie on a conic.

Some of the above are done for the case of hyperbola triads as well. Most of our results have been obtained through experimentation with dynamic geometry software first, and later confirmed geometrically and/or algebraically. See [6] for details.

Some long, symbolic proofs are omitted, with some expressions appearing in Section 6. Throughout the paper we will be using X_k notation for triangle centers, after [8].

Related Work

We have been inspired by the idea of erecting identical geometrical objects to the sides of a triangle, e.g., [3, 5, 10, 11]. Triads of “Artzt” parabolas, conceived in the XIX century, have been revisited in [4, 9, 15]. In [13], new properties of Artzt parabolas are detected via dynamic geometry software. Properties of conic triads with a shared focus are studied in [1]. A 6-point conic passing through the tangency point of the excircles (which turns out to coincide with the vertices of V-ellipses) is described in [2, 18]. A Construction of 3 “Soddy” hyperbolas (called here V-hyperbolas) with foci on vertices appears in [17, Sec. 12.4, p. 148]. Properties of a triad of circles tangent to the nine-point circle are studied in [12].

¹These refer to the intersection of a conic with the focal axis.

Article organization

Properties of triads of V-ellipses, P-ellipses, V-hyperbolas, and P-hyperbolas, are covered in Sections 2 to 5, respectively. In last section we pose to the reader a few open questions. The last section contains some long-form symbolic expressions for a construction appearing in Section 2.

2 A triad of V-ellipses

Referring to Figure 1:

Definition 1 (V-ellipses) Given a triangle $\triangle ABC$, a triad of V-ellipses $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$ have foci on (B, C) , (C, A) , (A, B) and pass through A , B , and C , respectively.

Proposition 1 The V-ellipses $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$ are centered at the midpoints of $\triangle ABC$'s sides. Their vertices¹ are the (external) tangency points of the excircles with triangle's sidelines and lie on a conic, \mathcal{Y} .

Proof. Let a, b, c be the sidelengths of $\triangle ABC$. Let $(I_a), (I_b), (I_c)$ the escribed circles and let $A_1, A_2, B_1, B_2, C_1, C_2$ their (external) tangency points with the lines BC, CA, AB , as shown in Figure 2. We shall prove that these points are the intersection of the V-ellipses with their focal axis BC, CA, AB , hence their vertices. Elementary properties of tangents from a point to a circle yield:

$$\begin{aligned} AC_2 = AB_1 = BA_2 = BC_1 = CA_1 = CB_2 = p, \\ BA_1 = CA_2 = p - a, \\ AB_2 = CB_1 = p - b, \\ AC_1 = BC_2 = p - c, \end{aligned} \quad (1)$$

where $p = (a + b + c)/2$ is the semi-perimeter. Hence:

$$\begin{aligned} A_1A_2 &= A_1B + BC + CA_2 = \\ &= (p - a) + a + (p - a) = 2p - a = b + c. \end{aligned}$$

Since $A_1B = A_2C$, and since points A_1, A_2, B and C are collinear, the former are precisely the two vertices of \mathcal{E}_a . Furthermore, the segments BC and A_1A_2 share their midpoint, the center of \mathcal{E}_a . The proof for \mathcal{E}_b and \mathcal{E}_c is similar.

In order to prove that their six vertices are on a conic, by Carnot's Theorem, it is enough to check that

$$\frac{AC_1}{BC_1} \cdot \frac{AC_2}{BC_2} \cdot \frac{BA_1}{CA_1} \cdot \frac{BA_2}{CA_2} \cdot \frac{CB_1}{AB_1} \cdot \frac{CB_2}{AB_2} = 1. \quad (2)$$

This claim is obtained by substituting (1) into (2). \square

Remark 1 *The fact that a conic passes through the six external tangency points with the excircles was discovered by Paul Yiu [18]. In [8] its center is labeled X_{478} .*

It can be shown that the Yiu conic \mathcal{Y} can never be a circle except when $\triangle ABC$ is an equilateral.

Proposition 2 *Each V-ellipse $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$ is respectively tangent at A, B, C to the sides of the excentral triangle.*

Proof. Referring to Figure 2, since I_a, I_b, I_c are the centers of the escribed circles, the lines I_bI_c, I_cI_a, I_aI_b are the external bisectors of $\angle BAC, \angle ACB,$ and $\angle BCA$; thus AI_a, BI_b, CI_c are altitudes in $\triangle I_aI_bI_c$ as well as (internal) bisectors of $\triangle ABC$. By the optic propriety of conics, lines I_bI_c, I_cI_a, I_aI_b are also the tangents in A, B, C to the ellipses $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$. \square

Referring to Figure 2, let $(A', A''), (B', B''),$ and (C', C'') denote the pairwise intersections between $(\mathcal{E}_b, \mathcal{E}_c), (\mathcal{E}_c, \mathcal{E}_a),$ and $(\mathcal{E}_a, \mathcal{E}_b),$ respectively.

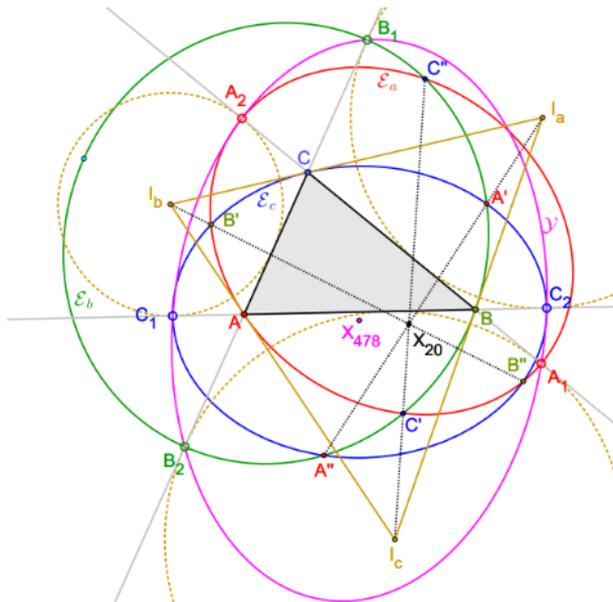


Figure 2: *Properties of a V-ellipses $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$ (red, green, blue) with respect to a $\triangle ABC$ (black). (i) Its vertices are at the tangency points of the excircles (dashed gold) with the sidelines; hence they lie on the Yiu conic (magenta) [18]. (ii) Each ellipse is tangent at A, B, C to a side of the excentral triangle $\triangle I_aI_bI_c$. (iii) The 3 chords $A'A'', B'B'',$ and $C'C''$ between the intersections of $(\mathcal{E}_b, \mathcal{E}_c), (\mathcal{E}_c, \mathcal{E}_a),$ and $(\mathcal{E}_a, \mathcal{E}_b)$ pass through $I_a, I_b, I_c,$ and concur at X_{20} .*

Proposition 3 *The lines through $A', A'', B', B'', C', C''$ pass through the 3 excenters $I_a, I_b, I_c,$ respectively, and concur at the de Longchamps' point X_{20} .*

Proof. It can be shown that the \mathcal{E}_a is given by the following implicit equation in barycentric coordinates $[x, y, z]$:

$$\begin{aligned} \mathcal{E}_a : & 4c(b+c)xy - (a-b-c)(a+b+c)y^2 + \\ & + 4b(b+c)xz + 2(a^2+b^2+2bc+c^2)yz - \\ & - (a-b-c)(a+b+c)z^2 = 0 \end{aligned}$$

$\mathcal{E}_b, \mathcal{E}_c$ can be obtained cyclically on a, b, c . The barycentrics for the vertices of \mathcal{E}_a are $A_1 = [0, a+b+c, a-b-c]$ and $A_2 = [0, a-b-c, a+b+c]$. Let S be twice the area of $\triangle ABC$. The two real intersections A', A'' between $\mathcal{E}_b, \mathcal{E}_c$ are given by:

$$\begin{aligned} A' = & [(a-b-c)(a+b-c)(a-b+c) \cdot \\ & \cdot (3a^2+2ab-b^2+2ac+2bc-c^2) + \\ & + 4(-2a^3-a^2b-b^3-a^2c+b^2c+bc^2-c^3)S, \\ & (a-b-c)(a-b+c)(a+b+c) \cdot \\ & \cdot (a^2-2ab-3b^2+2ac+2bc+c^2) + \\ & + 4(a^3+ab^2+2b^3+a^2c-b^2c-ac^2-c^3)S, \\ & (a-b-c)(a+b-c)(a+b+c) \cdot \\ & \cdot (a^2+2ab+b^2-2ac+2bc-3c^2) + \\ & + 4(a^3+a^2b-ab^2-b^3+ac^2-bc^2+c^3)S] \end{aligned}$$

and A'' is obtained as above but with $S \rightarrow -S$. The intersections B', B'' and C', C'' are obtained cyclically. The line $A'A''$ is then given by:

$$\begin{aligned} & -(b-c)(a+b+c)^2x - (a+b-c)^2(a+c)y + \\ & + (a+b)(a-b+c)^2z = 0 \end{aligned}$$

It can be shown this line passes through excenter I_a . The other lines can be obtained cyclically. It can also be shown these meet at X_{20} , whose first barycentric coordinate is given by [8]: $[-3a^4 + 2a^2(b^2 + c^2) + (b^2 - c^2)^2]$, with the other two obtained cyclically. \square

Referring to Figure 3:

Proposition 4 *When $\triangle ABC$ is a right triangle, the V-ellipses pass through the reflection of the orthocenter on the circumcenter, the de Longchamps point X_{20} .*

Proof. Let C denote the right-angle vertex of $\triangle ABC$, and C' its reflection about the circumcenter X_3 . We shall prove that each V-ellipse passes through C' . Due to central symmetry, this is trivially true for \mathcal{E}_c . Consider \mathcal{E}_a : since its foci are B, C and it passes through A , its major axis has length $|AC| + |AB|$. Since $ACBC'$ is a rectangle, $|AC| = |BC'|$, and $|BC| = |C'A|$. Hence $|C'B| + |CC'| = |AC| + |AB|$, which ensures that $C' \in \mathcal{E}_a$. Similarly $C' \in \mathcal{E}_b$. \square

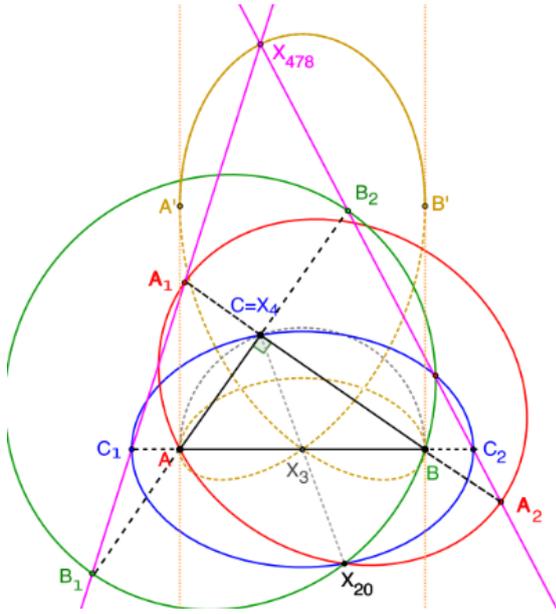


Figure 3: If $\triangle ABC$ is a right triangle, the Yiu conic \mathcal{Y} (magenta) is degenerate, and the V-ellipses intersect at X_{20} . Furthermore, over all C on a semicircle with AB as a diameter, the locus of the center X_{478} of \mathcal{Y} is an arc (solid gold) of a quartic (dashed gold). The lines through A, B perpendicular to AB (dotted gold) are tangent to the locus at its endpoints A', B' , and $|AA'| = |BB'| = |AB|$.

Degenerate six-point conic:

Still referring to Figure 3:

Proposition 5 \mathcal{Y} is degenerate iff $\triangle ABC$ is a right triangle.

Proof. Let $A_1, A_2, B_1, B_2, C_1, C_2$ the intersection points of the ellipses, with the lines BC, CA, AB , as in Figure 3. We shall prove that A_1, B_1, C_1 are collinear iff $\triangle ABC$ is right-angled. To do so, by Menelaus' theorem, we need to check that

$$\frac{A_1C}{A_1B} \cdot \frac{C_1B}{C_1A} \cdot \frac{B_1A}{B_1C} = 1. \tag{3}$$

Let $x = CA_1 = BA_2, y = AB_1 = CB_2, z = AC_1 = BC_2$.

Since the V- ellipses pass through one of triangle's vertices and have their foci into the other two, $a + 2x = b + c, b + 2y = a + c, c + 2z = a + b$, hence

$$x = p - a, y = p - b, z = p - c,$$

where $p = \frac{a+b+c}{2}$ is the semi-perimeter. Substituting this into (3), we obtain:

$$\frac{x}{p} \cdot \frac{p}{z} \cdot \frac{y}{p} = 1$$

hence $(p - a) \cdot (p - b) = p \cdot (p - c)$, which is equivalent to $c^2 = a^2 + b^2$. The result follows by Pythagoras' theorem. \square

Assume, without loss of generality, that $A = (1/2, 0)$ and $B = (-1/2, 0)$.

Proposition 6 Over C on the semicircle whose diameter is $AB, y > 0$, the locus of the center of the degenerate \mathcal{Y} is the arc of a quartic given by:

$$4(x^2 + y^2)^2 - 8y^3 - x^2 + 2y^2 = 0, y > 1$$

The semicircle with $y < 0$ produces a locus which is symmetric about the x -axis.

Proof. The claim was obtained via manipulation and simplification with a Computer Algebra System (CAS). \square

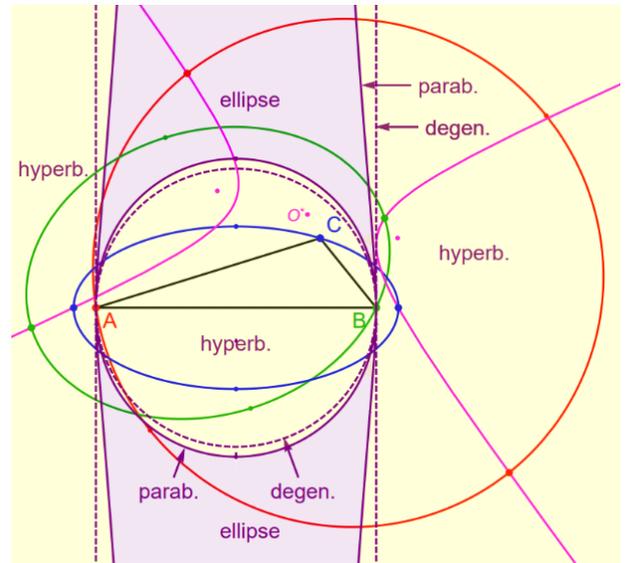


Figure 4: With A, B fixed, the solid (resp. dashed) purple lines are the locus of C such that the Yiu conic \mathcal{Y} is a parabola (resp. degenerate). As indicated, in between said boundaries, the conic is either an ellipse or a hyperbola. A particular $\triangle ABC$ is shown with C interior to the circumcircle, where \mathcal{Y} is a hyperbola (magenta).

Referring to Figure 4:

Proposition 7 With A, B fixed, the Yiu conic \mathcal{Y} of $\triangle ABC$ is (i) degenerate if C lies on the union of the circumcircle with the two lines tangent to it at A and B ; (ii) a parabola if C lies on a curve whose barycentrics satisfy the following degree-8 implicit equation:

$$a^8 + b^8 + c^8 - 2(a^4b^4 + a^4c^4 + b^4c^4) + 4abc(a^5 + b^5 + c^5 - a^4b - ab^4 - a^4c - ac^4 - bc^4 - b^4c + a^3bc + ab^3c + abc^3) = 0.$$

Proof. The claim was obtained via manipulation and simplification with a Computer Algebra System (CAS). \square

What about the co-vertices?

It turns out that for A, B fixed, there is a locus of C such that the 6 co-vertices of the V-ellipses lie on a conic. Without loss of generality, let $A = (-1, 0)$, and $B = (1, 0)$. Referring to Figure 5:

Proposition 8 *The locus of C such that the 6 co-vertices of $\mathcal{E}_a, \mathcal{E}_b,$ and \mathcal{E}_c lie on a conic is given by:*

$$\begin{aligned} & \left(x^6 - (2y^2 + 3)x^4 - (3y^4 - 8y^2 - 3)x^2 + 11y^4 - 6y^2 - 1\right) \rho_1 \rho_2 + \\ & + \left(-2x^6 - (22y^2 - 6)x^4 - (14y^4 - 36y^2 + 6)x^2 + 6y^6 + 22y^4 - \right. \\ & \quad \left. - 14y^2 + 2\right) (\rho_1 + \rho_2) + \\ & + 2x \left(x^6 + (3y^2 - 3)x^4 + (3y^4 - 2y^2 + 3)x^2 + y^6 - 7y^4 - y^2 - 1\right) \cdot \\ & \quad \cdot (\rho_1 - \rho_2) + \\ & + 2(x^2 + y^2 - 1) \left(5x^4 + 2(y^2 - 5)x^2 - 3y^4 - 14y^2 + 5\right) (x^2 - 1) = 0 \end{aligned}$$

where $\rho_1 = \sqrt{x^2 + y^2 + 2x + 1}$,
and $\rho_2 = \sqrt{x^2 + y^2 - 2x + 1}$.

Proof. Computer algebra system-based manipulation. \square

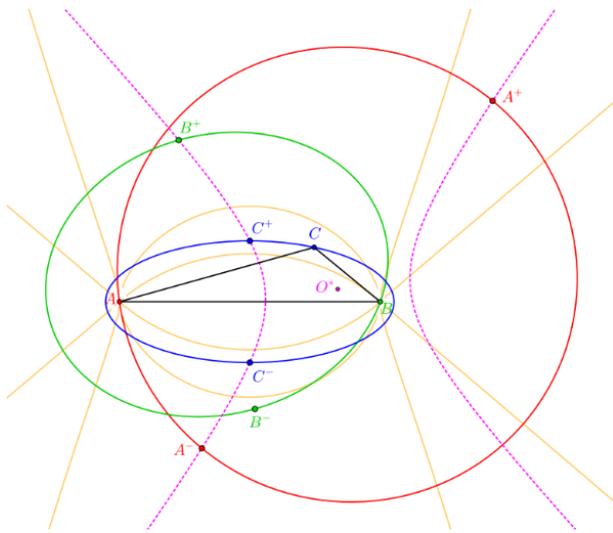


Figure 5: $\triangle ABC$ is shown, as well as its 3 P-ellipses (red, green, blue) with co-vertices $A^+, A^-, B^+, B^-, C^+, C^-$. Also shown is the locus of C (yellow) such that the co-vertices lie on a conic. Notice that for the triangle shown, C does lie on said locus. For illustration, a hyperbola is shown (dashed magenta) which passes through 5 co-vertices but misses B^- .

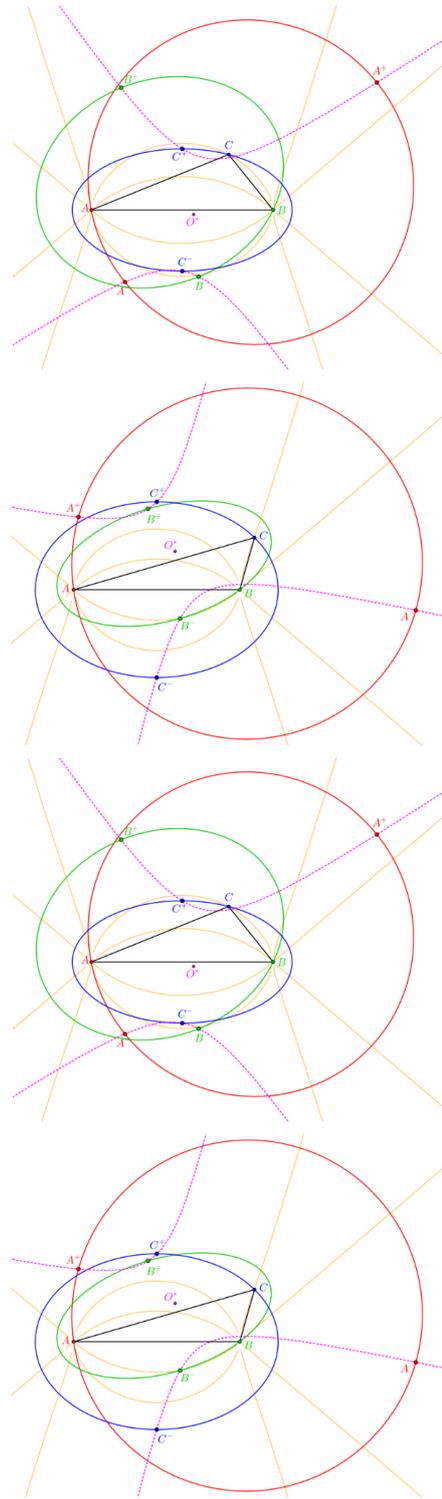


Figure 6: Four choices for C on the locus (yellow) such that the co-vertices $A^+, A^-, B^+, B^-,$ and C^+, C^- of V-ellipses $\mathcal{E}_a, \mathcal{E}_b,$ and \mathcal{E}_c of $\triangle ABC$ lie on a conic (dashed magenta). In the top two cases (resp. bottom two), the co-vertices are split 3x3 (resp. 5x1) on each branch of the conic.

Notice that a full 8 branches of the locus converge on either A or B . Also note that if one attempts to eliminate the square roots in the implicit, one obtains a degree-36 polynomial.

Examples of the 6-point co-vertex conic for different locations of C on the above locus appear in Figure 6, suggesting that (i) this conic is always a hyperbola, and that (ii) depending on the branch of the locus of C is on, co-vertices are split as 3:3 or 5:1 along the two branches of the hyperbola.

3 A triad of P-ellipses

Referring to Figure 7:

Definition 2 (P-ellipses) A triad of P-ellipses $\mathcal{E}_a^*, \mathcal{E}_b^*, \mathcal{E}_c^*$ have foci on $(B, C), (C, A), (A, B)$ and pass through a given point P .

Consider a triad of P-ellipses as in Definition 2.

Theorem 1 The six vertices of a triad of P-ellipses lie on a conic \mathcal{Y}^* .

Proof. Referring to Figure 7, let A_1, A_2 (resp. B_1, B_2 , and C_1, C_2) denote the vertices of \mathcal{E}_a^* (resp. \mathcal{E}_b^* and \mathcal{E}_c^*). Note that A_1A_2 shares its midpoint with BC , and so on cyclically. Therefore: $AC_2 = BC_1$, $BA_2 = A_1C$, and $CB_1 = B_2A$. To finish the proof, we apply Carnot’s theorem as in Proposition 1. \square

Let a, b, c denote the sidelengths of $\triangle ABC$. Let $\delta_a = |PB| + |PC|$, $\delta_b = |PC| + |PA|$, $\delta_c = |PA| + |PB|$. Referring to Figure 8 (left).

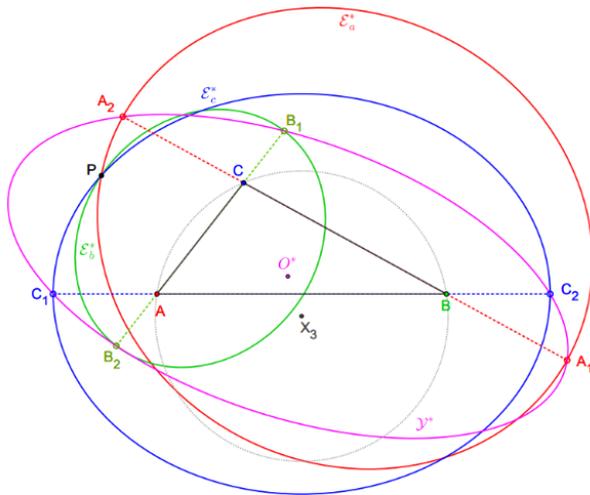


Figure 7: The six vertices A_1, A_2, B_1, B_2, C_1 , and C_2 of P-ellipses $\mathcal{E}_a^*, \mathcal{E}_b^*, \mathcal{E}_c^*$ are on a conic \mathcal{Y}^* . For reference, the circumcenter X_3 of $\triangle ABC$ and the center O^* of \mathcal{Y}^* are also shown.

Proposition 9 There is a unique point P^* such that \mathcal{Y}^* is a circle given by:

$$\begin{aligned} & [(a^2 - \delta_a^2)(c^2 - b^2 + \delta_b^2 - \delta_c^2)]^2 + \\ & [(b^2 - \delta_b^2)(a^2 - c^2 + \delta_c^2 - \delta_a^2)]^2 + \\ & [(c^2 - \delta_c^2)(b^2 - a^2 + \delta_a^2 - \delta_b^2)]^2 = 0 \end{aligned}$$

Furthermore, \mathcal{Y}^* is concentric with the circumcircle of $\triangle ABC$.

Level curves of the above function for a particular triangle are shown in Figure 8 (left). Interestingly, there is a straightforward way to construct a triangle whose \mathcal{Y}^* is a circle.

Definition 3 (anticevian triangle) Given $\triangle ABC$ and a point Q , the Q -anticevian $\triangle A'B'C'$ is such that $\triangle ABC$ is its Q -cevian [16].

Referring to Figure 8 (right), a first “needle in a haystack” find is:

Proposition 10 Given a reference triangle $\triangle ABC$, its X_3 is the P^* of its X_3 -anticevian $\triangle A'B'C'$. Furthermore, (i) \mathcal{Y}^* of the latter is concentric with its circumcircle, and (ii) its center lies on the X_4X_6 line of $\triangle ABC$.

Proof. This needle-in-a-haystack phenomenon was discovered experimentally and then verified using CAS. \square

Barycentric coordinates for the circumcenter X_3' of the X_3 -anticevian appear in Section 6.

While it can be shown that given a generic $\triangle A'B'C'$, there is always a triangle $\triangle ABC$ which is the former’s X_3 -cevian (map is invertible), we don’t yet have a geometric construction for the latter.

A degenerate 6-point conic:

As shown in Figure 9 (left), a simple condition renders \mathcal{Y}^* degenerate, namely:

Proposition 11 If P is on the circumcircle of $\triangle ABC$, \mathcal{Y}^* is degenerate (two straight lines).

Proof. Via CAS, it can be verified that the 3x3 discriminant of the homogeneous equation for the conic vanishes. \square

Referring to Figure 9 (left):

Proposition 12 Over P on the circumcircle, the locus of the center O^* of \mathcal{Y}^* is the union of arcs of three distinct ellipses $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$, all of which pass through the midpoints of ABC . The endpoints of \mathcal{L}_a are one vertex of V-ellipse \mathcal{E}_b and one of \mathcal{E}_c , and so on cyclically for the endpoints of $\mathcal{L}_b, \mathcal{L}_c$.

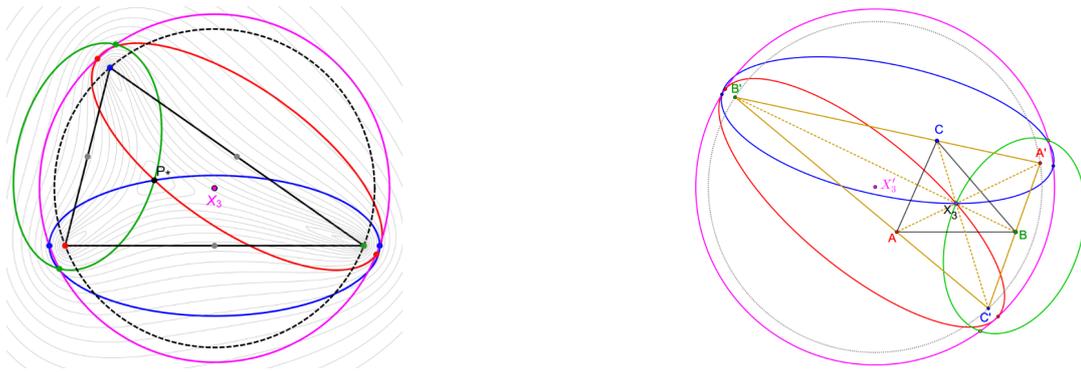


Figure 8: **Left:** Given a triangle, there is a unique P^* such that the 6-point conic (magenta) of a triad of P -ellipses is a circle. The latter is concentric with the circumcircle (dashed black). Also shown are level curves of the functional in Proposition 9: P^* is its unique zero. **Right:** The vertices of a P -ellipse with foci on vertices of the X_3 -anticevian $\triangle A'B'C'$ of $\triangle ABC$, and passing through the latter's X_3 lie on a circle (magenta). The latter is concentric with the circumcircle of $\triangle A'B'C'$ (dashed black) at X_3' .

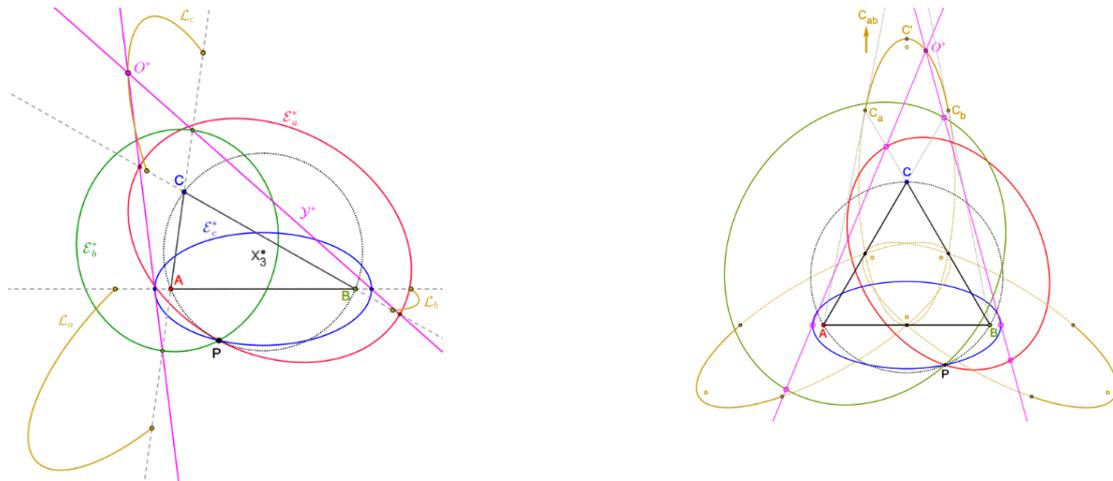


Figure 9: **Left:** If P lies on the circumcircle of $\triangle ABC$, the Yiu conic \mathcal{Y}^* (magenta) is degenerate. Over P on the circumcircle, the locus (gold) of the center O^* of the degenerate conic (magenta lines) is the union of three arcs of ellipse $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$. **Right:** Over all P on the circumcircle of an equilateral $\triangle ABC$, the locus of the center O^* of the degenerate 6-pt conic (magenta) is the union of 3 elliptic arcs (solid gold) centered on A, B, C , whose major axes are the altitudes of $\triangle ABC$. The major (resp. minor) semi-axes measure $|AB| = \sqrt{3}/2$ (resp. $|AB| = \sqrt{3}/6$).

Proof. Referring to Figure 10, that the endpoints of \mathcal{L}_c are a vertex A'' of \mathcal{E}_a and a vertex B'' of \mathcal{E}_b can be seen from the fact that the limit of \mathcal{E}_a^* (resp. \mathcal{E}_b^*) as P approaches A (resp. B) is \mathcal{E}_a (resp. \mathcal{E}_b) and that the center O^* of the degenerate \mathcal{Y}^* will approach the intersection of AA'' and BC . The same argument applies for the endpoints of $\mathcal{L}_a, \mathcal{L}_b$, cyclically. To show that the locus of O^* is the union of three elliptic arcs, we (i) restrict P to a given “third” of the circumcircle, e.g., the arc between A and B . Then (ii) we obtain, via a CAS, a (rather long) symbolic expression for the implicit function $f(x, y)$ representing the ellipse which passes through the 5 proposed points, namely, two vertices of V -ellipses and

the midpoints of the sides of $\triangle ABC$. We then (iii) obtain a parametric expression for O^* as a function of P and plug it into $f(x, y)$, and notice via a CAS, that this simplifies to zero, independent of P . (iv) The same can be repeated cyclically for the other 3 portions of the circumcircle. \square

Referring to Figure 9 (right):

Corollary 1 Let $\triangle ABC$ be an equilateral of side 1. Over P on the circumcircle, the locus of the center O^* of the degenerate \mathcal{Y}^* is the union of arcs of three congruent ellipses with semi-axes $a = \sqrt{3}/2$ and $b = \sqrt{3}/6$, centered on A, B, C .

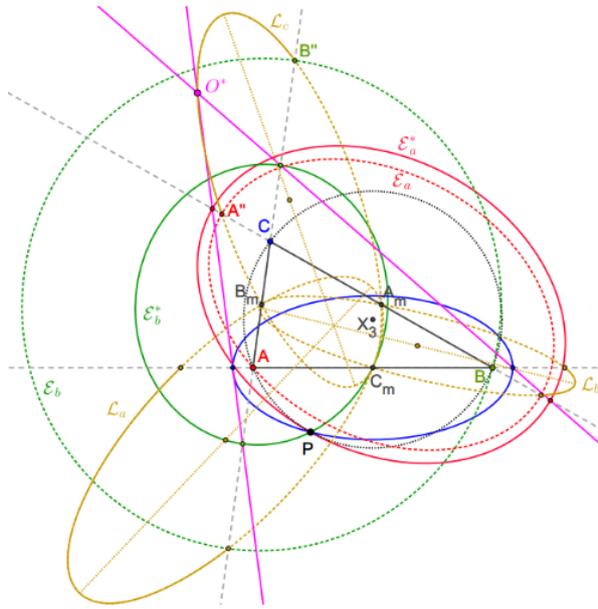


Figure 10: Definitions used in Proposition 12. The locus of O^* is the union of three arcs of ellipse (solid gold) $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$, each of which passes through the 3 midpoints A_m, B_m, C_m of $\triangle ABC$. The endpoints A'', B'' of \mathcal{L}_c are vertices of V-ellipses \mathcal{E}_a and \mathcal{E}_b (dashed red, green). The major axes (dashed gold) of the three loci nearly concur, though not exactly.

Let C_a and C_b denote the endpoints of the elliptic locus of O^* , over P on the arc of the circumcircle below AB . Let C' denote the locus' top vertex. Referring to Figure 9 (right), the following can be shown:

- C_a and C_b are the reflections of the midpoints of AC and BC about C
- lines AC_a and BC_b are tangent to the locus. Let C_{ab} denote their intersection.
- C' is the midpoint between C and C_{ab} .
- Therefore, $C_a C_b$ is the mid-base of $\triangle AC_{ab} B$, therefore the latter is 3 times the area of $\triangle ABC$.

Regions of conic type:

It turns out the type of \mathcal{Y}^* (ellipse, parabola, hyperbola, degenerate) depends on the position of P . The case of an equilateral $\triangle ABC$ is illustrated in Figure 11.

Remark 2 If $\triangle ABC$ is an equilateral, it can be shown that the portions of the locus of P such that \mathcal{Y}^* is: (i) degenerate (deltoid interior to $\triangle ABC$) are branches of 3 regular cubics; (ii) a parabola: branches of a degree-20 polynomial on x, y .

Remark 3 If \mathcal{Y}^* is a hyperbola it can never be a rectangular one.

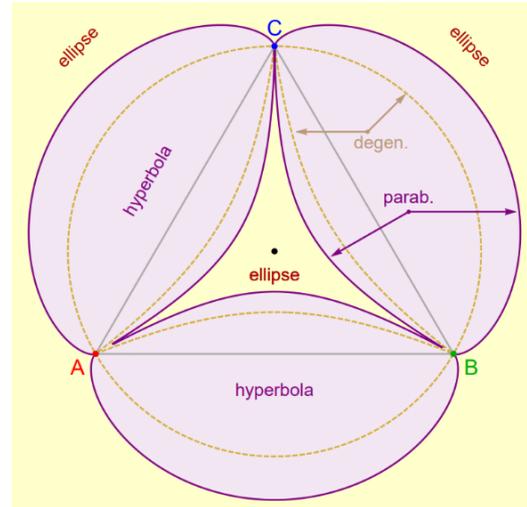


Figure 11: For $\triangle ABC$ an equilateral, the figure illustrated regions of P such that the \mathcal{Y}^* conic is of a given type.

What about the co-vertices?

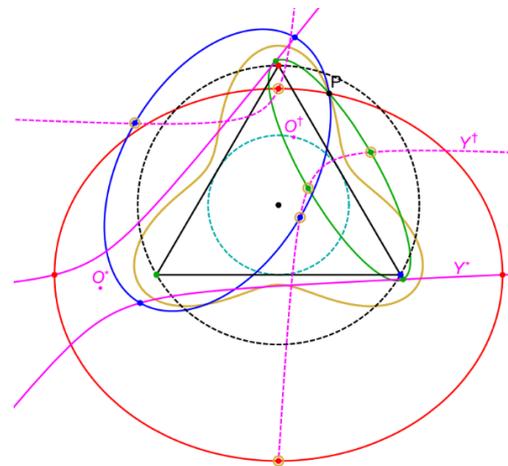


Figure 12: Given an equilateral (black), the locus for P such that the 6 co-vertices of the 3 P -ellipses lie on a conic is a degree-10 algebraic curve (gold) woven symmetrically about the equilateral (there is an isolated point at the centroid as well). The three P -ellipses (red, green, blue) are shown for a specific choice of P on said locus. Also shown are (i) the conic \mathcal{Y}^* (solid magenta, center O^*) through the major vertices, and (ii) the conic \mathcal{Y}^\dagger (dashed magenta, center O^\dagger) through the 6 co-vertices (highlighted by small gold circles). Notice that if P is on the locus, O^\dagger lies on the incircle of the equilateral.

It turns out that for given $\triangle ABC$, there is a 1d locus for P such that the 6 co-vertices lie on a conic. As before, let $\delta_a = |PB| + |PC|$, $\delta_b = |PC| + |PA|$, $\delta_c = |PA| + |PB|$.

Referring to Figure 12:

Proposition 13 *If $\triangle ABC$ is an equilateral, the locus for P such that the 6 co-vertices lie on a conic \mathcal{Y}^\dagger is given by:*

$$\delta_a^2 \delta_b^2 + \delta_a^2 \delta_c^2 + \delta_b^2 \delta_c^2 - 8(\delta_a^2 + \delta_b^2 + \delta_c^2) + 48 = 0$$

Furthermore, the center O^\dagger of \mathcal{Y}^\dagger lies on the incircle of the equilateral.

Note: if one eliminates all square roots involved in computing $\delta_a, \delta_b, \delta_c$, the above becomes a degree-10 equation on x, y .

4 A triad of V-hyperbolas

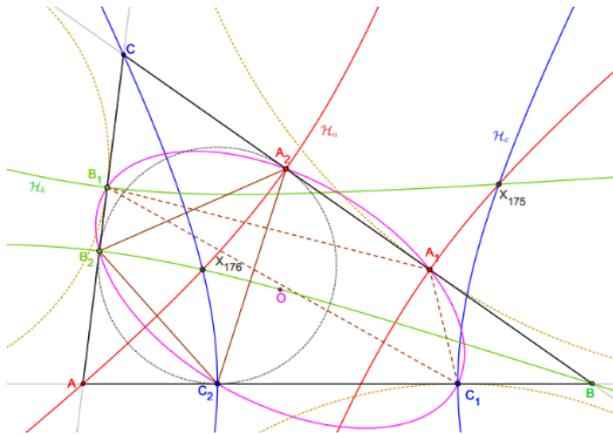


Figure 13: A triad of V-hyperbolas $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ (red, green, blue) is shown with foci on (B, C) , (C, A) , and (A, B) passing through A, B, C , respectively. Notice (i) their vertices taken as triples $A_1B_1C_1$ and $A_2B_2C_2$ are the vertices of the extouch (dashed brown) and intouch (solid brown) triangles; (ii) these 6 points are known to lie on the Privalov conic (magenta), whose center O is X_{5452} on [8]; (iii) the 3 hyperbolas pass through both the “isoperimetric” and “equal detour” points, i.e., X_{175} and X_{176} , respectively. Note: these coincide when the outer Soddy circle is external to the three mutually tangent circles.

In this section we describe properties – some old, some new – of a special triad of hyperbolas, described in [17, Sec. 12.4, p. 148] where they are called “Soddy” hyperbolas. Referring to Figure 13:

Definition 4 (V-hyperbolas) *Given a triangle $\triangle ABC$, a triad of V-hyperbolas $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ have foci on (B, C) , (C, A) , (A, B) and pass through A, B , and C , respectively.*

²These pass through the vertices of the intouch triangle.

Let A_1, A_2 be the vertices of \mathcal{H}_a . Define B_1, B_2 and C_1, C_2 for $\mathcal{H}_b, \mathcal{H}_c$, respectively. Recall the extouch (resp. intouch) triangle is where the 3 excircles (resp. incircle) touch a triangle’s sides.

Remark 4 *Let $\lambda_a = |AB| - |AC|$. In barycentric coordinates for the vertices of \mathcal{H}_a are given by: $A_1 = [0, a + \lambda_a, a - \lambda_a]$, and $A_2 = [0, a - \lambda_a, a + \lambda_a]$, with the others computed cyclically.*

Corollary 2 $\triangle A_1B_1C_1$ (resp. $\triangle A_2B_2C_2$) is the extouch (resp. intouch) triangle of $\triangle ABC$.

Recall that for any triangle, the intouch and extouch triangles have the same area [16, extouch triangle]. Referring to [8, X(5452)]:

Corollary 3 $A_1, A_2, B_1, B_2, C_1, C_2$ lie on the Privalov conic centered on X_{5452} , and whose barycentric coordinates x, y, z satisfy:

$$k_1 k_2 k_3 (x^2 + y^2 + z^2) + 2 [k_2 (k_4 - 2ab)xy + k_3 (k_4 - 2ac)xz - k_1 (k_4 - 2bc)yz] = 0$$

where a, b, c are the sidelengths of $\triangle ABC$, $k_1 = (a - b - c)$, $k_2 = (a + b - c)$, $k_3 = (a - b + c)$, and $k_4 = a^2 + b^2 + c^2$.

Remark 5 *When $\triangle ABC$ is isosceles, one of the V-hyperbolas is degenerate, namely, a pair of coinciding lines at the perpendicular bisector of the base. In this case, the Privalov conic is tangent to the base at its midpoint.*

Intersections between V-hyperbolas:

Referring to Figure 15, recall that given a triangle, one can construct² three “kissing” circles C_A, C_B , and C_C centered each on each vertex, and externally tangent to each other [14].

The Apollonius’ problem for this triple has (as usual) eight distinct solutions, two of which have the same tangency type (tangent externally or internally to all three circles).

Definition 5 (Soddy circles of a triangle) *The two solutions for the Apollonius’ problem with the same tangency type are the so-called “Soddy circles”. The inner Soddy circle is the one whose center is inside the triangle and whose interior does not intersect any of the three kissing circles; the other one is the outer Soddy circle.*

Note that the outer Soddy circle, always tangent to the 3 kissing circles, can either (i) contain them (see Figure 15), (ii) be a line tangent to them, or (iii) be externally tangent to them. For (ii) and (iii) see Figure 16.

The centers of Soddy circles correspond to a pair of triangle centers found in [8] and derived in [7]. Namely:

Definition 6 (Isoperimetric point) *The center of the outer Soddy circle (X_{175} in [8]). Equivalently, the unique point X such that:*

$$|XB| + |XC| \pm |BC| = |XC| + |XA| \pm |CA| = |XA| + |XB| \pm |AB|$$

where the positive (resp. negative) sign is chosen if the outer Soddy circle contains (resp. is external to) the three mutually tangent circles in Definition 5. As derived in [7], containment corresponds to:

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} < 2.$$

In [7] it is shown that if the sum of half-tangents is exactly 2, then the outer Soddy circle degenerates to a line. Referring to Figure 14, it can be shown that:

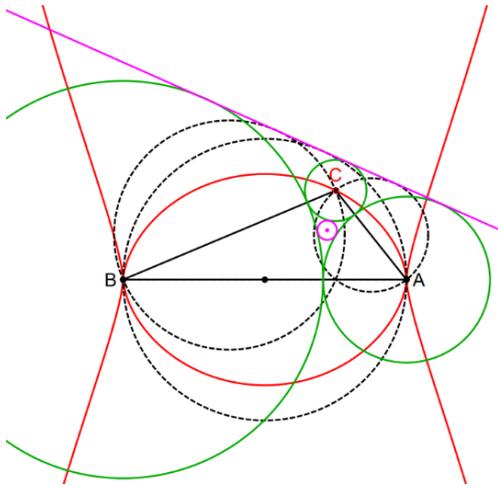


Figure 14: When A and B are fixed, the locus of C (red) such that the outer Soddy circle degenerates to a line (magenta) is given by the degree-6 implicit equation in Proposition 14. This line is also tangent to the 3 circles (dashed black) whose diameters are the sides of $\triangle ABC$.

Proposition 14 *Without loss of generality, let $A = (-1, 0)$, $B = (1, 0)$, the locus of C such that the sum of half-tangents of $\triangle ABC$ is 2 is given by the union of the following degree-6 polynomial and its reflection about the x -axis:*

$$-4x^6 - 4x^4(2y^2 + 2y + 1) - 4x^2(y^4 + y^3 - 4y - 5) + 4y^5 + 13y^4 + 20y^3 + 8y^2 - 8y - 12 = 0.$$

Definition 7 (Equal detour point) *The center of the inner Soddy circle (X_{176} in [8]), always internal to a triangle. Also the unique point X in $\triangle ABC$ such that:*

$$|XB| + |XC| - |BC| = |XC| + |XA| - |CA| = |XA| + |XB| - |AB|.$$

Proposition 15 *The three V -hyperbolas intersect at the centers of the two Soddy circles, i.e., X_{175} and X_{176} , respectively.*

Proof. Assume that $a > c > b$ as in Figure 13. Then $r_a < r_c < r_b$. C_a and C_b are two circles centered at A and B and of radii $r_a < r_b$, which are externally tangent at C_2 . The locus of the centers of the circles that are externally tangent to both C_a and C_b is the branch of the hyperbola with foci on A and B , that passes through their tangency point C_2 . The other branch contains the centers of the circles that are internally tangent (i.e., contain both). The internal Soddy circle is externally tangent to the three circles C_a , C_b , and C_c ; hence its center is necessarily the intersection of the three branches of hyperbolas passing through A_2, B_2, C_2 , the vertices of the intouch triangle. Since $r_a < r_c < r_b$, we may specify those branches as $\mathcal{H}_a^+ = \{P : |PB| - |PC| = r_b - r_c\}$, $\mathcal{H}_b^+ = \{P : |PC| - |PA| = r_c - r_a\}$, and $\mathcal{H}_c^+ = \{P : |PA| - |PB| = r_a - r_b\}$.

Thus, if a point $P \in \mathcal{H}_a^+ \cap \mathcal{H}_c^+$ then $PA - PC = r_a - r_c$ hence it is on \mathcal{H}_b^+ as well. Since $r_c + r_a = b$ and $r_b + r_a = c$, P verifies the equal-detour definition of X_{176} .

The points on the other branches contain centers of circles that are internally tangent to the other two; therefore, if two branches, say \mathcal{H}_a^- and \mathcal{H}_b^- have a common point P , then, as above, P is also on the third branch and is the (unique) center of an external Soddy circle, that contains C_a , C_b , and C_c . In this case, P verifies the isoperimetric definition of X_{175} .

In contrast, if \mathcal{H}_a^- and \mathcal{H}_b^- do not intersect, then there will be no “negative branch” intersection. In this case, the three positive branches will intersect in two distinct points: the centers of the inner and outer Soddy circles. Note that each pair $(\mathcal{H}_a, \mathcal{H}_b)$, $(\mathcal{H}_b, \mathcal{H}_c)$, and $(\mathcal{H}_c, \mathcal{H}_a)$ have one common focus C, A, B respectively; hence they necessarily have four (real) intersections. This guarantees the existence of both detour and isoperimetric points. \square

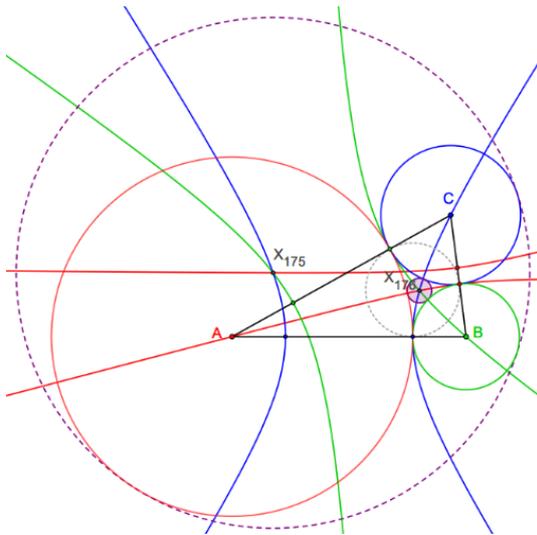


Figure 15: A construction found in [17, Sec. 12.4, p. 148]: 3 mutually-tangent circles (red, green, blue) of $\triangle ABC$ touch at the contact points of the incircle (dashed gray) with the sides. In turn, these coincide with a vertex of each the 3 V-hyperbolas. Notice the latter intersect at the centers X_{176} and X_{175} of the inner (shaded purple), and outer (dashed purple) Soddy circles, respectively.

Referring to Figure 17:

Proposition 16 The \mathcal{H}_a V-hyperbola passes through the intersections A' and A'' of V-ellipses \mathcal{E}_b , and \mathcal{E}_c . The same holds for $\mathcal{H}_b, \mathcal{H}_c$, cyclically.

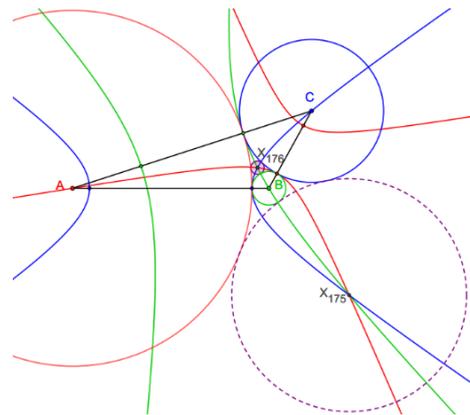
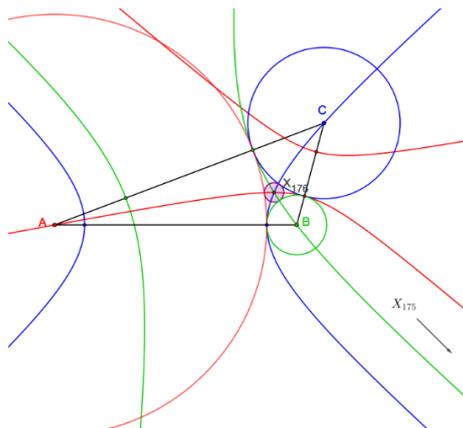


Figure 16: Two cases of $\triangle ABC$ such that the external Soddy circle (dashed purple) is: (left) a straight line ($\sum \tan(\theta_i) = 2$), and (right) does not contain the three kissing circles. Notice that in both cases the three V-hyperbolas (red, green and blue) intersect at the center X_{176} of the inner Soddy circle (shaded purple), interior to the triangle. In the first case their second intersection is at infinity (in the direction perpendicular to the Soddy line), while in the second case they intersect along the same branches where their X_{176} intersection lies.

Proof. Referring to Figure 17, let A'' denote an intersection of \mathcal{E}_b with \mathcal{E}_c . Then:

$$|A''A| + |A''B| = |CA| + |CB|, \quad |A''A| + |A''C| = |BA| + |BC|.$$

Subtracting, $|A''C| - |A''B| = |BA| - |CA|$, meaning that A'' lies on the branch of hyperbola \mathcal{H}_a not containing A . \square

Still referring to Figure 17, let A_1 and A_2 denote the two 2-branch intersections between \mathcal{H}_b and \mathcal{H}_c , define B_1, B_2 and C_1, C_2 cyclically.

Proposition 17 The three lines A_1A_2, B_1B_2 and C_1C_2 concur at the Nagel point X_8 of $\triangle ABC$.

Proof. Let a, b, c denote the sidelines. The barycentrics of A_1 are given by:

$$A_1 : [(L-2a)(3a^2 + 2ab - b^2 + 2ac - 2bc - c^2 + 2(b-c)\gamma), \\ (L-2c)(3a - 3b + c)L + 2(a^2 - c^2 - 2b^2 + ab - bc)\gamma, \\ (L-2b)(3a + b - 3c)L - 2(a^2 - b^2 - 2c^2 + ac + bc)\gamma]$$

where $L = a + b + c$ and $\gamma = 3a^2 + 2ab - b^2 + 2ac - 2bc - c^2$. The barycentrics of A_2 are obtained by replacing γ with $-\gamma$. The barycentrics of points on A_1A_2 satisfy:

$$(-2a^2 + ab + b^2 + ac - 2bc + c^2)x + \\ + (a - c)(L - 2a)y + (a - b)(L - 2a)z = 0$$

and cyclically for B_1B_2 and C_1C_2 . It can be shown that the 3 lines pass through X_8 , whose barycentric coordinates are $[b + c - a, a + c - b, a + b - c]$, see [8]. \square

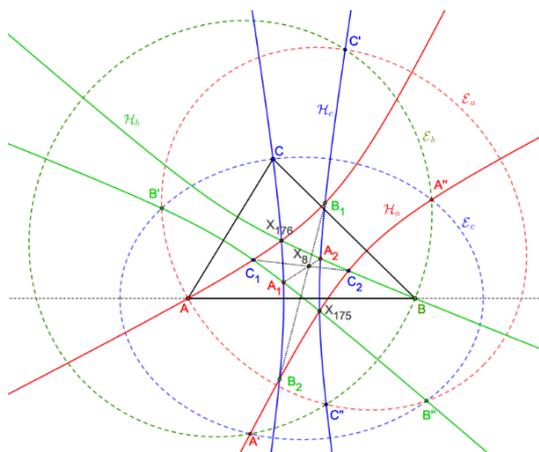


Figure 17: Each V-hyperbola passes through the two intersections between pairs of V-ellipses. As an example, consider A'' , common to \mathcal{H}_a , \mathcal{E}_b , and \mathcal{E}_c . Also shown is the fact that the 3 segments A_1A_2 , B_1B_2 , and C_1C_2 connecting opposing 2-branch intersections of the 3 V-hyperbolas concur at X_8 .

Referring to Figure 18:

Remark 6 On side BC there lie the 2 vertices of \mathcal{E}_a and the 2 of \mathcal{H}_a . Consider the degenerate cubic which is the union of the sidelines of $\triangle ABC$. It is a 15-point cubic since it passes through (i) the three vertices of the triangle, (ii) the 6 vertices of the V-ellipses, and (iii) the 6 vertices of the V-hyperbolas.

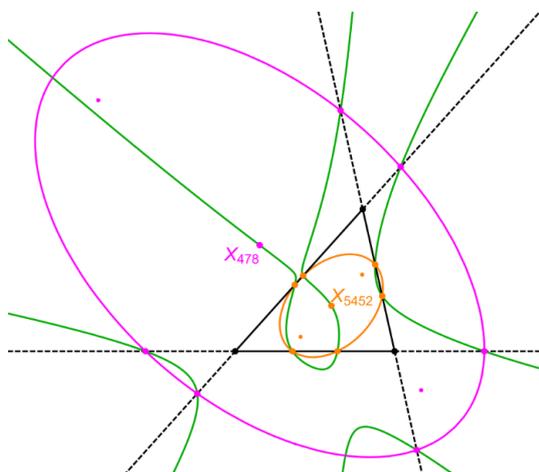


Figure 18: Given a triangle (black), the union of the 3 sidelines (dashed black) can be regarded as a 15-point degenerate cubic. It passes through (i) the triangle vertices, (ii) the 6 points on the Yiu conic (magenta), and (iii) the 6 points on the Privalov conic (orange). Just for fun, also shown are branches of the 14-point quartic (green) that passes through the 12 points on the Yiu+Privalov as well as their centers X_{478} and X_{5452} , respectively.

Referring to Figure 14:

Proposition 18 When the external Soddy circle degenerates to a line, the three circles whose diameters are the sides of $\triangle ABC$ are also tangent to it.

Proof. Let A_T, B_T be the tangency points of the degenerate Soddy circle \mathcal{L} (a line) with circles C_a, C_b , and let T be the tangency point between the latter two. The perpendicular dropped from T onto line AB meets \mathcal{L} at M . Then, owing to properties of tangents from a point to a circle, $|MT| = |MA_T| = |MB_T|$. Since B_T, M, A_T are collinear, then $\angle A_T T B_T = 90^\circ$. On the other hand, since MT and MA_T are tangents from M to C_a , $MA \perp TA_T$ and similarly, $MB \perp TB_T$. $\angle AMB = 90^\circ$. Hence, if O is the midpoint of AB then $OM = AO = OB$. Finally, the quadrilateral $[ABB_T A_T]$ is a trapezium (AA_T, BB_T are perpendicular to \mathcal{L}) and OM is its mid-base. Hence OM is also perpendicular to \mathcal{L} at the midpoint M of $A_T B_T$. Therefore the circle of diameter AB is tangent to \mathcal{L} at M , and so on cyclically for (C_b, C_c) and (C_a, C_c) . \square

5 A triad of P-hyperbolas

We now extend V-hyperbolas to a trio with respect to a point P . Referring to Figure 19:

Definition 8 (P-hyperbolas) A triad of P-hyperbolas $\mathcal{H}_a^*, \mathcal{H}_b^*, \mathcal{H}_c^*$ with respect to $\triangle ABC$ have foci on (B, C) , (C, A) , (A, B) and pass through a given point P .

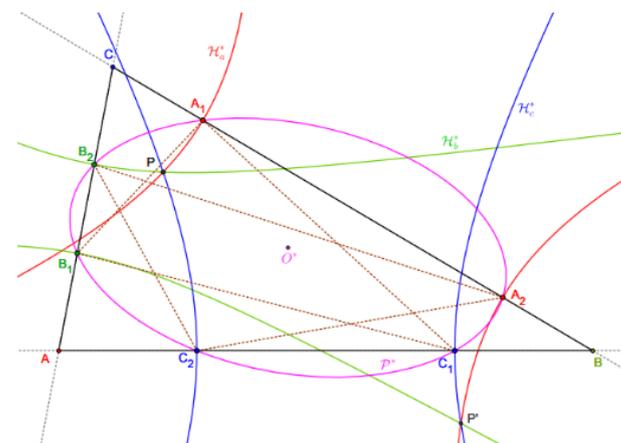


Figure 19: Given a point P , (i) the triad of P-hyperbolas $\mathcal{H}_a^*, \mathcal{H}_b^*, \mathcal{H}_c^*$ has a second common point P' ; (ii) their vertices $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a conic \mathcal{P}^* (magenta), not necessarily an ellipse; (iii) $\triangle A_1 B_1 C_1$ has the same area as $\triangle A_2 B_2 C_2$.

Still referring to Figure 19:

Proposition 19 Besides P , the triad of P -hyperbolas meets at a second real point P' .

Proof. Let \mathcal{H}_a^{*+} , \mathcal{H}_b^{*+} , \mathcal{H}_c^{*+} denote the three branches that pass through P . We need to prove that the other three branches \mathcal{H}_a^{*-} , \mathcal{H}_b^{*-} , \mathcal{H}_c^{*-} also meet at some point.

First, let us show that two other branches \mathcal{H}_b^{*-} and \mathcal{H}_c^{*-} must intersect. To prove it, we perform a polar dual with respect to a circle centered at their common focus A , as shown in Figure 20. The polar dual of each hyperbola will be a circle, whose diameter is delimited by the inverses of hyperbola vertices. By polarity, the intersection points of the original hyperbolas are sent to the common tangents of their reciprocal circles and vice-versa; since, by hypothesis, \mathcal{H}_b^* and \mathcal{H}_c^* intersect at a point P , these reciprocal circles admit (at least) one common tangent. Hence, they are either externally tangent or secant. Therefore, these circles admit at least two common tangents. One of these tangents is precisely the polar of P ; the other one, passing through the same homothety center, is the polar of a point P' which is the intersection of the other two branches, \mathcal{H}_b^{*-} and \mathcal{H}_c^{*-} . Similarly, branches \mathcal{H}_a^{*-} and \mathcal{H}_c^{*-} also intersect. Now, as in Figure 19, if a point $P' \in \mathcal{H}_b^{*-} \cap \mathcal{H}_c^{*-}$, then it satisfies $P'C - P'A = B_1B_2$ and $P'A - P'B = C_1C_2$. Hence, by adding these two relations, we obtain $P'C - P'B = B_1B_2 + C_1C_2$. Nevertheless, by hypothesis P is the common point of three branches: \mathcal{H}_a^{*+} , \mathcal{H}_b^{*+} , \mathcal{H}_c^{*+} . Then three similar relations can be written for P : $PA - PC = B_1B_2$, $PB - PA = C_1C_2$, and $PB - PC = A_1A_2$. By adding the first two, we obtain $PB - PC = B_1B_2 + C_1C_2$, hence $B_1B_2 + C_1C_2 = A_1A_2$. The later relation ensures that $P'C - P'B = A_1A_2$, hence $P' \in \mathcal{H}_a^{*-}$ finishing the proof. \square

Proposition 20 The 6 vertices of the 3 P -hyperbolas lie on a conic \mathcal{P}^* .

Proof. Referring to Figure 19, by definition, the center of the P -hyperbola \mathcal{H}_a^* is at the midpoint of BC , and so on cyclically. Hence:

$$|A_1C| = |A_2B| = x, \quad |B_1A| = |B_2C| = y, \quad |BC_1| = |AC_2| = z. \tag{4}$$

We obtain the claim using Carnot's theorem. \square

Recall the classic result that for any triangle, the intouch and extouch triangles have the same area (we saw this in Colloraly 2 in the context of V -hyperbolas). The analogous result for P -hyperbolas still holds:

Proposition 21 Let A_1, A_2 denote the vertices of \mathcal{H}_a^* , and B_1, B_2, C_1, C_2 , those of \mathcal{H}_b^* and \mathcal{H}_c^* , respectively. Then $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ have the same area.

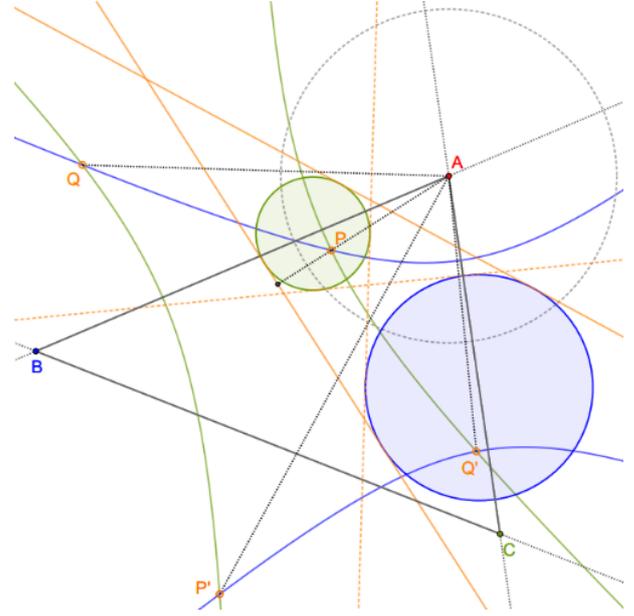


Figure 20: Two P -hyperbolas (green, blue) sharing a focus at A are shown as well as their reciprocals (shaded green and blue circles) with respect to an inversion circle centered at A (dashed black). The branches closer (resp. further) to A always intersect; their intersection points P and P' are the poles of the common external tangents to their reciprocal circles. When these circles are disjoint (as in the figure), the poles of the internal common tangents are intersections between alternate branches.

Proof. This is again a consequence of (4). Specifically, let $\vec{a} = \vec{BC}$, $\vec{b} = \vec{CA}$, $\vec{c} = \vec{AB}$. Let α, β , and γ be such that:

$$\vec{BA}_1 = \alpha\vec{a}, \quad \vec{CB}_1 = \beta\vec{b}, \quad \vec{AC}_1 = \gamma\vec{c};$$

In order to prove that $S_{A_1B_1C_1} = S_{A_2B_2C_2}$, we simply show that they represent the same fraction of $S = S_{ABC}$. In fact, $S_{A_1B_1C_1} = S - [S_A + S_B + S_C]$, where $S_A = S_{AB_1C_1}$, $S_B = S_{BA_1C_1}$, and $S_C = S_{CA_1B_1}$. A direct computation yields:

$$\begin{aligned} S_A &= S_{AB_1C_1} = \\ &= \frac{1}{2} \|\vec{AB}_1 \times \vec{AC}_1\| = \frac{1}{2} \|(1 - \beta)\vec{b} \times (\gamma\vec{c})\| = \gamma(1 - \beta)S. \end{aligned}$$

Cyclically, $S_B = \alpha(1 - \gamma)S$, and $S_C = \beta(1 - \alpha)S$. Therefore:

$$\begin{aligned} S_{A_1B_1C_1} &= S - [S_A + S_B + S_C] = \\ &= S[1 - \gamma(1 - \beta) - \alpha(1 - \gamma) - \beta(1 - \alpha)]. \end{aligned}$$

Similarly:

$$S_{A_2B_2C_2} = S - [S'_A + S'_B + S'_C]$$

where $S'_A = S_{AB_2C_2}$, $S'_B = S_{BA_2C_2}$, $S'_C = S_{CA_2B_2}$. Then:

$$S'_A = S_{AB_2C_2} = \frac{1}{2} \|\vec{AB}_2 \times \vec{AC}_2\| = \frac{1}{2} \|\beta \vec{b} \times ((1-\gamma)\vec{c})\| = \beta(1-\gamma)$$

and cyclically for S'_B, S'_C . Thus, the area of $S_{A_2B_2C_2}$ can be computed as $S_{A_1B_1C_1}$, where α, β, γ are replaced at each occurrence by $(1-\alpha), (1-\beta), (1-\gamma)$. Thus:

$$S_{A_2B_2C_2} = S - [S'_A + S'_B + S'_C] = S[1 - (1-\gamma)\beta - (1-\alpha)\gamma - \alpha(1-\beta)].$$

Referring to Figure 21:

Proposition 22 Given a $\triangle ABC$ there is a unique pair of distinct points P^* and Q^* such that the 6-point conic \mathcal{P}^* is a circle. These are a pair of common intersections of the triad of P-hyperbolas. It can be shown their barycentrics satisfy:

$$\begin{aligned} & [(c^2 - \lambda_c^2)(-a^2 + b^2 + \lambda_a^2 - \lambda_b^2)]^2 + \\ & [(b^2 - \lambda_b^2)(-a^2 + c^2 + \lambda_a^2 - \lambda_c^2)]^2 + \\ & [(a^2 - \lambda_a^2)(-b^2 + c^2 + \lambda_b^2 - \lambda_c^2)]^2 = 0 \end{aligned}$$

where $\lambda_a = |PB| - |PC|$, $\lambda_b = |PC| - |PA|$, and $\lambda_c = |PA| - |PB|$.

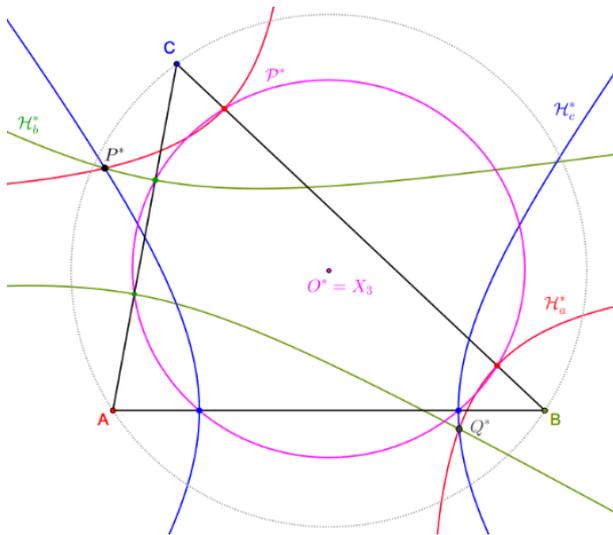


Figure 21: Given $\triangle ABC$, there is a pair P^* and Q^* such that \mathcal{P}^* (magenta) is a circle. Furthermore the latter is concentric with the circumcircle (dashed black) of $\triangle ABC$.

Definition 9 (reflection triangle) The reflections A', B', C' of a point Q on the sides of $\triangle ABC$ are the vertices of the Q -reflection triangle.

Surprisingly, we can construct a triangle such that the vertices of the 6 P-hyperbolas lie on a circle. In [8], center X_{55} is the internal center of similitude of the incircle and circumcircle.

Referring to Figure 22, experimental evidence supports the following “needle in a haystack” phenomenon:

Conjecture 1 Let T' be the X_{55} -reflection triangle of a reference triangle T . The 6 vertices of the P-hyperbolas of T' passing through X_{55} -of- T lie on a circle, concentric with the circumcircle of T' which coincides with X_7 -of- T

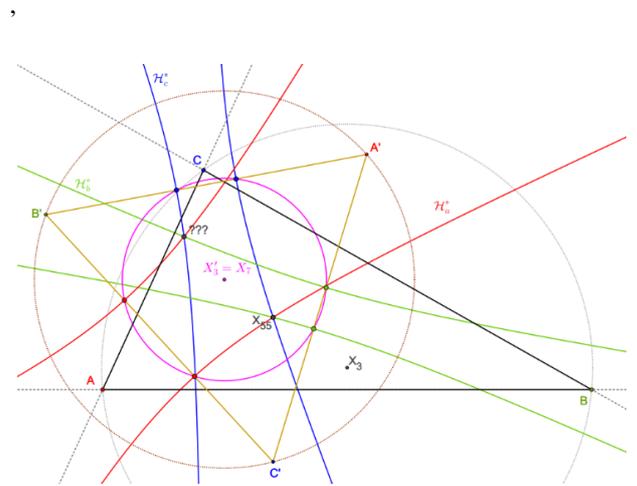


Figure 22: The vertices of P-hyperbolas \mathcal{H}_a^* , \mathcal{H}_b^* , and \mathcal{H}_c^* (red, green, blue) passing through X_{55} of $\triangle ABC$, with foci on pairs of vertices of the X_{55} -reflection triangle $\triangle A'B'C'$ (gold), lie on a circle (magenta), concentric with the circumcircle (dotted brown) of $\triangle A'B'C'$, whose circumcenter is the Gergonne point X_7 of the reference. Note that said 3 hyperbolas meet at a second mystery point “???”.

Referring to Figure 23:

Proposition 23 The \mathcal{H}_a^* P-hyperbola passes through the non-P intersection A' between P-ellipses \mathcal{E}_b^* , and \mathcal{E}_c^* . The same holds for $\mathcal{H}_b^*, \mathcal{H}_c^*$, cyclically.

Proof. Referring to Figure 23, let $A' \in \mathcal{E}_b^*$; then $A'A + A'C = PA + PC$. If A' is also contained in \mathcal{E}_c^* , then $A'A + A'B = PA + PB$. Subtracting $A'B - A'C = PB - PC$, i.e., both A' and P lie on the same branch of $\in \mathcal{H}_a^*$. \square

As before, let a, b, c be the sidelengths, and λ_a, λ_b , and λ_c as above. As shown in Figure 24, the plane of a $\triangle ABC$ can be split into zone where \mathcal{P}^* is an ellipse, a hyperbola, or a parabola. In particular:

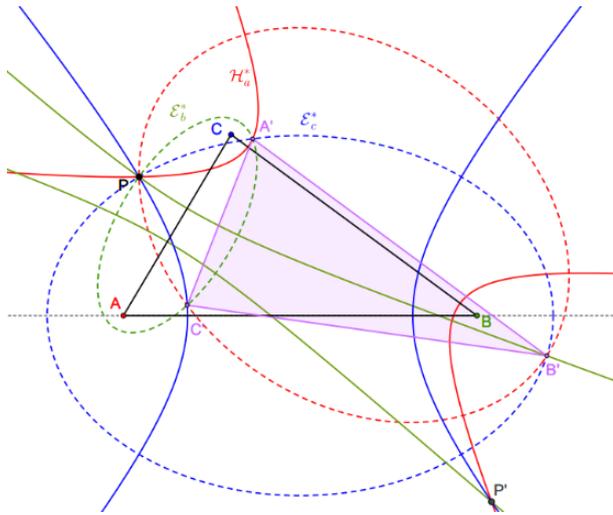


Figure 23: The 3 P -hyperbolas (solid red, green, blue) also pass through the 3 non- P intersections A', B', C' between pairs of P -ellipses, e.g., \mathcal{H}_a^* passes through the intersection A' of \mathcal{E}_b^* and \mathcal{E}_c^* . The triangle with vertices on A', B', C' is shown (purple).

Proposition 24 The conic \mathcal{P}^* through the 6 vertices of the P -hyperbolas is a parabola if:

$$(c^2\lambda_a^2\lambda_b^2)^2 + (b^2\lambda_a^2\lambda_c^2)^2 + (a^2\lambda_b^2\lambda_c^2)^2 + \mu(\mu(-3 + 4(\lambda_a^2/a^2 + \lambda_b^2/b^2 + \lambda_c^2/c^2)) + 2\lambda_a^2\lambda_b^2\lambda_c^2(6 - \lambda_a^2/a^2 - \lambda_b^2/b^2 - \lambda_c^2/c^2) - 6(c^2\lambda_a^2\lambda_b^2 + b^2\lambda_a^2\lambda_c^2 + a^2\lambda_b^2\lambda_c^2)) = 0$$

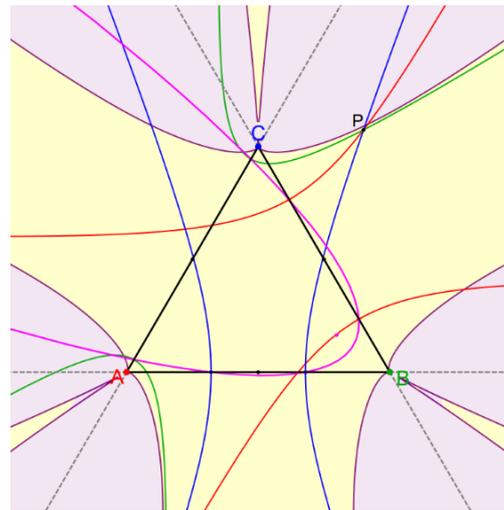
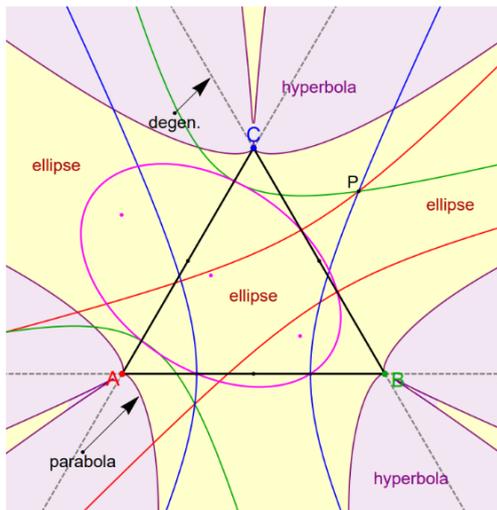


Figure 24: The 6-point conic \mathcal{P}^* (magenta) through the vertices of P -hyperbolas (red, green, blue) is an ellipse if P lies in the yellow (resp.) purple region. It is degenerate if P is on any sideline (dashed black). In the left (resp. right) P is in the yellow region (at the interface) and therefore \mathcal{P}^* (magenta) is an ellipse (resp. parabola).

where $\mu = (abc)^2$. Furthermore, \mathcal{P}^* is degenerate if P lies on either (infinite extension) of the sidelines of the triangle.

6 Open Questions

- Figure 4: what is the locus of the focus of the Yiu conic over C along the parabola locus?
- Figure 6: prove the 6-point co-vertex conic is always a hyperbola and explain why there are two (3:3 and 5:1) distributions of co-vertices over the branches of the conic.
- Figure 8 (left): Prove P^* is unique.
- Figure 8 (right): given $\triangle A'B'C'$ can one always find an inscribed $\triangle ABC$ such that the former is its X_3 -anticevian triangle?
- Figure 11: how do the zones of 6-vertex conic type deform as one moves C away from the equilateral configuration? What is the locus of the center O^* of the degenerate conic over P on the 3 branches of the inner deltoid? What is the locus of the focus of the conic over the 6 arcs where the conic is a parabola? Prove if a hyperbola, said conic can never be rectangular.
- Figure 12: Prove that if P is on the locus, the center O^\dagger of the co-vertex conic is on the incircle. What does the locus of P look like if $\triangle ABC$ is not an equilateral? Over P on said locus, what is the locus of O^\dagger ?

- prove Conjecture 1. Provide an expression for the second triple intersection point of the 3 P-hyperbolas.
- Figure 18: What are interesting loci for C (with A, B fixed) with respect to properties and/or degeneracies of the 14-point quartic?
- Figure 19: describe the map $P \rightarrow P'$ and/or $P \rightarrow O^*$? What is the image of a lattice under it?
- Figure 22: given $\triangle A'B'C'$ can one always find a $\triangle ABC$ such that the former is its X_{55} -reflection triangle?

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Long Barycentric Equations

Here we provide barycentric equations and coordinates (with respect to the reference $\triangle ABC$) of various associated objects. Let a, b, c be the reference’s sidelengths. Let \mathcal{E}_a^\dagger denote the ellipse with foci on $B'C'$, and through A of the reference. Note: the long expression below were kept verbatim so as to facilitate copy-paste.

A-ellipse:

The barycentrics $[x, y, z]$ of \mathcal{E}_a^\dagger satisfy:

$$8b^4c^4(a^2+b^2-c^2)(a^2-b^2+c^2)(a^6b^4-3a^4b^6+3a^2b^8-b^{10}+3a^4b^4c^2-6a^2b^6c^2+3b^8c^2+a^6c^4+3a^4b^2c^4+6a^2b^4c^4-2b^6c^4-3a^4c^6-6a^2b^2c^6-2b^4c^6+3a^2c^8+3b^2c^8-c^{10})x^2+4b^2c^4(a^2+b^2-c^2)^2(a^2-b^2+c^2)(a^8b^2-2a^6b^4+2a^2b^8-b^{10}-a^8c^2-2a^6b^2c^2+10a^4b^4c^2-10a^2b^6c^2+3b^8c^2+4a^6c^4+4a^4b^2c^4+10a^2b^4c^4-2b^6c^4-6a^4c^6-6a^2b^2c^6-2b^4c^6+4a^2c^8+3b^2c^8-c^{10})x^2+y^2(a^2-b^2-c^2)(a^2+b^2+c^2)(a^12-7a^8b^4+16a^6b^6-21a^4b^8+16a^2b^{10}-5b^{12}-6a^{10}c^2-4a^8b^2c^2+12a^6b^4c^2+24a^4b^6c^2-38a^2b^8c^2+12b^{10}c^2+15a^8c^4+16a^6b^2c^4+6a^4b^4c^4+32a^2b^6c^4-5b^8c^4-20a^6c^6-24a^4b^2c^6-20a^2b^4c^6-8b^6c^6+15a^4c^8+16a^2b^2c^8+9b^4c^8-6a^2c^{10}-4b^2c^{10}+c^{12})y^2-4b^4c^2(a^2+b^2-c^2)(a^2-b^2+c^2)^2(a^8b^2-4a^6b^4+6a^4b^6-4a^2b^8+b^{10}-a^8c^2+2a^6b^2c^2-4a^4b^4c^2+6a^2b^6c^2-3b^8c^2+2a^6c^4-10a^4b^2c^4-10a^2b^4c^4+2b^6c^4+10a^2b^2c^6+2b^4c^6-2a^2c^8-3b^2c^8+c^{10})xyz-2a^2b^2c^2(a^2-b^2-c^2)(a^2+b^2+c^2)(a^2-b^2+c^2)(a^{10}-3a^8b^2+2a^6b^4+2a^4b^6-3a^2b^8+b^{10}-3a^8c^2+8a^6b^2c^2-14a^4b^4c^2+16a^2b^6c^2-7b^8c^2+2a^6c^4-14a^4b^2c^4-26a^2b^4c^4+6b^6c^4+2a^4c^6+16a^2b^2c^6+6b^4c^6-3a^2c^8-7b^2c^8+c^{10})y^2+z^2(a^2-b^2-c^2)(a^2-b^2+c^2)(a^{12}-6a^{10}b^2+15a^8b^4-20a^6b^6+15a^4b^8-8a^2b^{10}+b^{12}-4a^8c^2+16a^6b^2c^2-24a^4b^4c^2+16a^2b^6c^2-4b^8c^2-7a^8c^4+12a^6b^2c^4+6a^4b^4c^4-20a^2b^6c^4+9b^8c^4+16a^6c^6+24a^4b^2c^6+2a^2b^4c^6-8b^6c^6-6a^4c^8-38a^2b^2c^8-5b^4c^8+16a^2c^{10}+12b^2c^{10}-5c^{12})z^2 = 0$$

Major vertices:

Let S be twice the area of the reference and:

$$rt = \sqrt{(a^6-3a^2b^4+2b^6+6a^2b^2c^2-2b^4c^2-3a^2c^4-2b^2c^4+2c^6)}$$

The two major vertices of \mathcal{E}_a^\dagger are given by:

$$[(a(a^2-b^2-c^2)((b^2-c^2)(a^4b^2-2a^2b^4+b^6+a^4c^2+4a^2b^2c^2-b^4c^2-2a^2c^4-b^2c^4+c^6)+/-2a^3S\sqrt{rt})/(a(a^4b^2-2a^2b^4+b^6+a^4c^2+4a^2b^2c^2-b^4c^2-2a^2c^4-b^2c^4+c^6)+/-2(b^2-c^2)S\sqrt{rt}), b^2(-a^2+b^2-c^2), -(c^2(-a^2-b^2+c^2))]$$

P-ellipse 6-point circle

The center X_3' of the 6-point circle of Proposition 10 lies on the Van Aubel line (X_4X_6) of the reference. It can be regarded as the circumcenter of the X_3 -anticevian and is given by barycentrics $[f(a, b, c), f(b, c, a), f(c, a, b)]$ where $f(a, b, c)$ is given by:

$$(a^{14}-5a^{12}b^2+9a^{10}b^4-5a^8b^6-5a^6b^8+9a^4b^{10}-5a^2b^{12}+b^{14}-5a^{12}c^2+10a^{10}b^2c^2-13a^8b^4c^2+28a^6b^6c^2-31a^4b^8c^2+10a^2b^{10}c^2+b^{12}c^2+9a^{10}c^4-13a^8b^2c^4-30a^6b^4c^4+22a^4b^6c^4+21a^2b^8c^4-9b^{10}c^4-5a^8c^6+28a^6b^2c^6+22a^4b^4c^6-52a^2b^6c^6+7b^8c^6-5a^6c^8-31a^4b^2c^8+21a^2b^4c^8+7b^6c^8+9a^4c^{10}+10a^2b^2c^{10}-9b^4c^{10}-5a^2c^{12}+b^2c^{12}+c^{14})a^2$$

P-hyperbolas

Let $La = |PB| - |PC|$, $Lb = |PC| - |PA|$, and $Lc = |PA| - |PB|$. Points on the \mathcal{H}_a^* P-hyperbola satisfy:

$$(2(b^2-c^2-La^2)*p*q+(a^2-La^2)*q^2-2(b^2-c^2+La^2)*p*r-2(a^2+La^2)*q*r+(a^2-La^2)*r^2)*x^2-2(b^2-c^2-La^2)*p^2*x*y-(a^2-La^2)*p^2*y^2+2*(b^2-c^2+La^2)*p^2*x*z+2*(a^2+La^2)*p^2*y*z-(a^2-La^2)*p^2*z^2 = 0$$

The conic \mathcal{P}^* through the vertices of the 3 P-hyperbolas is given by:

$$x^2+y^2+z^2-(2(a^2+La^2)*y*z)/(a^2-La^2)-(2*(b^2+Lb^2)*z*x)/(b^2-Lb^2)-(2*(c^2+Lc^2)*x*y)/(c^2-Lc^2)=0$$

The first barycentric coordinate for its center X_{5452} is given by:

$$(a^2*((-2*La^2)/(a^2-La^2)+(b^2+Lb^2)/(b^2-Lb^2)+(c^2+Lc^2)/(c^2-Lc^2)))/(a^2-La^2)$$

With the other two computed cyclically.

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