Parabola-Inscribed Poncelet Polygons
Derived from the Bicentric Family

ABSTRACT
We study loci and properties of a Parabola-inscribed family of Poncelet polygons whose caustic is a focus-centered circle. This family is the polar image of a special case of the bicentric family with respect to its circumcircle. We describe closure conditions, curious loci, and new conserved quantities.

Key words: Poncelet, closure, porism, parabola, bicentric, conservation, invariants

MSC2020: 37M05, 00A72, 51N20, 37-04

1 Introduction
This is a continuation of our investigation of Euclidean phenomena of Poncelet families [11, 12, 14, 20]. Recall Poncelet’s porism: specially-chosen pairs of conics $C, C'$ admit a one-parameter family of polygons inscribed in $C$ while simultaneously circumscribed about $C'$ [5, 7, 8].

Here we consider a certain family such that $C$ is a parabola $P$ while $C'$ is a circle centered on the focus of $P$. As shown in Figure 1, this is simply the polar image of the bicentric family (interscribed between two circles) with respect to its circumcircle, see Appendices A and B for construction details. We derive closure conditions for this new family for $N = 3, 4, 5, 6$ cases ($N$ is the number of sides) and describe some of its properties and loci of associated points. Also considered is its polar image with respect to $P$.

Main results

- The loci of vertex and area centroids of polar polygons are straight lines, whereas that of the perimeter centroid is a non-conic.
- In the $N = 3$ case, the locus of the orthocenter is a straight line as are those of many triangle centers of the polar family. The Euler line of the polar family always passes through the parabola’s focus.
- Several centers of the $N = 3$ polar family are stationary and/or sweep circles. In the latter case, they all belong to a single parabolic pencil.
- We prove that the quantity $\sum \sin \theta_i / 2$ is conserved, where $\theta_i$ are the interior angles of parabola-inscribed polygons. In fact, this quantity is conserved by any conic-inscribed polar image of the bicentric family.

Most of the above properties were first noticed via simulation [25], and later proved with a computer-algebra system (CAS) [17], using the explicit parametrizations given in Appendix A. For brevity, we omit any CAS-based proofs.
Related work

We can roughly divide it into three groups: (i) the study of point loci over certain triangle families [18, 19, 27], (ii) proving that loci of certain Poncelet triangle families are of a given curve type [9, 13, 21, 23], and (iii) proving properties and invariants over \( N \geq 3 \) Poncelet families [2, 4, 6, 22]. Also related is the Steiner-Soddy Poncelet family which are the polar image of the so-called Brocard porism with respect to the circumcircle [10].

In Appendix A we provide explicit parametrizations for the vertices of both the \( N = 3 \) and \( N = 4 \) families, as well as their respective polar families. In Appendix B we explore the relation of parabola-inscribed families with the traditional bicentric family.

2 Loci of parabola-inscribed triangles

Referring to Figure 2, consider a Poncelet family \( T \) of triangles inscribed in a parabola \( \mathcal{P} \), and circumscribed about a focus-centered circle. Let \( F = [-f,0] \) and \( V = [0,0] \) denote focus and vertex, respectively, where \( f \) is the focal distance. Consider a circle \( C \) centered at \( F \) with radius \( r \).

Figure 2: A Poncelet triangle (green) is shown inscribed in a parabola \( \mathcal{P} \) (gold), circumscribed about a focus-centered circle \( C' \) (brown). Over the family, \( X_4 \) sweeps a line (solid green) parallel to the directrix (dashed gold). The loci of barycenter \( X_2 \), circumcenter \( X_3 \), and Spieker center \( X_{10} \) are coaxial parabolas (blue); their foci are labeled \( F_2, F_3, \) and \( F_{10} \), respectively. Notice the latter is on an intersection of \( C' \) and the axis of the \( \mathcal{P} \) (dashed gray). Since the family circumscribes a circle centered on \( F, F_2, X_2, X_{10} \) are collinear (dashed blue) and \( X_{10} = F + (3/2)(X_2 - F) \), see Remark 7.

Proposition 1 \( \mathcal{P} \) and \( C \) will admit a Poncelet family of triangles if, and only if, \( r/f = 2(\sqrt{2} - 1) \).

Proof. Consider the Poncelet triangle with two parallel sides shown in Figure 3, inscribed in the parabola \( y = x^2/(4f) \), where \( f \) is the focal length. At \( x = r \) the parabola must be at \( y = f - r \), i.e., \( f - r = r^2/(4f) \), and the result follows.
Proposition 2

Over $T$, the locus of the orthocenter $X_4$ is the line parallel to $D$ given by $x = (5 - 2\sqrt{2})f$, with $y \in \mathbb{R}\setminus\{\pm(2\sqrt{2} - 1)f\}$. 

The proof below was kindly contributed by Alexey Zaslavsky [26]:

**Proof.** Let $C$ be the unit circle in the complex plane and $A, B, C$ the touching points with the sides of the parabola-inscribed triangles. The polar transformation with center $F$ maps the parabola to a circle with center $I$ passing through $F$ and touching $AB, BC, CA$. Using Euler's formula $|FI|^2 = r^2 = R(R - 2r)$ [24], with $R = 1$ its radius, and $r = \sqrt{2} - 1$. Consider the line $FI$ as the real axis.

Since $I$ is self-conjugated with respect to $ABC$, we have $a + b + c = 2\sqrt{2} - 2 + (3 - 2\sqrt{2})abc, ab + bc + ca = 3 - \sqrt{2} + (2\sqrt{2} - 2)abc$. The polar images of the altitudes of the original triangle are the common points of $BC, CA, AB$ with the lines passing through $F$, and perpendicular to $FA, FB, FC$ respectively. We have to calculate the common point of the line passing through these three points and the real axis. The coordinate functions of this point are symmetric functions in $a, b, c$, so we can express them as elementary symmetric functions on said variables, and verify that they are constant. The restriction on $y$ coordinates are poles in the parametric equation that describes the locus. □

Note that in [10], Section 4 a more general result was proved: the locus of the orthocenter $X_4$ of any Poncelet triangle family inscribed in a parabola $P$ whose caustic is a circle centered on the axis of $P$, is a straight line parallel to the directrix of $P$.

In Appendix B we describe how the parabola-inscribed family is the polar image of the bicentric family with respect to its circumcircle. Referring to Figure 4, Proposition 2 is actually a special case of:

**Proposition 3** The locus of $X_4$ of an family which is the polar image of $N = 3$ bicentrics with respect to its outer circle is an ellipse, straight line, or hyperbola if the circumcenter of the bicentric triangle lies in the interior, on top, or outside its incircle.
2.2 Three parabolic loci

Referring to Figure 2, we show below that over $\mathcal{T}$, the loci of the barycenter, circumcenter, and Spieker centers are all parabolas. The first and last correspond to the vertex and perimeter centroids of a triangle. This is curious since, in general, the locus of the perimeter centroid of a Poncelet family is not a conic [22].

**Proposition 4** Over $\mathcal{T}$, the locus of the barycenter $X_2$ is a parabola coaxial with $\mathcal{P}$, with focus $F_2 = [-f/3,0]$, and vertex $V_2 = [2f(1-2\sqrt{2})/3,0]$.

**Proposition 5** Over $\mathcal{T}$, the locus of the circumcenter $X_3$ of $T$ is a parabola coaxial with $\mathcal{P}$, with focus $F_3 = [-f(2\sqrt{2}-3)/2,0]$, and vertex $V_3 = [-f(2\sqrt{2}+3)/2,0]$.

**Proposition 6** Over $\mathcal{T}$, the locus of the Spieker center $X_{10}$ is a parabola coaxial with $\mathcal{P}$, with focus $F_{10} = [(1-2\sqrt{2})f,0]$ and vertex $V_{10} = [(3/2-2\sqrt{2}),0]$. In particular, $F_{10} = [-f-r,0]$, i.e., it lies on the left extreme of $C$.

Note that $X_{10}$ is the perimeter centroid of a triangle, while $X_2$ doubles up as both the vertex and area centroid. A. Akopyan has reminded us of the following general fact:

**Remark 1** If a polygon circumscribes a circle (let its center be $O$), then $C_1,C_2,O$ are collinear and $(C_1-O) = (3/2)(C_2-O)$.

Therefore:

**Corollary 1** Over $\mathcal{T}$, $X_{10}$ is collinear with $X_2$ and $X_{10} = F + (3/2)(X_2-F)$.

3 The polar $N = 3$ family

Referring to Figure 5, let $T'$ denote the polar triangle of a triangle $T$ in $\mathcal{T}$, i.e., whose sidelines are the polars of $T$ with respect to $\mathcal{P}$. Since $\mathcal{T}$ is inscribed in $\mathcal{P}$ these are simply the tangents.

Recall some known properties of the polar triangle with respect to any parabola [13]: (i) the circumcircle of $T'$ passes through the focus $F$; (ii) the orthocenter of $T'$ is on the directrix; (iii) its area is half that of the reference triangle.

**Proposition 7** The $T'$ family is Ponceletian. It is circumscribed about $\mathcal{P}$ and is inscribed in a hyperbola $\mathcal{H}$ with center $[f,0]$. Its axes are the axis and directrix of $\mathcal{P}$. Its implicit equation reads

$$\mathcal{H} : \left(\sqrt{2} + \frac{3}{2}\right)(x-f)^2 - \frac{y^2}{2} - 2f^2 = 0.$$
The locus of $X'_1$ is bounded by two lines parallel to the directrix and approximately $f/850$ apart, see Figure 6.

Proposition 11 The locus of the symmedian point $X'_6$ of $T'$ is a line parallel to $\mathcal{D}$ and parametrized by

$$X'_6 = \left[ (5-3\sqrt{2})f, \frac{(3\sqrt{2}+4)(2\sqrt{2}y^2-28f^2+y^2)}{14(\sqrt{2}y-2f+y)(\sqrt{2}+2f+y)} \right].$$

Proposition 12 The locus of $X'_{10}$ of the polar family is an algebraic curve of degree four given by

$$x^4 - 4(\sqrt{2}+12)x^3 + 8(2\sqrt{2}+1)yx^2 + 24(5\sqrt{2}+8)x^2y + 4(9\sqrt{2}+8)f^2x - \left(\sqrt{2}+4\right)f^2y^2 + 7f^4 = 0$$

This locus is tightly bound by the following two lines parallel to the directrix:

$$x = \left(\sqrt{2} - 1 + \frac{\sqrt{10}-7\sqrt{2}}{2}\right)f \quad \text{and} \quad x = \left(\sqrt{2} - 2^{-1/4}\right)f.$$ 

The distance between these lines is approx. $f/1700$.

Figure 6: Left: The locus of the polar incenter $X'_1$ is the branch of a quartic which visually is a straight line. It fits within to lines parallel to the directrix and at a distance of $f/850$. In the figure the curve is shown at aspect ratio of 28,000. Right: The locus of the polar Spieker center $X'_{10}$ (perimeter centroid) is an algebraic curve of degree at least four, bounded by two vertical lines separated by $f/1700$. The aspect ratio of the figure is 56,000.

Still referring to Figure 5:

Proposition 9 The locus of the barycenter $X'_2$ of $T'$ is a line parallel to $\mathcal{D}$ and parametrized by

$$X'_2 = \frac{1}{3} \left[ (2\sqrt{2}-1)f, \frac{(4-8\sqrt{2})f^2y+y^3}{(8\sqrt{2}-12)f^2+y^2} \right].$$

Proposition 10 The locus of the circumcenter $X'_3$ of $T'$ is a line parallel to $\mathcal{D}$ and parametrized by

$$X'_3 = \left[ (\sqrt{2}-1)f, \frac{(3\sqrt{2}+2)(2\sqrt{2}y^2-28f^2+y^2)y}{14(\sqrt{2}y-2f+y)(\sqrt{2}+2f+y)} \right].$$

Referring to Figure 7, the above expressions for $X'_2$ and $X'_3$ yield:

Corollary 2 The (varying) Euler line $X'_2X'_3$ of the polar family passes through the focus $F = [-f,0]$ of $\mathcal{P}$.

Still referring to Figure 7, the next 4 propositions were obtained from experimental evidence and verification by CAS:

Figure 7: Over the polar family (red), the Euler line (dashed magenta) will always pass through the focus $F$ of the parabola-inscribed family (green). $X'_{26}$ (resp. $X'_{68}$ and $X'_{110}$) remain stationary at the focus $F$ (resp. the two vertices of the hyperbola to which the polar family is inscribed). Experimentally, $X'_{61}$ is stationary at the intersection of the caustic with the parabola axis farthest from the latter’s vertex. Also shown is the Kiepert inparabola (magenta), whose focus is $X'_{110}$ and directrix is the Euler line. Thus the polar family simultaneously inscribes the original parabola (gold) and the Kiepert (magenta). Finally, the figure depicts the circular locus of Steiner point $X'_{99}$ of the polar family.
3.2 Stationary points

The circumcenter of the tangential triangle appears as X_{26} on [16].

**Proposition 13** Point X'_{26} of T' is stationary at the focus F of \( P \).

Note X_{26} does not lie in general on the circumcircle of a reference triangle. In our case it does since, as mentioned above, the circumcircle of the polar contains the focus.

The Kiepert parabola of a triangle is an inscribed parabola whose focus is labeled X_{110} on [16]. Its directrix is the Euler line [24]. Referring to Figure 7:

**Proposition 14** The focus X'_{110} of the Kiepert parabola (resp. the Prasolov point X'_{68}) of the polar family is stationary at the vertex of H farthest (resp. closest) to the focus of \( P \). Furthermore, X'_{161} is stationary at the intersection of the incircle with the parabola axis farthest from the parabola vertex, i.e., at \([ (1 - 2\sqrt{2})f, 0 ]\).

**Observation 1** Over the polar family, the vertex of its Kiepert parabola sweeps a circle.

3.3 Linear loci galore

Referring to Figure 8:

**Observation 2** Over the first 1000 triangle centers in [16], the following triangle centers of T' sweep linear loci parallel to D: X_k', k = 2, 3, 4, 5, 6, 20, 22, 23, 24, 25, 49, 51, 52, 54, 64, 66, 67, 69, 74, 113, 125, 140, 141, 143, 146, 154, 155, 156, 159, 182, 184, 185, 186, 193, 195, 206, 233, 265, 323, 343, 368, 370, 373, 376, 378, 381, 382, 389, 394, 399, 403, 427, 428, 468, 546, 547, 548, 549, 550, 567, 568, 569, 575, 576, 578, 597, 599, 631, 632, 858, 895, 973, 974.

3.4 A pencil of circular loci

Referring to Figure 7:

**Proposition 15** The locus of the Steiner point X'_99 is a circle whose center O'_99 lies on the axis of \( P \) of radius R'_99 such that at its right endpoint it touches X'_{110}. Explicitly,

\[ O'_99 = \left( (6\sqrt{2} - 7) f, 0 \right), \quad R'_99 = 2f \sqrt{17 - 12\sqrt{2}}. \]

Referring to Figure 9:

**Observation 3** Over the first 1000 triangle centers in [16], the following triangle centers of T' sweep circular loci with centers on the axis of \( P \) and passing through X'_{110}: X_k', k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935.

Figure 8: Many triangle centers of the polar family sweep lines parallel to the directrix. The following are shown: X_k, k = 2, 3, 4, 5, 6, 20, 22, 24, 25, 49, 51, 52, 54, 64, 66, 67, 69, 74.

Figure 9: Over the polar family we find that if a certain triangle center sweeps a circular locus, said locus will be an element of a parabolic pencil with X_{110} as their common point (not labeled). In the figure the circular loci of X_k, k = 99, 107, 112, 249, 476, 691, 827, 907, 925, 930, 933, 935 are shown. Notice all lie on the dynamically-moving circumcircle (dashed red) except for X_{249}.

This gives credence to:

**Conjecture 1** If the locus of X'_k is a circle with nonzero radius, it is in the parabolic pencil with X_{110} as a common point.
4 Parabola-inscribed quadrilaterals

Referring to Figure 10, consider a Poncelet family \( Q \) of quadrilaterals inscribed in a parabola \( \mathcal{P} \), and circumscribed about a focus-centered circle \( C \) of radius \( r \). As before, let \( f \) denote the parabola’s focal distance, and \( V = [0,0] \), \( F = [-f,0] \), its vertex and focus, respectively.

**Proposition 16** \( \mathcal{P} \) and \( C \) will admit a Poncelet family of convex quadrilaterals if, and only if, \( r/f = 2\sqrt{5} - 2 \).

**Proof.** Referring to Figure 11, consider the symmetric Poncelet quadrilateral \( P_i = [x_i,y_i] \), \( i = 1, \ldots, 4 \), inscribed in the parabola \( y = x^2/(4f) \), i.e., \( x = 2\sqrt{y} \). Clearly, \( y_1 = f - r \), and \( y_2 = f + r \). Requiring that \( P_1P_2 \) be tangent to \( C \) yields the quartic \( r^2 + 4f\sqrt{r^2 - r^2} = 0 \). The claim is the one positive root of this quartic. □

![Figure 10: A Poncelet quadrilateral (green) is shown inscribed in a parabola \( \mathcal{P} \) (gold) and circumscribed about a focus-centered circle (brown). Over the family, (i) the intersection \( W \) of its diagonals (dashed green) is stationary; (ii) the loci of vertex \( C_0 \), perimeter \( C_1 \), and area \( C_2 \) centroids sweep 3 distinct parabolas (blue) coaxial with \( \mathcal{P} \) with foci on \( F_0 \), \( F_1 \), and \( F_2 \). Notice the vertex of \( C_0 \) is \( F \) and that of \( C_1 \) is \( F_0 \). (iv) As predicted by Remark 4, \( C_1 \) is collinear with \( F \) and \( C_2 \) (dashed black); (v) \( C_0, C_2, W \) are collinear (dashed blue). Also shown is the polar quadrilateral \( \mathcal{Q} \) (red) with respect to \( \mathcal{P} \), inscribed in a hyperbola (dashed, red) centered at \([f,0]\). One observes that: (a) its diagonals (dashed red) also intersect at \( W \); (b) the loci of its vertex \( C_0 \) and area \( C_2 \) centroids are lines (dashed orange) perpendicular to the axis of \( \mathcal{P} \); (c) \( C_0, C_2, W \) are collinear (dashed red); (d) the locus of the polar perimeter centroid \( C'_1 \) is algebraic and of degree 10.

![Figure 11: Construction used to derive \( r/f \) in for parabola-inscribed convex quadrilaterals in Proposition 16](image)

4.1 The three centroids

Referring to Figure 10, let \( C_0 \), \( C_1 \), and \( C_2 \) denote the vertex, perimeter, and area centroids of the quadrilaterals in \( Q \), respectively.

**Proposition 18** Over the family, \( C_0 \), \( C_2 \), and \( W \) are collinear.

**Proposition 19** Over the Poncelet family, the loci of \( C_0 \), \( C_2 \) are parabolas coaxial with \( \mathcal{P} \), whose foci and vertices locations are listed in Table 1.

From Remark 4

**Corollary 3** The locus of \( C_1 \) is a 3/2-scaled version of the locus of \( C_2 \) with \( F \) as the homothety center.

<table>
<thead>
<tr>
<th>centroid (N=4)</th>
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<th>vertex ( x/f )</th>
<th>vts. ( x/f ) (num)</th>
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Table 1: Location of centroids \( C_0, C_1, C_2 \) in the convex \( N = 4 \) family.
4.2 The polar quadrilateral

Referring to Figure 10, consider the polar quadrilateral whose sides are the tangents to \( \mathcal{P} \) at the vertices of the original family. Let \( P_i, i = 1, \ldots, 4 \) denote its vertices and \( C_0, C_1, \) and \( C_2 \) denote its vertex, perimeter, and area centroids.

**Proposition 20** The locus of the polar quadrilateral’s vertices is the hyperbola \( \mathcal{H} \) given by

\[
\mathcal{H} : \frac{(x - f)^2}{4(\sqrt{5} - 2)f^2} - \frac{y^2}{4f^2} = 1 = 0.
\]

with center at \([f, 0]\) and foci \([f(1 \pm 2\sqrt{5} - 1)], 0]\).

Let \( W \) be defined as in Proposition[17] The next two propositions result from visual (and numerical) detection, followed by verification by CAS.

**Proposition 21** The two diagonals of the polar quadrilateral intersect at \( W \).

**Proposition 22** Over the polar quadrilateral family, \( C_0, C_1, \) and \( W \) are collinear.

**Proposition 23** Over \( Q_1 \) the loci of \( C_0 \) and \( C_1 \) are lines parallel to the parabola’s directrix and given by \( C_0 : x = (3 - \sqrt{3})f/2 \), and \( C_1 : x = (4 - \sqrt{3})f/3 \).

Rather laborious CAS manipulation yields:

**Proposition 24** Over \( Q_1 \) the locus of \( C_1 \) is one connected component of an algebraic curve of degree ten, given by the following equation:

\[
C_1 = - \left( 1457008 \sqrt{5} + 3257968 \right) f x^2 y^2 + \left( 122156 \sqrt{3} + 273148 \right) f^2 x^4 y^4
+ \left( 465164 \sqrt{5} + 1040132 \right) f^3 x^6 y^6
+ \left( 119256 \sqrt{3} + 266664 \right) f^4 x^8 y^8
+ \left( 505052 \sqrt{3} + 1129268 \right) f^5 x^{10} y^{10}
+ \left( 8564 \sqrt{3} + 19204 \right) f^6 x^{12} y^{12}
+ \left( 43955 \sqrt{3} + 98289 \right) f^7 x^{14} y^{14}
+ \left( 24568 \sqrt{3} + 54936 \right) f^8 x^{16} y^{16}
+ \left( 7250 \sqrt{3} + 16210 \right) f^9 x^{18} y^{18}
+ \left( 445706 \sqrt{3} + 9967712 \right) f x^2
- \left( 7787152 \sqrt{3} + 17412608 \right) f^2 x^4
+ \left( 5470436 \sqrt{3} + 12323344 \right) f^3 x^6
- \left( 1690535 + 755997 \sqrt{3} \right) f^4 x^8
+ \left( 330322 \sqrt{3} + 739868 \right) f^5 x^{10}
- \left( 228 \sqrt{3} + 672 \right) f^6 x^{12}
+ \left( 2750 \sqrt{3} + 7150 \right) f^7 x^{14}
+ \left( 16145 \sqrt{3} + 36103 \right) f^8 x^{16}
- \left( 84196 \sqrt{3} + 188268 \right) f^9 x^{18}
+ \left( 544928 \sqrt{3} + 1218496 \right) f^{10} x^{20} = -726 f^{10} = 0.
\]

Furthermore, \( C_1 \) is bound by the following two lines parallel to the directrix and approximately \( f/25 \) apart:

\[
x = \left( 5 + \sqrt{2} - \sqrt{3} \right) f/2, \quad \text{and} \quad x = \left( \sqrt{3} \sqrt{2} - \sqrt{3} - 2 \sqrt{2} + 3 \right) f/2.
\]

5 Parabola-inscribed pentagons

Referring to Figure 12, consider a family of pentagons inscribed in a parabola \( \mathcal{P} \) of focal distance \( f \), and circumscribed about a focus-centered circle \( C \) of radius \( r \).

![Figure 12: Parabola-inscribed pentagons (green), and their polar polygon (red). The loci of vertex \( C_0 \), perimeter \( C_1 \), and area centroids \( C_4 \) are parabolas (blue) coaxial with \( \mathcal{P} \) (gold). Over the polar family, \( C_0 \) and \( C_2 \) are straight lines (dashed orange) perpendicular to the directrix (dashed black). Though the locus of the perimeter centroid \( C_1 \) is indistinguishable from a straight line, it is an algebraic curve of degree likely much higher than 10 (since that is the degree for \( C_1 \) on \( N = 4 \)).

**Proposition 25** The pair \( \mathcal{P}, C \) will admit a Poncelet family of pentagons if, and only if, \( r/f \) is the only positive root of the following sextic polynomial (\( r/f \approx 0.995219 \)):

\[
x^6 + 12x^5 - 28x^4 + 32x^3 + 112x^2 - 64x - 64 = 0.
\]

**Proof.** Referring to Figure 13, without loss of generality, let \( \mathcal{P} \) be the unit parabola \( y = x^2 \) with focus \( F = [0, 1/4] \).
and let $C$ be a circle of radius $r$ centered at $F$. Consider the Poncelet pentagon $P_i$, $i = 1, \ldots, 5$ with $P_4$ at infinity, and $P_1P_2$ horizontal and tangent to $C$ at $[0, 1/4 - r]$. Compute the next Poncelet vertex $P_3 = [x_3, y_3]$ as the intersection of a tangent to $C$ from $P_2$ with $P$. By requiring that $x_3 = r$, we obtain the sextic in the claim.

Referring to Figure 12:

**Conjecture 2** Over the parabola-inscribed pentagon family, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $P$.

**Conjecture 3** Over the family of polar polygons to parabola-inscribed pentagons, the locus of vertex and area vertices are lines perpendicular to the axis while that of the perimeter centroid is an algebraic curve of degree at least four.

### 6 Parabola-inscribed hexagons and summary

#### 6.1 Hexagons and summary

Referring to Figure 14, we can also consider a family of parabola-inscribed hexagons.

![Figure 14: Hexagons (green) inscribed in a parabola $P$. As before, the loci of $C_0$, $C_1$, and $C_2$ are parabolas (blue) coaxial $P$. Over the polar family (red), the loci of $C'_0$, $C'_2$ are lines perpendicular to the axis while that of $C'_1$ is algebraic, and though visually a straight line, its degree is likely much higher than 10 (since that is the degree for $C'_1$ on $N = 4$). An analogous construction (based on symmetric configurations) was used to obtain $r/f$ required for convex $N = 6$. A summary of all $r/f$ thus obtained appears in Table 2.](image)

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</tbody>
</table>

Table 2: Table of $r/f$ required for closure of convex $N$-gons inscribed in a parabola, and circumscribed about a focus-centered circle. Algebraic expressions (2nd column) are only possible for $N = 3, 4$. The last column shows the number of possible solutions for $r/f$ if one were to include cases where circle and parabola intersect (the Poncelet polygon may be self-intersecting and/or non-convex). For Cayley’s conditions in the general case, see [8].

### 7 Generalizing centroidal loci

Let $R$ be a Poncelet family of $N$-gons inscribed to a parabola $P$, and circumscribed about a focus-centered circle...
As in Appendix B, let we propose the following generalizations (reader contributions are encouraged):

**Conjecture 4** Over $\mathcal{R}$, for any $N \geq 3$, the loci of vertex, perimeter, and area centroids are parabolas coaxial with $\mathcal{P}$.

**Conjecture 5** Over $\mathcal{R}$ and for any $N \geq 3$, the loci of vertex and area centroids of the polar polygons with respect to $\mathcal{P}$ are straight lines parallel to the directrix of $\mathcal{P}$.

Let $\mathcal{B}'$ be the conic-inscribed polar image of a generic bicentric family of $N$-gons with respect to the bicentric circumcircle (see Appendix B).

Recall that the locus of vertex and area centroids $C_0, C_2$ are conics over any Poncelet family, while that of the perimeter centroid $C_1$ is not, in general, a conic $\mathcal{C}$. A consequence of Remark 1, analogously exploited in Corollary 2, is that:

**Corollary 4** Over $\mathcal{B}'$, the locus of the perimeter centroid is a conic.

Let $\mathcal{P}'$ be the conic to which $\mathcal{B}'$ is inscribed.

**Conjecture 6** Over the polar polygons of $\mathcal{B}'$ with respect to $\mathcal{P}'$, the locus of the perimeter centroid is never a conic.

## 8 A conserved quantity

As in Appendix B, let $\mathcal{B}$ denote a bicentric family of $N$-gons inscribed to a circle $\mathcal{C} = (O, R)$, and circumscribed about a second, nested circle $\mathcal{C}'$. Let $d_i$ denote the perpendicular distance from the bicentric circumcenter $O$ to side $P_iP_{i+1}$. Referring to Figure 15:

**Lemma 1** Over $\mathcal{B}$, the quantity $\sum d_i$ is conserved.

The argument below was kindly provided by A. Akopyan [1].

**Proof.** The above statement is equivalent to stating that over $\mathcal{B}$ the sum of unit vectors from a point $P$ in the direction perpendicular to bicentric sides is constant. In turn, the latter is a corollary of the well-known fact that over $\mathcal{B}$, the centroid of the touchpoints of sidelines with $\mathcal{C}'$ is stationary. □

Let $\theta_i, i = 1, \ldots, N$, denote the angles interior to a polygon $\mathcal{B}$.

![Figure 15: An $N = 4$ bicentric polygon is shown (blue). Without loss of generality, in the case shown the circumcenter $O$ is interior to the incircle, i.e., the polar family (green) is ellipse-inscribed. Also shown is the pedal polygon (pink) with respect to a point $P$ in the interior of the circumcircle and the unit vectors (brown) along each perpendicular dropped from $P$ onto the sides.](image)
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Appendix A. Vertex parametrizations

A.1. Parabola-inscribed triangles

A 3-periodic orbit $P_t = [x_t, y_t] = [-\sqrt{2}/(4f), y_t]$ is such that

$$y_2 = \frac{2\left(1 - \sqrt{2}\right)(4f y_1 + \Delta) f}{8 f^2 \sqrt{2} - 12 f^2 + y_1^2},$$
$$y_3 = \frac{2\left(\sqrt{2} - 1\right)f \Delta}{8 f^2 \sqrt{2} - 12 f^2 + y_1^2},$$

where $\Delta = \sqrt{16(8\sqrt{2} - 11)f^4 + 8f^2y_1^2 + y_1^4}$.

A.2. Hyperbola-inscribed polar triangles

A 3-periodic orbit $Q_t = [q_{1,t}, q_{2,t}]$ is such that

$$Q_1 = (1 + \sqrt{2}).$$
$$Q_2 = \left[\frac{(4f y_1 + \Delta)y_1}{2(2\sqrt{2} + 3)y_1^2 - 8f^2}, \frac{(1 + \sqrt{2})y_1^4 - 4(1 + \sqrt{2})f^2 y_1 - 2\Delta f}{2(2\sqrt{2} + 3)y_1^2 - 8f^2}\right],$$
$$Q_3 = (1 + \sqrt{2}).$$
$$Q_3 = \left[\frac{(5 - \sqrt{2})((1 + 2\sqrt{2})y_1^2 - 28f^2) f}{7((3 + 2\sqrt{2})y_1^2 - 4f^2)}, \frac{8f^2 y_1}{(3 + 2\sqrt{2})y_1^2 - 4f^2}\right],$$

where $\Delta = \sqrt{y_1^4 + 8f^2y_1^2 + 16(8\sqrt{2} - 11)f^4}$.

A.3. Parabola-inscribed quadrilaterals

A 4-periodic orbit $P_t = [x_t, y_t] = [-\frac{1}{17}y_1^2, y_t]$ is such that:

$$y_2 = \frac{2\sqrt{5} - 2\Delta + 4f y_1 (3 - \sqrt{5})}{4f^2 \sqrt{5} - 8f^2 - y_1^2},$$
$$y_3 = \frac{4(2 - \sqrt{5}) f^2}{y_1},$$
$$y_4 = \frac{2\sqrt{5} - 2\Delta + 4f y_1 (\sqrt{5} - 3)}{4f^2 \sqrt{5} - 8f^2 - y_1^2},$$

where $\Delta = \sqrt{y_1^4 + 8f^2y_1^2 + 16(9 - 4\sqrt{5})f^4}$.

A.4. Hyperbola-inscribed polar quadrilaterals

A 4-periodic orbit $P_t = [p_{1,t}, q_{1,t}]$ is such that:

$$p_1 = \frac{\sqrt{5} - 2\left(\Delta_1 + 6f y_1 \sqrt{\sqrt{5} - 2} + 14f y_1 \sqrt{\sqrt{5} - 2}\right)}{4y_1^2 + 2\sqrt{3} - 8f^2},$$
$$q_1 = \frac{2\sqrt{5} + 2\Delta_1 f + 4f^2 y_1 y_1^2}{32f^4 - 32f^2 y_1^2 - 2y_1^4},$$
$$p_2 = \frac{2\sqrt{5} - 2\left(\Delta_1 + 2f(y_1^2 - y_1^2)\right)}{16f^4 - 16f^2 y_1^2 - y_1^4},$$
$$q_2 = \frac{2\sqrt{5} + 2\left(\Delta_1 + 2\sqrt{3} + 2f^2 y_1^2 - 8\left(y_1^2 - 2\right)^3 f^3\right)}{y_1^2 (16f^4 - 16f^2 y_1^2 - y_1^4)},$$
$$q_3 = \frac{-2\sqrt{5} - 2\left(y_1^2 \left(\Delta_1 - 2\sqrt{2} + 4f^2 y_1 y_1^2 + 8\left(y_1^2 - 2\right)^3 f^3\right)\right)}{y_1^2 (16f^4 - 16f^2 y_1^2 - y_1^4)},$$
$$q_4 = \frac{\sqrt{5} - 2\left(y_1^2 \left(\Delta_1 - 2\sqrt{2} + 4f^2 y_1 y_1^2 + 8\left(y_1^2 - 2\right)^3 f^3\right)\right)}{y_1^2 (16f^4 - 16f^2 y_1^2 - y_1^4)}.$$
Appendix B. Relation to the bicentric family

Referring to 16, the bicentric family \( \mathcal{B} \) of \( N \)-gons is a family of Poncelet \( N \)-gons inscribed in a circle \( C = (O, R_b) \), and circumscribed about another circle \( C' = (O', r_b) \). Let \( d = |O - O'| \). Relations between \( d, R, r_b \) are known for many "low \( N \)" and are listed in [24, Poncelet’s porism].

![Figure 16](image1.png)

**Figure 16:** The bicentric family is a family of Poncelet polygons interscribed between two circles. Shown are the \( N = 4 \) (left) and \( N = 5 \) (right) convex cases.

**Definition 1 (Polar polygon)** Given a polygon \( P \), its polar polygon \( P' \) with respect to a conic \( C \) is bounded by the tangents to \( C \) at the vertices of \( P \).

**Proposition 27** The polar family \( \mathcal{B}' \) of \( \mathcal{B} \) with respect to \( C \) is an ellipse, parabola, or hyperbola-inscribed if \( d \) is smaller, equal, or greater than \( R' \), respectively (\( O \) is interior, on the boundary, or exterior to \( C' \), respectively). Furthermore, one of the foci coincides with \( O' \).

As shown in Figure 17, when the polar family is hyperbola-inscribed, there are two layouts for its vertices: either (i) all lie on the branch of the hyperbola closest to the incenter of the family, or (ii) all but one lie on said branch, while the remaining one lies on the “other” branch.

![Figure 17](image2.png)

**Figure 17:** If the circumcenter \( O \) is exterior to the incircle of a bicentric polygon (blue), the polar (i.e., tangential) family will be hyperbola (gold) inscribed. Over the family there are two configurations: (i) solid green: all vertices lie on one branch of the hyperbola; (ii) dashed green: all but one vertex lie on the branch proximal to the incenter, while a lone one lies on the opposite branch.

**Proposition 28** The parabola \( \mathcal{P} \) which is the polar image of \( \mathcal{B} \) with \( d = r_b \), has focal distance \( f = R_b^2/(2r_b) \).

**Proof.** Let \( O = (0, 0) \). Consider a polygon in \( \mathcal{B} \) with a vertical side \( P_1P_2 \) tangent to the incircle at \( (2r_b, 0) \). The vertex \( V \) of \( \mathcal{P} \) is the pole of said side which can be obtained as the inversion of point \( (2r_b, 0) \) with respect to the circumcircle. This yields the result. \( \square \)

References


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