A Triple of Projective Billiards

ABSTRACT
A projective billiard is a polygon in the real projective plane with a circumconic and an inconic. Similar to the classical billiards in conics, the intersection points between the extended sides of a projective billiard lie on a family of conics which form the associated Poncelet grid. We extend the projective billiard by the inner and outer billiard and disclose various relations between the associated grids and the diagonals, in particular other triples of projective billiards.

Key words: ellipse, billiard, caustic, Poncelet grid, billiard motion

MSC2020: 51N35

1 Introduction

A billiard is the trajectory of a mass point in a domain called billiard table with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses (see Figures 1, 2, 8) and their projectively equivalent counterparts have attracted the attention of mathematicians, beginning with J.-V. Poncelet [4] and C.G.J. Jacobi [3] and continued, e.g., by S. Tabachnikov, who addresses in his book [10] a wide variety of themes around this topic. Computer animations carried out recently by D. Reznik [5] stimulated a new vivid interest in these well studied objects.

We focus on projective generalizations called projective billiards. This term stands for planar polygons $P_1P_2P_3\ldots$ with a circumconic $e$ and an inconic $c$ called caustic. Not all projective billiards are projectively equivalent to Euclidean billiards (see, e.g., Figure 9), and in not all cases exist periodical polygons between the conics $e$ and $c$. However, in all cases the intersection points between extended sides define a family of conics which form the associated Poncelet grid. The goal of this paper is to demonstrate that in a quite natural way any given projective billiard defines two more projective billiards with associated Poncelet grids. It will be demonstrated that not only the conics of these grids, but also configurations of related lines deserve our interest.

It needs to be pointed out, that the computation of the billiards’ vertices can only be carried out either iteratively or with the help of Jacobian elliptic functions (see, e.g., [8]). Therefore, it is not straightforward to obtain results on vertices and their respective $j$-th followers for any given integer $j > 1$. Often such assertions are equivalent to identities in terms of elliptic functions (see, e.g., [9, Section 5]).

Structure of the article. In Section 2 we introduce the three Poncelet grids associated respectively with a projective billiard and its inner and outer polygons. Section 3 is devoted to the conics $e^{(j)}$, $c^{(j)}$, and $r^{(j)}$ of the three grids. In Section 4 we recall results on the envelopes of diagonals and determine the points of contact. Finally in Section 5, we study the configuration of the $l$-th diagonals of the projective billiards inscribed respectively in $e^{(j)}$, $c^{(j)}$, and $r^{(j)}$. 
2 A triple of Poncelet grids

Let $P_1 P_2 P_3 \ldots$ be a polygon with circumconic $e$ and inconic $c$ in the real projective plane. Then there exists an associated Poncelet grid. We follow the notation in [7] and denote intersection points between extended sides\footnote{Note that $XY$ denotes the segment bounded by the points $X$ and $Y$, while $[X,Y]$ denotes the connecting line.} for $i, j = 1, 2, \ldots$ as

\[ S_t^{(j)} := \begin{cases} [P_{i-k-1}, P_{i-k}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k, \text{ and} \\ [P_{i-k}, P_{i-k+1}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k-1. \end{cases} \tag{1} \]

For fixed $j$, the points $S_1^{(j)}, S_2^{(j)}, \ldots$ are located on a conic $e^{(j)}$ which belongs to the dual pencil (range, in brief) spanned by $e$ and $c$. This is due to a result of M. Chasles in 1843 (note, e.g., [7, Theorem 3.5]).

If the polygon $P_1 P_2 \ldots$ is $N$-periodic, then we can confine to $1 \leq j \leq \left\lfloor \frac{N-3}{2} \right\rfloor$, since for even $N$ the locus $e^{(j)}$ with $j = N/2$ is a line which has the same pole with respect to (w.r.t., for short) $e$ and $c$. Under the billiard motion of $P_1 P_2 \ldots$, i.e., the variation of the vertices along the circumconic $e$ while $c$ remains fixed, each conic $e^{(j)}$ of the Poncelet grid remains fixed as well (note [7, Theorem 3.6]).\footnote{Beside the conics $e^{(j)}, j = 1, 2, \ldots$, the Poncelet grid contains a second family of conics. In the case of classical billiards with ellipses $e$ and $c$, the remaining conics are confocal hyperbolas (see, e.g., [7, Figures 5 or 6]) which vary under the billiard motion. However, here we focus only on $e^{(j)}$.}

In the classical case of a Euclidean billiard $P_1 P_2 \ldots$ in a conic $e$, the conics $e^{(j)}$ are confocal with $e$ and the caustic $c$ (Figure 2). If for a given ellipse $e$ the caustic $c$ is an ellipse, then the billiard is called elliptic and the conics $e$ and $c$ intersect in two pairs of complex conjugate points. Otherwise we obtain a hyperbolic billiard with a hyperbola as caustic (Figures 6 and 7). Then the two conics share four real points.

2.1 The outer polygon

The tangents $t_{r_1}, t_{r_2}, \ldots$ to the circumconic $e$ at the vertices $P_1, P_2 \ldots$ of a projective billiard define a polygon $R_1 R_2 \ldots$ called outer polygon in [5]. This polygon is polar to $P_1 P_2 \ldots$ w.r.t. $e$ and therefore inscribed in a conic $r$ which is polar to $c$ w.r.t. $e$ (Figure 1). Similar to (1), the vertices $R^{(j)}$ of the associated Poncelet grid are points of intersection between tangents to $e$ and denoted for $j = 1, 2, \ldots$ as given below:

\[ R^{(j)} := \begin{cases} t_{r_{j+1}} \cap t_{r_{j+1}} & \text{for } j = 2k, \text{ and} \\ t_{r_{j+1}} \cap t_{r_{j+1}} & \text{for } j = 2k-1. \end{cases} \tag{2} \]

hence $k = \left\lfloor \frac{j+1}{2} \right\rfloor$ (note Figure 2).

2.2 The inner polygon

Beside the Poncelet grids associated with the pairs of conics $(e,c)$ and $(r,e)$, there is a third Poncelet grid. This time we focus on the polygon of contact points $Q_1, Q_2 \ldots$ of the sides of $P_1 P_2 \ldots$ with the caustic $c$. The polygon $Q_1 Q_2 \ldots$
is called inner polygon in [5]. The vertices of the associated Poncelet grid are defined as

\[
Q^{(j)} := \begin{cases} 
[Q_{j-k-1}, Q_{j-k}] \cap [Q_{j+k-1}, Q_{j+k}] & \text{for } j = 2k, \\
[Q_{j-k-1}, Q_{j-k}] \cap [Q_{j+k-1}, Q_{j+k}] & \text{for } j = 2k-1
\end{cases}
\]

(note Figure 4).

The extended sides of the polygon \(Q_1Q_2\ldots\) envelop a conic \(q\) which is polar to \(e\) w.r.t. \(c\). The line \([Q_{i-1}, Q_i]\) contacts \(q\) at the \(c\)-pole \(F_i\) of the tangent \(t_{P_i}\) to \(e\) at \(P_i\). Therefore, in the case of a Euclidean billiard it is the point of intersection between the chord \([Q_{i-1}, Q_i]\) and the normal to \(e\) at \(P_i\) (Figure 1). The latter is the locus of poles of the tangent \(t_{P_i}\) w.r.t. the conics of the confocal family.

**Lemma 1** Referring to the previous notation, the circumconic \(c\) of the polygon \(R_1R_2\ldots\) with sides tangent to \(e\) at \(P_i\) is polar to \(c\) w.r.t. \(e\). The conic \(q\) of the polygon \(Q_1Q_2\ldots\) with circumconic \(c\) is polar to \(e\) w.r.t. \(c\). In the billiard case (Figure 1), \(R_iQ_i\) is orthogonal to \(c\) at \(Q_i\), and \(F_iP_i\) is orthogonal to \(e\) at \(P_i\).

Lemma 1 reveals that also the conics \(q\) and \(r\) are invariant under the billiard motion along \(e\). Clearly, if the original projective billiard \(P_1P_2\ldots\) is periodic, then \(Q_1Q_2\ldots\) and \(R_1R_2\ldots\) are periodic, too.

A polygon with circumconic \(e\) and inconic \(c\) can be periodic even when the two conics share two real and two complex conjugate points. An example is depicted in Figures 5 and 9 with the two conics as circles. Such polygons \(P_1P_2\ldots\) are called bicentric. They were first treated in 1828 by Jacobi [3] in the case where \(c\) lies in the interior of \(e\). In [6] various invariants of bicentric polygons are proved for the case that the circles \(e\) and \(c\) are either nested or disjoint.

### 3 More projective billiards in the three Poncelet grids

In the case of Euclidean billiards \(P_1P_2\ldots\) in the plane or on the sphere (see [7, Fig. 7]), the tangents to \(e\) at \(P_i\) and those to \(e^{(j)}\) at \(S_{j}^{(1)}\) are angle bisectors of extended sides of \(P_1P_2\ldots\). Therefore, the net of extended sides of \(P_1P_2\ldots\) is circular with the points \(R_{j}^{(i)}\) as centers of incircles of quadrilaterals (Figure 2). This result dates back to [1] in 2018. Below we present a generalization.

**Theorem 1** Given a projective billiard \(P_1P_2\ldots\), then for each \(j = 1,2,\ldots\) the vertex \(R_{j}^{(i)}\) of the Poncelet grid associated with the outer polygon \(R_1R_2\ldots\) is located on the tangents to \(e^{(j)}\) at \(S_{j}^{(1)}\) and \(S_{j}^{(1)}\). The points \(R_{j}^{(i)}, R_{j}^{(i)}\ldots\) belong to a conic \(r^{(j)}\) which is contained in the range spanned by \(e\) and \(r\). The polar conic of \(r^{(j)}\) w.r.t. \(e^{(j)}\) is the envelope of the extended sides of the polygon \(S_{2}^{(1)}S_{2}^{(1)}\ldots\)

---

**Figure 3:** \(N\)-periodic billiard with \(N = 8\). In the proof of Theorem 1 we focus on the quadrilateral formed by the tangents from \(S_{2}^{(1)}\) and \(S_{2}^{(1)}\) to the caustic \(c\).

**Figure 4:** The contact points of the sides of the polygon \(S_{2}^{(1)}S_{2}^{(1)}\ldots\) with their envelope \(e^{(j)}\) are the vertices \(Q^{(j)}\) of the Poncelet grid associated with \(Q_1Q_2\ldots\). In other words, the projective billiard \(S_{2}^{(1)}S_{2}^{(1)}\ldots\) has \(Q^{(j)}Q^{(j)}\ldots\) as its inner billiard.

**Proof.** According to (1), the extended sides \([P_{i+j}, P_{i+j+2}]\) through \(S_{j}^{(1)}\) for \(k = \left\lfloor \frac{j+1}{2} \right\rfloor\) and \([P_{i-1}, P_{i}]\) and \([P_{i+j}, P_{i+j+1}]\) through \(S_{j}^{(1)}\) for \(k = j\) form a quadrilateral with \(P_{i+j}, P_{i+j+1}\) in \(e\) and \(S_{j}^{(1)}, S_{j}^{(1)}\) in \(e^{(j)}\) as pairs of opposite vertices (see the case \(j = 2, N = 8\) and \(i = 7\) in Figure 3). All four sides are tangents of the caustic \(c\), while the conics and \(e, e^{(j)}\) and \(c\) belong to a range. According to the
mentioned result by Chasles and its extension in [7, Theorem 3.5]), the tangents to \(e\) at \(P_i\) and \(P_{i+j+1}\) and the tangents to \(e^{(j)}\) at \(S_{i+j+1}^{(j)}\) and \(S_{i+j}^{(j)}\) are concurrent. By (2), their meeting point is \(R_{i+j+1}^{(j)}\) (see Figure 2). After increasing all subscripts by 1, we obtain the analogue result for \(R_{i+k}^{(j)}\).

The Poncelet grid associated with \(R_1 R_2 \ldots\) contains conics \(r^{(j)}\) passing through the vertices \(R_1^{(j)}, R_2^{(j)}\). All conics \(r^{(j)}\) belong to the range spanned by \(e\) and \(r\) and are motion invariant, too. Since the polar line of \(R_i^{(j)} \in r^{(j)}\) w.r.t. \(e^{(j)}\) is the line \(S_i^{(j)}S_{i+k}^{(j)}\), the polar conic \(e^{(j)}\) of \(r^{(j)}\) w.r.t. \(e^{(j)}\) envelops the polygon \(S_i^{(j)} S_{i+k}^{(j)}\).

**Theorem 2** Referring to the previous notation, the sides of the polygon \(S_1^{(j)} S_2^{(j)} \ldots\) contact the enveloping conic \(e^{(j)}\) at the vertices \(Q_1^{(j)}, Q_2^{(j)}\). Hence, the envelope \(e^{(j)}\) coincides with the conic of the Poncelet grid associated with \(Q_1Q_2\ldots\) (Figure 4).

**Proof.** We replace the polygon \(P_1 P_2 \ldots\) inscribed in \(e\) and circumscribed to \(b\) by the polygon \(R_1 R_2 \ldots\) inscribed in \(r\) and circumscribed to \(e\). Then by virtue of Theorem 1, the side \([R_i^{(j)}, R_{i+1}^{(j)}]\) contacts the envelope \(e^{(j)}\) at the point \(S_i^{(j)}\). This implies for our original polygon \(P_1 P_2 \ldots\) that \([S_i^{(j)}, S_{i+1}^{(j)}]\) contacts the envelope \(e^{(j)}\) at the vertex \(Q_i^{(j)}\) of the Poncelet grid associated with the \(j\)-th diagonals of \(Q_1 Q_2 \ldots\).

---

**Corollary 1** Let \(P_1 P_2 \ldots\) be a projective billiard with \(R_1 R_2 \ldots\) and \(Q_1 Q_2 \ldots\) as respective outer and inner polygons. Then for fixed \(j \in \{1, 2, \ldots\}\), the vertices \(S_1^{(j)}, S_2^{(j)}\) on the conic \(e^{(j)}\) of the Poncelet grid associated with \(P_1 P_2 \ldots\) form another projective billiard with the polygons \(R_1^{(j)} R_2^{(j)}\) as outer billiard with circumconic \(r^{(j)}\) and...
$Q_{i}^{(j)}, Q_{2}^{(j)} \ldots$ as inner billiard with the inconic $c^{(j)}$, which is polar to $r^{(j)}$ w.r.t. $e^{(j)}$.

The Figures 5–7 illustrate that the triples $(e^{(j)}, e^{(j)}, r^{(j)})$ can look quite different in comparison with $(e^{(1)}, e^{(1)}, r^{(1)})$ or $(e^{(2)}, e^{(2)}, r^{(2)})$ in Figure 4.

As shown at the hyperbolic billiard in Figure 6, the conic $r^{(1)}$ passes through the intersection points of the hyperbola $e^{(1)}$ with $e$. This follows from particular poses with a twofold covered billiard (see Figure 7): When $P_1 \in e$ is specified at an intersection point$^5$ with the caustic $c$, then $P_2$ coincides with $P_{10}$ as well as with $S_1^{(j)}$ and $R_{1}^{(j)}$. There is a general statement in the background:

**Theorem 3** Referring to the previous notation, for each $j = 1, 2, \ldots$ the conics $r^{(j)}$, $e^{(j)}$ and $e$ belong to a pencil. The same is true for the three conics $e^{(j)}$, $e^{(j)}$, and $c$ (Figure 5).

**Proof.** We argue with help of the complex extension of the real projective plane. Whenever the point $R_{i}^{(j)} = P_{i} \cap P_{i+j}$ for $k = \left[\frac{j+1}{2}\right]$ is located on $e$, then follows $R_{i}^{(j)} = P_{i} = P_{i+j}$ and consequently $S_{j+k} = [P_{j-1}, P_{j}] \cap [P_{j+1}, P_{j+1}] = R_{i}^{(j)}$. This means that each point of intersection between $e$ and $r^{(j)}$ belongs also to $e^{(j)}$. Therefore, if $e$ and $r^{(j)}$ share four mutually different points, then $e^{(j)}$ belongs to the pencil spanned by $r^{(j)}$ and $e$.

The remaining cases with intersection points of higher order between $r^{(j)}$ and $e$ can be seen respectively as a limit where some of the four intersection points tend to coincidence. It cannot happen that in the limit the symmetric coefficient matrices of the three conics become linearly independent when everywhere else in the neighborhood they are linearly dependent.

The second statement follows just by replacing the triple $(r^{(j)}, e^{(j)}, e)$ by $(e^{(j)}, e^{(j)}, c)$.

## 4 Diagonals

In view of the envelopes of the $j$-th diagonals $[P_1, P_{j+1}]$ of our polygon $P_1P_2P_3 \ldots$, we can refer from [9] a result which was first stated in 1822 by V.-P. Poncelet [4] and reproved in 1828 by C.G.J. Jacobi for the case of nested circles $e$ and $c$. Moreover, we recall from [9] how to find the enveloping points. However, the proofs of the Theorems 1 and 2 in [9] cover only the cases of elliptic and hyperbolic billiards, where affine scalings are available between involved conics. The following theorem addresses the general case.

**Theorem 4** Let $P_1, P_2P_3 \ldots$ be a polygon inscribed in the conic $e$ and circumscribed to the conic $c$ with contact points $Q_1, Q_2, Q_3 \ldots$. Then for fixed $j = 1, 2, \ldots$, the envelope of the $j$-diagonals $[P_1, P_{j+1}]$ is a conic $h_{e[j]}$ included in the pencil spanned by $e$ and $c$, provided that in the particular case of $N$-periodic billiards with even $N$ holds $j \leq \left[\frac{N-1}{2}\right]$.

The diagonal $[P_1, P_{j+1}]$ contacts $h_{e[j]}$ at the intersection with the adjacent $j$-th diagonals $[Q_{j-1}, Q_{j+1}]$ and $[Q_j, Q_{j+1}]$ of the inner billiard $Q_1Q_2Q_3 \ldots$ (Figures 8 or 9).

**Proof.** (i) According to (1), the extended sides $[P_1, P_{j+1}]$ and $[P_{j+1}, P_{j+2}]$ intersect at the point $S_{j+k}^{(i)}$, $k := \left[\frac{j}{2}\right]$, on the conic $e^{(j)}$, which belongs to the range spanned by $e$ and $c$. The restriction on $j$ in the periodic case as mentioned in Theorem 4 excludes the case where $e^{(j)}$ is a line.

The polarity in the caustic $e$ transforms this into the following statement: The connecting lines $[Q_1, Q_{j+1}]$ envelop a conic $h_{e[j]}$ which belongs to the pencil spanned by $c$ and the polar conic $q$ of $e$ w.r.t. $c$ (Figures 1 and 8). In order to obtain the first part of our statement, it is sufficient to replace the polygon $Q_1Q_2Q_3 \ldots$ inscribed in $c$ and circumscribed to $q$ by the original polygon $P_1P_2P_3 \ldots$ with the circumconic $e$ and the inconic $c$.

**Figure 8:** Envelopes $h_{e[1]}$, $h_{e[2]}$ and $h_{e[3]}$ of the diagonals of the periodic elliptic billiard $P_1P_2 \ldots P_k$ and of its inner and outer polygons $Q_1Q_2 \ldots$ and $R_1R_2 \ldots$. Triples of these diagonals together with that of $F_1F_2 \ldots$ meet at 15 points in the interior of $P_1P_2 \ldots$.

$^5$Twofold covered poses of projective billiards arise when one vertex is specified either as a point of intersection between the circumconic $e$ and the inconic $c$ or as the contact point with a common tangent between $e$ and $c$ (note the gray pose in Figure 5).
Figure 9: In the bicentric case with circumcircle $e$ and in-
nersecting incircle $c$ (blue) the envelope of the first di-
gonal (green solid) of the periodic polygon $P_1 P_2 \ldots P_n$ is
the circle $h_{c|1}$ (green) with contact points $T_1, T_2, \ldots$.

The envelope $R$ $h_{c|1}$ (pink) and the diameter $e$ $(2)$ belong to the
associated Poncelet grid.

(ii) The point of contact between $[Q_i, Q_{i+j+1}]$ and the
envelope $h_{c|j}$ is the $c$-pole of the tangent to $e^{(j)}$ at $S^{(j)}_{i+k+1}$. By
virtue of Theorem 1, this tangent passes through $R^{(j)}_{i+k}$ and
$R^{(j)}_{i+k+1}$. Hence, the requested point of contact is the meet-
ing point of the polar lines of $S^{(j)}_{i+k+1}, R^{(j)}_{i+k}$ and $R^{(j)}_{i+k+1}$ w.r.t.
c.

The $c$-polar line of $S^{(j)}_{i+k+1}$ is the diagonal $[Q_i, Q_{i+j+1}]$.

Since by (2) the point $R^{(j)}_{i+k}$ is the intersection of the tan-
gents to $e$ at $P_i$ and $P_{i+j+1}$, the $c$-polar of $R^{(j)}_{i+k}$ connects
the contact points $F_i$ and $F_{i+j+1}$ of respective sides of the
polygon $Q_1 Q_2 \ldots$ with its envelope $q$. After increasing all subscripts by 1, we obtain $[F_{i+1}, F_{i+j+2}]$ as the $c$-polar of
$R^{(j)}_{i+k}$.

In order to prove the second claim, it is sufficient to replace
the polygon $Q_1 Q_2 \ldots$ with the inconic $q$ by the polygon
$P_1 P_2 \ldots$ with the inconic $c$ and the contact point $F_{i+1}$ of the
diameter $[Q_i, Q_{i+1}]$ by the contact point $Q_i$ of the side $[P_i, P_{i+1}]$.

In Figure 8, the particular case $j = 1$ is depicted along with
the configuration of the $j$-th diagonals of $R_1 R_2 \ldots.\ P_1 P_2, \ldots, Q_1 Q_2, \ldots$, and $F_1 F_2, \ldots$ with triples of concurrent
lines. The depicted enveloping conics $h_{c|1}, h_{c|1}$ and $h_{c|1}$ of
the $j$-th diagonals of $R_1 R_2 \ldots, P_1 P_2, \ldots$ and $Q_1 Q_2, \ldots$ in
Figure 8 reveal that we obtain a sequence of triples of con-
ics like $(r, e, c)$. This reminds on sequences of billiards as
presented in [2].

**Corollary 2** Let $P_1 P_2 \ldots$ be a projective billiard with
$R_1 R_2 \ldots$ and $Q_1 Q_2 \ldots$ as outer and inner polygon, which
$F_1, F_2, \ldots$ are the contact points of the inner polygon with
its inconic $q$. Then the $j$-th diagonals of $Q_1 Q_2 \ldots$ are the
sides of another projective billiard, where the $j$-th diag-
nals of $P_1 P_2 \ldots$ are the sides of the outer polygon and that
of $F_1 F_2 \ldots$ sides of the inner polygon (Figure 8).

For later use we record a consequence of the Theorems 2
and 4:

**Lemma 2** Referring to the previous notation, the conic
$h_{e|j}$ is polar to $e^{(j)}$ w.r.t. the caustic $c$. The enveloping
point of $[S^{(j)}_1, S^{(j)}_{i+1}]$ is the $c$-pole

$$
\begin{cases}
Q^{(j)}_1 & \text{of } d := [P_{i-k}, P_{i+k+1}] \text{ for } j = 2k, \\
Q^{(j)}_{i+1} & \text{of } d := [P_{i-k+1}, P_{i+k+1}] \text{ for } j = 2k - 1.
\end{cases}
$$

The line $d$ is a $j$-th diagonal of $P_1 P_2 P_3 \ldots$ and a diagonal
of the quadrilateral consisting of the tangents drawn from
$S^{(j)}_i$ and $S^{(j)}_{i+1}$ to the caustic $c$.

The composition of the polarities in $c$ and $e$ is a collinear
transformation $\kappa$. It takes $Q_i$ to $R_i$ by (3) and (2) $Q^{(j)}_i$
to $R^{(j)}_i$ for all $i$. Moreover, it sends $c$ to $r$ and $e^{(j)}$ via $h_{e|j}$ to
$r^{(j)}$ and the envelope of the $j$-th diagonals of $Q_1 Q_2 \ldots$ to
the envelope of $j$-th diagonals of $R_1 R_2 \ldots$ (Figure 8). Lines
with equal poles w.r.t. $e$ and $c$ remain fixed under $\kappa$ as for
example the axes of symmetry of $e$ in the case of classical
billiards.

## 5 Configurations of lines related to the Pon-
celet grids

The term ‘Poncelet grid’ usually stands for a configuration
of conics, which are confocal in the particular case of Eu-
clidean billiards. Below we demonstrate that a Poncelet
grid is also combined with a configuration of lines.

The following theorem deals with the $l$-th diagonals of the
polygon $S^{(j)}_1 S^{(j)}_2 \ldots$ inscribed in the conic $e^{(j)}$ of the
Poncelet grid associated with $P_1 P_2 \ldots$ and circumscribed to the
conic $c^{(j)}$. Note that in the case $l = j$ we obtain extensions
of the sides of the original billiard $P_1 P_2 \ldots$.

**Theorem 5** The $l$-th diagonal $[S^{(j)}_1, S^{(j)}_{l+1}]$ of the polygon
$S^{(j)}_1 S^{(j)}_2 \ldots$ inscribed in $e^{(j)}$ contains three meeting points
of at least five $l$-th diagonals of other polygons of the three
involved grids (Figure 10):

(i) The contact point with the $l$-th diagonal of the polygons
of $S^{(j)}_1 S^{(j)}_2 \ldots$ is common to $[Q^{(j)}_{i-1}, Q^{(j)}_{i+1}], [Q^{(j)}_j, Q^{(j)}_{j+1}]$ as
well as for $j = 2k$ to $[Q_{i-k}, Q_{i+k+1}]$ and $[Q_{i+k}, Q_{i+k+1}]$
and for $j = 2k - 1$ to $[Q_{i-k}, Q_{i-k+1}]$ and $[Q_{i+k}, Q_{i+k+1}]$.

(ii) The intersection point with the preceding diagonal

---

**H. Stachel: A Triple of Projective Billiards**

---

49
\[ S_{i+1}^{(j)} \] belongs also to \([R_i^{(j)}]_{l+1} \), as well as for even \( j \) to \([P_{i-k-1}, P_{i+k+1}] \) and \([P_{i+k}, P_{i+k+1}] \). A similar result holds for the follower \([S_{i+1}^{(j)}, S_{i+2}^{(j)}] \).

**Proof.** (i) The first statement is a direct consequence of Theorem 4, applied to the projective billiard \( S_1^{(j)} S_2^{(j)} \ldots \) with the circumconic \( e^{(j)} \) and the inconic \( c^{(j)} \).

In order to prove the second statement of (i), we apply Lemma 2 to the polygon \( S_1^{(j)} S_2^{(j)} \ldots \), which is formed by \( l \)-th diagonals of \( S_1^{(j)}, S_2^{(j)} \ldots \), but also by diagonals of a certain type in the polygon (or the union of polygons) with the caustic \( c \) and the side lines \([S_i^{(j)}], [S_{i+1}^{(j)}] \). Hence, the contact point of \([S_i^{(j)}, S_{i+1}^{(j)}] \) with the envelope of the \( l \)-th diagonals is the \( c \)-pole of a diagonal in the quadrilateral formed by the tangents drawn from \( S_i^{(j)} \) and \( S_{i+1}^{(j)} \). According to (1), these tangents contact \( c \) respectively

\[
\begin{cases}
\text{for } j = 2k & \text{at } Q_{i-k-1}, Q_{i+k} \text{ and } Q_{i-k+1}, Q_{i+k+1}, \\
\text{for } j = 2k-1 & \text{at } Q_{i-k}, Q_{i+k+2} \text{ and } Q_{i-k+2}, Q_{i+k}.
\end{cases}
\]

Due to the rules of the polarity w.r.t. \( c \), the requested pole is the intersection of the connections of respective contact points, i.e., \([Q_{i-k-1}, Q_{i-k+1}] \cap [Q_{i-k}, Q_{i+k+1}] \) for even \( j \) and \([Q_{i-k}, Q_{i+k+1}] \cap [Q_{i-k+1}, Q_{i+k+1}] \) for odd \( j \).

(ii) From Theorem 4 applied to \( r^{(j)} \) and \( e^{(j)} \) follows that the contact point of \([R_i^{(j)}], R_{i+l}^{(j)} \) with the envelope of the \( l \)-th diagonals of \( R_1^{(j)} R_2^{(j)} \ldots \) is common to \([S_i^{(j)}, S_{i+1}^{(j)}] \) and \([S_i^{(j)}, S_{i+1}^{(j)}] \).

In order to prove the second statement, we replace in Lemma 2 the pair of conics \((c, e^{(j)})\) by \((e, r^{(j)})\) and apply this result to the polygons \( R_1^{(j)} R_2^{(j)} \ldots \) formed by \( l \)-th diagonals of \( R_1^{(j)} R_2^{(j)} \ldots \). Hence, the contact point of \([R_i^{(j)}, R_{i+l}^{(j)}] \) with the envelope of these \( l \)-th diagonals is the \( e \)-pole of a diagonal \( d \) in the quadrilateral formed by the tangents drawn from \( R_1^{(j)} \) and \( R_2^{(j)} \ldots \) to \( e \). According to (2), the requested diagonal \( d \) of the quadrilateral connects the points

\[
\begin{cases}
T_{i,k-1} \cap T_{l,k+l} & \text{for } j = 2k, \\
T_{i,k-2} \cap T_{l,k+l+1} & \text{for } j = 2k-1.
\end{cases}
\]

The \( e \)-pole of \( d \) is the intersection of the connections of respective contact points with \( e \), which confirms the claim.

\[ \square \]

**Figure 10:** Each \( l \)-th diagonal \([S_i^{(j)}, S_{i+1}^{(j)}] \) of the projective billiard \( S_1^{(j)} S_2^{(j)} \ldots \) in \( e^{(j)} \) contains three meeting points with at least four other \( l \)-th diagonals of involved polygons (Theorem 5). Here the case \( j = 2 \) and \( l = 1 \) of a periodic elliptic billiard \( P_1 P_2 \ldots P_6 \) is depicted; note the diagonal \( S_1^{(2)} S_3^{(2)} \) (red).
References


Helmut Stachel
orcid.org/0000-0001-5300-4978
e-mail: stachel@dmg.tuwien.ac.at
Vienna University of Technology
Institute of Discrete Mathematics and Geometry
Wiedner Hauptstraße 8-10, A-1040 Wien, Austria