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János Bolyai's Angle Trisection Revisited

Dedicated to Professor Hellmuth Stachel on the occasion of his 80th birthday

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ABSTRACT

J. Bolyai proposed an elegant recipe for the angle trisection via the intersection of the arcs of the unit circle with that of an equilateral hyperbola c. It seems worthwhile to investigate the geometric background of this recipe and use it as the basic idea for finding the n^{th} part of a given angle. In this paper, we shall apply this idea for the trivial case $n=4$, and for 5. Following Bolyai in the case 5, one has to intersect the unit circle with cubic curve c. There, and in the cases $n \geq 5$, we find only numerical solutions, which shows the limitation of Bolyai's method. Therefore, we propose another construction based on epicycloids inscribed to the unit circle. By this method is even possible to construct the $(\frac{n}{m})^{th}$ part of a given angle.

Key words: angle trisection, angle n-section, equilateral hyperbola, cubic, epicycloid

MSC2020: 51-03, 51M04, 51M15, 51N20, 53A04, 53A17

Ponovno razmatranje trisekcije kuta metodom Jánosa Bolyaija

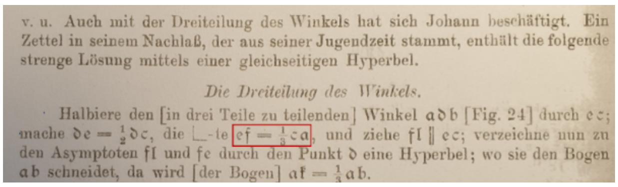
SAŽETAK

J. Bolyai je predložio elegantnu metodu za trisekciju kuta određivanjem sjecišta lukova jedinične kružnice s lukovima jednakostranične hiperbole c. Vrijedno je istražiti geometrijsku pozadinu ovog postupka te ga koristiti kao temeljnu ideju za pronalaženje n-tog dijela zadanog kuta. U ovom radu primijenit ćemo navedenu ideju u trivijalnom slučaju $n=4$, te za $n=5$. Slijedeći Bolyaija, u slučaju $n=5$ jediničnu kružnicu treba presjeći kubnom krivuljom c. U tom slučaju, kao i u slučajevima $n \geq 5$, nalazimo samo numerička rješenja, što pokazuje ograničenost Bolyaijeve metode. Stoga predlažemo i drugu konstrukciju, ovaj put utemeljenu na epicikloidi upisanoj jediničnoj kružnici. Ovom metodom moguće je čak konstruirati $\frac{n}{m}$-ti dio zadanog kuta.

Ključne riječi: trisekcija kuta, n-sekcija kuta, jednakostranična hiperbola, kubna krivulja, epicikloida

1 Angle trisection according to János Bolyai

P. Staeckel mentions in his book [3] about the geometric investigations of Wolfgang and Johann Bolyai on page 234 that "J. Bolyai delt with the angle trisection, as can be found on a slip of paper dating back to the early days of him". We present this passage from Staeckel's book in Figure 1a, b:



v. u. Auch mit der Dreiteilung des Winkels hat sich Johann beschäftigt. Ein Zettel in seinem Nachlaß, der aus seiner Jugendzeit stammt, enthält die folgende strenge Lösung mittels einer gleichseitigen Hyperbel.

Die Dreiteilung des Winkels.

Halbiere den [in drei Teile zu teilenden] Winkel adb [Fig. 24] durch ec; mache $de = \frac{1}{2}dc$, die ⌊-te $\boxed{ef = \frac{1}{3}ca}$, und ziehe $fl \parallel ec$; verzeichne nun zu den Asymptoten fl und fe durch den Punkt d eine Hyperbel; wo sie den Bogen ab schneidet, da wird [der Bogen] $af = \frac{1}{3}ab$.

Figure 1a: *Reproduction of the text concerning the angle-trisection of [3, p.234].*

A translation of the text in Figure 1a would read as follows:

The trisection of an angle
Halve the angle adb (Fig. 24) [to be divided into three parts] by ec; make now $de = \frac{1}{2}dc$, *(make) the (normal)* ⌊ $\boxed{ef = \frac{1}{3}ca}$ *and draw* $fl \parallel ec$; *draw now a hyperbola through point d and with asymptotes* fl *and* fe; *where it intersects the arc ab, the arc ak becomes* $\frac{1}{3}ab$.

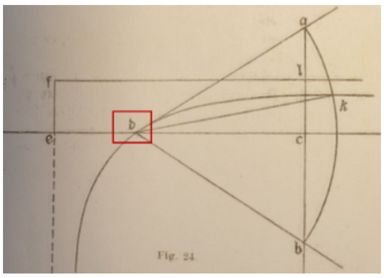


Figure 1b: *Reproduction of Fig. 24 of [3, p.234] concerning the angle-trisection.*

Figure "Fig. 24" does not exactly correspond with the"text", so there are misprints, as indicated by red rectangles. In "Fig. 24" point b, the centre of arc ab, should be labelled as "d", and in the text the term $ef = \frac{1}{3}ca$ should be replaced by $ef = \frac{1}{2}ca$.

Besides the mentioned recipe there is no further explanation or justification for it. János Bolyai (∗1802,†1860) was familiar with some Projective Geometry and the properties of conics. Therefore, one can suppose that, among geometers and mathematicians of his time, these subjects were generally known and a detailed explanation of the recipe could have been omitted. Nowadays, as mathematicians more or less disregard Classical Geometry, analytical treatment of Bolyai's construction can prove that the recipe is correct, but such proof does not show, why it is correct and how it was invented. The following chapter presents one possible idea, that J. Bolyai could have had in mind as a basis for his recipe.

2 Presumable geometric background of János Bolyai's angle trisection

We start with the unit circle u in the Eucliden plane, which, as "Gauss plane", also models the affine line of complex numbers $\mathbb{C}$. Let an angle $\angle AOB$, with O the center of u and $A,B \in u$, have the measure $\measuredangle AOB = 3\alpha$, and we use halve line OA as "real axis" in the Gauss plane. Then the complex number $z := \cos\alpha + i.\sin\alpha$ describes point B, and the cubic roots of z become $\sqrt[3]{z}_p := \cos(\frac{3\alpha}{3} + p.\frac{2\pi}{3}) + i.\sin(\frac{3\alpha}{3} + p.\frac{2\pi}{3})$, $p = 0,1,2$. These three complex numbers describe points $P_0, P_1, P_2 \in u$ forming an equilateral triangle and solving the demanded trisection of $\angle AOB$. Having the idea to intersect c with an algebraic curve through P_0, P_1, P_2 one could use a conic for this purpose. There exist a two-parameter set of conics through P_0, P_1, P_2, and we can choose one, which is somehow connected with the givens. For example, choosing orthogonal asymptote-directions in addition to P_0, P_1, P_2 selects equilateral hyperbolae h in that set. Equilateral hyperbolae have the well-known nice property, that with any three points P_0, P_1, P_2 of such a hyperbola h the orthocentre O of triangle P_0, P_1, P_2 is also a point of h, see e.g. [2, p. 54]. This theorem seems to be stated first by Charles Brianchon (∗1783,†1864) and Jean Poncelet (∗1788,†1867), which were contemporaries of J. Bolyai. So, he could have been familiar with this theorem. Within the pencil of equilateral hyperbolae h we take that one having line OA as one of the asymptote-directions, see Figure 2. Therewith h is described by

$$xy - ax - by = 0. \tag{1}$$

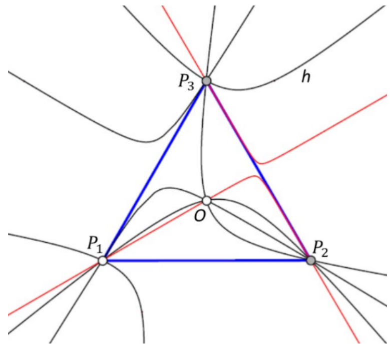


Figure 2: *Equilateral hyperbolae h through vertices of an equilateral triangle and its center O.*

Because of $\cos 3\alpha = 4(\cos\alpha)^3 - 3\cos\alpha$ and $-\sin 3\alpha = 4(\sin\alpha)^3 - 3\sin\alpha$, what we abbreviate by $V := 4x^3 - 3x$, resp. $-W := 4y^3 - 3y$, $(x := \cos\alpha, y := \sin\alpha, V^2 + W^2 = 1, x^2 + y^2 = 1)$, it follows that the intersection of h with the unit circle u must fulfil the conditions

$$(4x^3 - 3x - V)(x - S) = 0 \wedge (4y^3 - 3y + W)(y - T) = 0. \tag{2}$$

Thereby the additional fourth intersection point Q has the coordinates (S,T) with $S^2 + T^2 = 1$. We express y resp. x in (1) by $y = \frac{ax}{x-b}$ resp. $x = \frac{by}{y-a}$ and put these expressions into the equation of the unit circle u receiving the fourth order equations

$$\begin{aligned} &y^4 - 2ay^3 - y^2(a^2 + b^2 - 1) + 2ay - b^2 = 0 \\ &x^4 - 2bx^3 - x^2(a^2 + b^2 - 1) + 2by - a^2 = 0). \end{aligned} \tag{3}$$

Comparing coefficients of (3) with those of (2) delivers

$$T = 2b, W = 2b, S = 2a, V = -2a, \tag{4}$$

such that h has midpoint $M = (-\frac{1}{2}\cos 3\alpha, \frac{1}{2}\sin 3\alpha)$. The fourth intersection point has the coordinates $(S,T) = (-\cos 3\alpha, \sin 3\alpha)$ and is therefore diameter endpoint opposite to O. From the polar form of (1), and specialising with the coordinates of the origin $O = (0,0)$, it follows for the tangent t_O of h in O that

$$t_O \ldots by = -ax \ldots y = x\tan 3\alpha, \tag{5}$$

such that t_O has exactly the slope of the given angle, which is trisected by h.

We collect and visualise the mentioned properties of h in Figure 3. In connection with angle trisection we shall call this special equilateral hyperbola h the "Bolyai-hyperbola".

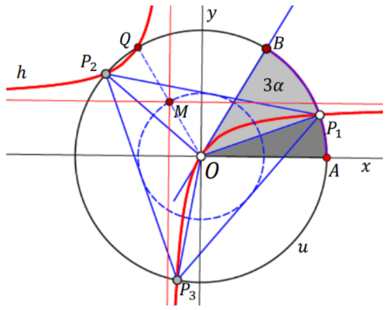


Figure 3: *Angle trisection as the intersection of the unit circle u with the "Bolyai-hyperbola" h.*

A consequence of the properties of the Bolyai-hyperbola h follows

Theorem 1 ([2, p.55]) *A circle c with its midpoint at an arbitrary point P of an equilateral hyperbola h and passing through the opposite point Q of P on h intersects h, besides in Q, in vertices of an equilateral triangle.*

It is not quite clear, who discovered the property of equilateral hyperbolae mentioned in Theorem 1. For example, it seems to be known already to H. Brocard in [1], but nowadays it cannot be considered as "widely known".

3 The $n-$section of an angle based on Bolyai's method

We follow the idea of J. Bolyai, when looking for the n^{th} part of a given angle α. As in chapter 2 we start with the n^{th} root of a complex unit number $z := \cos\alpha + i.\sin\alpha$ describing a point B at the unit circle u. The n^{th} roots of z become

$$\sqrt[n]{z}_p := \cos(\frac{\alpha}{n} + p.\frac{2\pi}{n}) + i.\sin(\frac{\alpha}{n} + p.\frac{2\pi}{n}), \quad p = 0, 1, \ldots, n-1. \tag{6}$$

These n complex numbers describe points $P_0, P_1, \ldots, P_{n-1} \in u$ forming a regular $n-$gon. If n is the product of primes and their powers, it is obvious that one proceeds consecutively. For example, let $n = n_1 \cdot n_2 \cdot \ldots \cdot n_j$ (with $n_1 \geq n_2 \geq \ldots \geq n_j$), then the first stage delivers an n_1-gon with vertices P_i, the next stage delivers n_2-gons to each angle defined by P_i. In total one receives an $n_1 n_2-$gon with vertices $P_{i,j}$, and so on until we finally get an $n_1 n_2 \ldots n_j-$gon with vertices $P_{i,j,\ldots,k}$.

Even so it is trivial, we shall deal with "halving an angle" as a first example. According to (6) there will be two solutions P_1, P_2, which are opposite points of the unit circle u. Following J. Bolyai we need an algebraic curve, which intersects u in the two points P_1, P_2. As a suitable curve of minimal degree we can take a line g, which passes through the centre O of u, see Figure 4. Given an angle 2α we put

$$\begin{aligned} &g \ldots y.\cos\alpha = x\sin\alpha, \\ &\cos 2\alpha = 2\cos^2\alpha - 1 =: V, \\ &\sin 2\alpha = 2\sin\alpha\cos\alpha =: W. \end{aligned} \tag{7}$$

Therefore, as $u \ldots x^2 + y^2 = 1$, the intersection points $g \cap u = \{P_1, P_2\}$ with

$$\begin{aligned} P_1 &= \left(+\sqrt{\frac{1}{2}(1+V)}, +\sqrt{\frac{1}{2}(1-W)}\right) =: (V_1, W_1) \\ P_2 &= \left(-\sqrt{\frac{1}{2}(1+V)}, -\sqrt{\frac{1}{2}(1-W)}\right) =: (-V_1, -W_1) \end{aligned} \tag{8}$$

and the solution angles are $\alpha_1 = \angle AOP_1$, $\alpha_2 = \angle AOP_2$, (with $A = (1, 0)$).

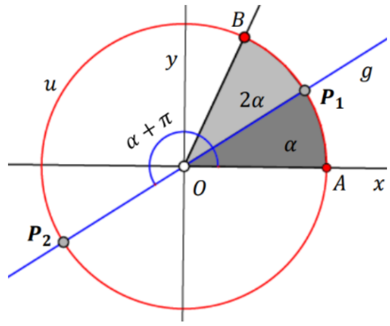


Figure 4: *Angle bisection as intersections of the unit circle u with the "Bolyai-line" g.*

If $n = 4 = 2 \cdot 2$, we can continue applying the halving procedure for the angles α_1, α_2. But we could also try a direct approach, too. We know already that the four solution points P_j must form a square inscribed to u. As a simple curve c, which intersects u in these points, we still can use a conic. The square of points P_j defines a pencil of concentric and coaxial conics. Along the lines of case $n = 3$, we can choose a "clever" conic within that pencil, namely, the degenerate one forming the diagonals of the square, see Figure 5.
Thus c is the product of the equations of two orthogonal lines:

$$c \ldots (y - kx)\left(y + \frac{1}{k}x\right) = 0, \quad k \in \mathbb{R}. \tag{9}$$

From (6) we get

$$\begin{aligned} &\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1 =: V \\ &\sin 4\alpha = 4\sin\alpha\cos\alpha(2\cos^2\alpha - 1) =: W, \end{aligned} \tag{10}$$

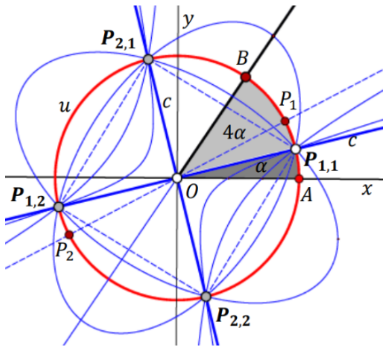


Figure 5: *Angle quadri-section as the intersection of the unit circle u with the degenerate "Bolyai-conic" c. The intermediate angle bisection and the pencil of conics through the solution points $P_{i,j}$ is shown, too.*

and we immediately can see that $\sin 4\alpha = 2\sin 2\alpha \cos 2\alpha$, $cos 4\alpha = \cos^2 2\alpha - sin^2 2\alpha$. Even so it could be calculated in a much shorter way, we want to show the general principle with this example. Rewriting (9) and (10) we get

$$
\begin{aligned}
&c \ldots x^2 + Kxy - y^2 = 0, \quad (K := (k^2-1)/k),\\
&V = 8x^4 - 8x^2 + 1\\
&W = 4xy(2x^2-1)\\
&u \ldots x^2 + y^2 = 1. \qquad (11)
\end{aligned}
$$

From these four equations we calculate K resp. k:
From $K = -(2x^2-1)/x\sqrt{1-x^2}$ follows

$$K^2 = \frac{4x^4 - 4x^2 + \frac{1}{2} + \frac{1}{2}}{x^2 - x^4 - \frac{1}{8} + \frac{1}{8}} = \frac{4(V+1)}{1-V}, \quad K_{1,2} = \pm 2\sqrt{\frac{1+V}{1-V}},$$

whereof we finally get four values for the slopes k. By considering the third equation of (11) we combine the correct sine-values y_j to the four cosine-values x_i, such that the points $P_i = (x_i, y_i)$ indeed will form a square.
We see that in the case of $n = 4$ the calculation of the algebraic problem is reducible and leads to consecutively extracting two roots. This means that the essential procedure concerns prime numbers n, as already noticed at the beginning of this chapter.

4 Finding the fifth of a given angle with Bolyai's method

As a non-trivial example, we now shall deal with the case $n = 5$. Here we expect a regular pentagon as the solution inscribed to the unit circle u. As five points already define a single conic, in our case the circle u, a low-degree curve through this pentagon surely must be at least a cubic c. There occurs an additional intersection point $Q = (S, T) \in u$, which, for special cases of the given angle 5α, might coincide with a point P_i. The set of planar cubic curves is 9-dimensional. This means that cubics through a pentagon still form a four-dimensional set and the first task would be to find a "clever" specimen within this set for our intersection purpose.

4.1 Cubics through the origin

Let $B = (\cos 5\alpha, \sin 5\alpha) = (V, W)$, and let, as a first try, c pass through the origin O. Let one ideal point U of c be the that of the $y-$axis. This means that c has an asymptote parallel to y. The consequences are some simplifications of the general equation of c:

$$x^3 + bx^2y + cxy^2 + ex^2 + fxy + gy^2 + hx + jy = 0. \qquad (12)$$

Because of

$$
\begin{aligned}
&\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha,\\
&\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha, \qquad (13)
\end{aligned}
$$

and by putting $\cos\alpha := x$, $\sin\alpha =: y$, ($x^2 + y^2 = 1$, which is the equation of u), we finally must compare (12) with

$$
\begin{aligned}
&(16x^5 - 20x^3 + 5x - V)(x - S) = 0\\
&(16y^5 - 20y^3 + 5y - W)(x - T) = 0. \qquad (14)
\end{aligned}
$$

We can eliminate y in (12) by replacing y^2 in (12) by $1 - x^2$, by separating the terms, where y occurs linearly, from the others, and finally squaring the resulting equation:

$$
\begin{aligned}
&y(bx^2 + fx + j) =\\
&= -x^3 - cx(1-x^2) - g(1-x^2) - ex^2 - hx =\\
&= x^3(c-1) + x^2(g-e) - x(c+h) - g. \qquad (15)
\end{aligned}
$$

By squaring, and again replacing y^2 by $1 - x^2$, we receive an equation of degree 6 in x. Thereby we abbreviate $c - 1 =: C$, $g - e =: E$, $c + h =: H$. The left side of (15) becomes

$$
\begin{aligned}
&(1-x^2)(b^2x^4 + 2bfx^3 + (f^2 + 2bj)x^2 + 2fjx + j^2) =\\
&= -b^2x^6 - 2bfx^5 + (b^2 - 2bj - f^2)x^4 + 2f(b-j)x^3 +\\
&\quad + (f^2 + 2bj - j^2)x^2 + 2fjx + j^2.
\end{aligned}
$$

The right side of (15) becomes

$$
\begin{aligned}
&C^2x^6 + 2CEx^5 + (E^2 - 2CH)x^4 - (2EH + 2Cg)x^3 +\\
&+ (H^2 - 2Eg)x^2 + 2Hgx + g^2.
\end{aligned}
$$

Both sides together deliver the equation

$$
\begin{aligned}
&(b^2 + C^2)x^6 + 2(CE + bf)x^5 +\\
&+ (E^2 - 2CH - b^2 + 2bj + f^2)x^4 -\\
&- (2EH + 2Cg - 2bf + 2fj)x^3 +\\
&+ (H^2 - 2Eg - f^2 - 2bj + j^2)x^2 +\\
&+ 2(Hg - fj)x + (g^2 - j^2) = 0, \qquad (16)
\end{aligned}
$$

and now we can compare (16) with

$$16x^6-16Sx^5-20x^4+20Sx^3+5x^2-(5S+V)x+VS=0. \tag{17}$$

The same way we eliminate x in (12) and square the following equation:

$$x(y^2(1+c)+fy+h)=by^3+y^2(e-g)-y(b+j)-e.$$

Abbreviating $g-e=:E$, $b+j=:J$ we get

$$\begin{aligned}&(b^2+C^2)y^6+2(Cf-bE)y^5+\\&+(-C^2+E^2+2hC+f^2)y^4+\\&+(2EJ-2be-fC+fh)y^3+\\&+(2eE-2Ch+J^2-f^2+h^2)y^2+\\&+2(eJ-fh)y+(e^2-h^2)=0\end{aligned} \tag{18}$$

and can compare it with

$$16y^6-16Ty^5-20y^4+20Ty^3+5y^2-(5T+W)y+WT=0. \tag{19}$$

We collect the coefficients of equations (16) to (19) in the Table 1.
The 14 equations are not independent, the equations (0)-(6) resp. (0')-(6') alone allows us to express the coefficients $b,\dots,j$ of the cubic c as functions of V,W and S,T.
We see that the conditions of type (0)...(6') are quadratic equations in the unknowns b,c,e,f,g,h,j,S,T and the givens V,W. They can be interpreted as hyperquadrics $Q_j^{(2)}$ in an 11-dimensional affine space $\mathbf{A}^{11}$, whereby $Q_0^{(2)}$ and $Q_1^{(2)}$ are the hypercylinders with equations $V^2+W^2=1$ resp. $S^2+T^2=1$.
When we use $V,W(V)$ as parameters, we finally will get a curve as intersection $Q_j^{(2)}\cap\cdots\cap Q_k^{(2)}$, which represents a one-parameter set of cubics c. Obviously, to a fixed parameter pair $V_0,W(V_0)$ there will be, in the algebraic sense, up to 32 solutions of cubics $c(V_0)$. Figure 6 (left) shows an example of one solution of the cubics c belonging to the given angle $5\alpha=98°$. To each given angle 5α the calculations must be performed individually.
A similar calculation is performed for cubics having the ideal point of the $y-$axis as inflection point. Figure 6 (right) shows an example of this kind calculated to an angle $5\alpha=60°$.

Equ. No.	Degr.	Coeff. (16)	Coef. (17)	Equ. No.	Degr.	Coeff. (18)	Coef. (19)
(6)	x^6	$b^2+(1-c^2)$	16	(6')	y^6	$b^2+(c-1)^2$	16
(5)	x^5	$2((1-c)(e-g)+bf)$	$-16S$	(5')	y^5	$2((c-1)f-b(g-e))$	$-16T$
(4)	x^4	$(e-g)^2+2(1-c)(c+h)$ $-b^2+2bj+f^2$	-20	(4')	y^4	$-(c-1)^2+(g-e)^2$ $+2(c-1)(h+1)$ $-2b(b+j)+f^2$	-20
(3)	x^3	$2(e-g)(c+h)$ $+2(1-c)g-2bf+2fj$	$20S$	(3')	y^3	$2(g-e)(b+j)$ $-2be-2f(c-1)$ $+f(h+1)$	$20T$
(2)	x^2	$2(c+h)^2+2g(e-g)$ $-f^2-2bj+j^2$	5	(2')	y^2	$2e(g-e)$ $2(h+1)(c+1)$ $+(b+j)^2-f^2$ $+(h+1)^2$	5
(1)	x^1	$2((c+h)g-fj)$	$-5S-V$	(1')	y^1	$2(e(b+j)-f(h+1))$	$-5T-W$
(0)	x^0	g^2-j^2	VS	(0')	y^0	e^2-h^2	WT

Table 1: *Coefficients of (16) - (19) for the comparing procedure*

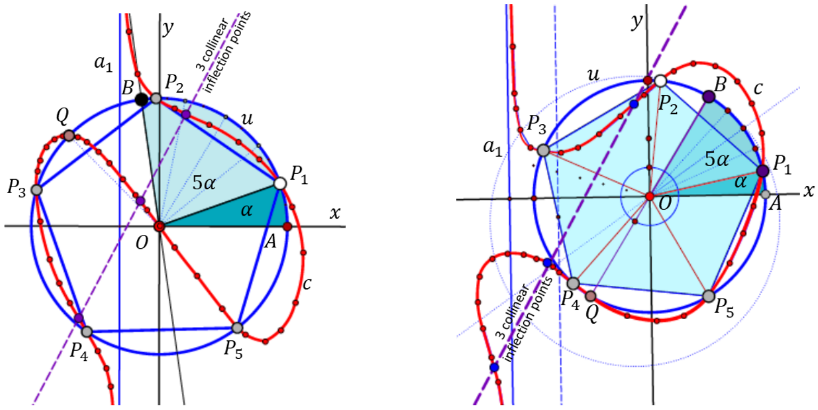


Figure 6: *Numerically gained solutions of "quinti-sectioning" the given angle* $\angle AOB = 5\alpha$.

4.2 Cubics with three given ideal points

One of the key-conditions of Bolyai's recipe is that all the "Bolyai hyperbolas" have the same ideal points and therefore are similar. In a *new approach* we focus at cubics $c(V)$ with the same triplet of ideal points. Note that the ideal points of $c(V)$ must be different from those of the unit circle. Let us try with three real ideal points $U_1 = (0,1,0)\mathbb{R}$, $U_2 = (0,0,1)\mathbb{R}$, $U_3 = (0,-1,1)\mathbb{R}$, (here we use homogeneous coordinates $(x_0,x_1,x_2)\mathbb{R}$ instead of affine coordinates $(1,x,y) =: (x,y)$). The equation of the general cubic c through these ideal points U_i becomes

$$x^2y + xy^2 + ex^2 + fxy + gy^2 + hx + jy + k = 0. \tag{20}$$

The tangents a_i at U_i, i.e. the asymptotes of c, are

$$a_1 \ldots y = -e, \quad a_2 \ldots x = -g, \quad a_3 \ldots y = -x + e - f + g. \tag{21}$$

When we demand that U_3 shall be an inflection point of c, i.e. a_3 is an inflection asymptote, the coefficients $e, \ldots, k$ fulfil the conditions

$$e^2 - g^2 - ef + fg + h - j = 0 \ldots Q^{(2)}_{a_3}, \tag{22}$$

$$g(e-f)^2 + 2(e-f)g^2 + g^2 + (e-f+g)j + k = 0 \ldots Q^{(3)}_{a_3}.$$

We shall compare (20) with (17) based on the condition $x^2 + y^2 = 1$. From (20) follows

$$(x^2 + fx + j) = x^3 - x^2(e-g) - x(1+h) - (g+k). \tag{23}$$

We abbreviate $e - g =: G$, $h + 1 =: H$, $g + k =: K$ and square (23), we finally receive

$$2x^6 + 2(f-G)x^5 + (G^2 - 2H - 1 + f^2 + 2j)x^4 +$$
$$+ 2(GH - K - f + fj)x^3 + (H^2 + 2GK - f^2 + j^2 - 2j)x^2 +$$
$$+ (2HK - 2fj)x + (K^2 - j^2) = 0. \tag{24}$$

Similarly, we shall compare (19) with (20) based on condition $x^2 + y^2 = 1$. From (20) follows now $-x(y^2 + fy + h) = (1-y^2)y + e(1-y^2) + gy^2 + jy + k$ and when we abbreviate again $e - g =: G$, $j + 1 =: J$, $e + k =: E$, we get

$$x(y^2 + fy + h) = y^3 + Gy^2 - Jy - E, \tag{25}$$

and by squaring this it becomes

$$2y^6 + 2(f-G)y^5 + (G^2 - 2J - 1 + f^2 + 2h)y^4 +$$
$$+ 2(GJ - E - f + fh)y^3 + (J^2 - 2EG - f^2 + h^2 - 2h)y^2 +$$
$$+ 2(EJ - fh)y + (E^2 - h^2) = 0. \tag{26}$$

By putting "(4)=(4')" and "(2)=(2')" we get $h = j$ and $g = e$ such that (20) simplifies to

$$x^2y + xy^2 + ex^2 + fxy + ey^2 + hx + hy + k = 0,$$

but again, we only get numerical solutions, (see Figure 7), as we must intersect hyperquadrics (and hyperplanes) in an 11-dimensional affine space. None of the results are such that there exists a one-parameter set of similar cubics. This allows at least to

Conjecture 1 *There is no irreducible cubic carrying a one-parameter set of regular pentagons.*

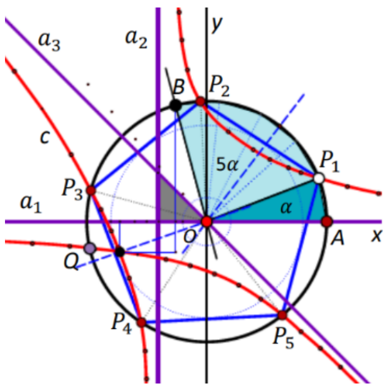


Figure 7: *Numerically gained solutions of "quinti-sectioning" the given angle* $\angle AOB = 5\alpha = 105°$ *with help of a cubic with three real asymptotes.*

Equ. No.	Degr.	Coeff. (24)	Coef. (17)	Equ. No.	Degr.	Coeff. (26)	Coef. (19)
(6)	x^6	2	16	(6')	y^6	2	16
(5)	x^5	$2(f-G)$	$-16S$	(5')	y^5	$2(f-G)$	$-16T$
(4)	x^4	$G^2-2H-1+f^2+2j$	-20	(4')	y^4	$G^2-2J-1+f^2+2h$	-20
(3)	x^3	$2(GH-K-f+fj)$	$20S$	(3')	y^3	$2(GJ-E-f+fh)$	$20T$
(2)	x^2	$H^2+2GK-f^2$ $+j^2-2j$	-5	(2')	y^2	$2(J^2-2EG-f^2$ $+h^2-2h)$	-5
(1)	x^1	$2(HK-fj)$	$-5S-V$	(1')	y^1	$2(EJ-fh)$	$-5T-W$
(0)	x^0	K^2-j^2	SV	(0')	y^0	E^2-h^2	TW

Table 2: *Coefficients of (24), (26) for the comparing procedure with (17), (19)*

4.3 Reducible cubics through regular pentagons

We try now with a conic c, which should pass through four points of the regular solution pentagon, and a line l through its fifth point. Thereby c and l shall have the ideal point $(0,0,1)\mathbb{R}$ of the $y-$axis in common. Here we will get, in general, five solutions, as there are five possibilities for l (and for c). But here, too, the explicit calculation turns out to become lengthy and results in numerical gained solutions, see Figure 8 showing solutions with a reducible cubic splitting into a hyperbola c and a line l parallel to the $y-$axis.

We note that within the pencil of conics $P_2,\ldots,P_5$ there is a special hyperbola c passing through the origin O. It is symmetric to OP_1, its asymptotes include $120°$, and its midpoint's M distance from the origin is one-third of the radius of (unit) circle u, (see Figure 9 showing c in the standard position $5\alpha=0$). We used it already in Figure 8, but now we will add line OA as the linear component l of the reducible cubic c in standard position and rotate this cubic c according to the given angle 5α.

Hyperbola c and line l in standard position have the equations

$$c\ldots 3x^2+2x-y^2=0, \quad l\ldots y=0. \tag{27}$$

We rotate by angle φ, and we abbreviate $\sin\varphi =: s$, $\cos\varphi =: t$ to get the formulas shorter and better readable. We know already that φ must turn out to become the solution angle α. The rotated version of (27) reads as

$$c\ldots(3t^2-s^2)x^2+8stxy-(2s^2-t^2)y^2+2tx+2sy=0,$$
$$l\ldots sx-ty=0. \tag{28}$$

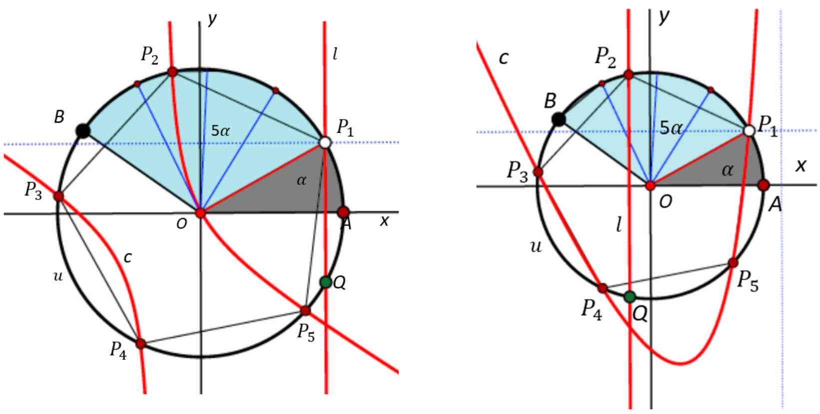


Figure 8: *Numerically gained solutions of "quinti-sectioning" the given angle $\angle AOB=5\alpha$ with help of a reducible cubic with a line component l parallel y.*

Equ.	x^6	x^5	x^4	x^3	x^2.	x^1	x^0
(17)	16	$-16U$	-20	$+20U$	$+5$	$-(5U+V)$	$+UV$
"$(29)^2$"	16	$16t$	...	...	...	...	$-t^2(16t^4-20t^2+5t)$

Table 3: *Comparison of the coefficients of the squared equation (29) with those of (17)*

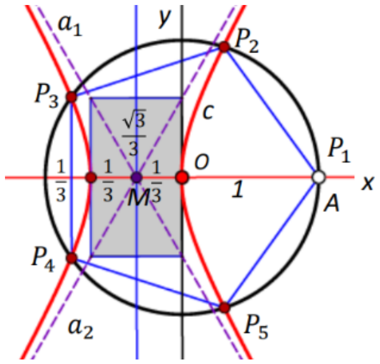


Figure 9: *Hyperbola c through $O, P_2, \ldots, P_5$ in standard position.*

With $3t^2 - s^2 = 4t^2 - 1$ and $3s^2 - t^2 = 4s^2 - 1$ we find the equation of the reducible cubic c as

$$s(4t^2-1)x^3 + 3t(4s^2-1)x^2y - 3s(4t^2-1)xy^2 - \\ -t(4s^2-1)y^3 + 2stx^2 + 2(1-2t^2)xy - 2sty^2 = 0. \quad (29)$$

Following the procedure used in the former chapters, we again put $y^2 = 1 - x^2$ and separate terms containing y from those containing only x^k, $k = 0, \ldots, 3$. By squaring this equation, we finally receive an equation of 6th degree in the variable x and with coefficients being functions of s and t, which are the only unknowns. Therefore, it is enough to consider the coefficients of x^6, x^5, and x^0 for connecting s, t with the given cosine V of polynomial (17), see table 3.
From the coefficients of x^6 follows that the proportionality factor of the two equations (17) and "$(29)^2$" is 1. From x^5 we get $U = -t$, as expected, and from x^0 we receive the polynomial (13). Therefore, we end up with a tautology: To get the rotation angle we have to solve the original equation (13) $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$.
All the presented attempts to get sort of a standard cubic to solve the 5-section of an angle failed.
Result: The $n-$section ($n \geq 5$) of an angle $n\alpha$ based on Bolyai's method to intersect the unit circle with an algebraic curve c of suitable degree leads to calculating the coefficients of an equation of c individually to each given angle $n\alpha$.

Remark 1 *Angle trisection, as one of the classical cubic problems, is only graphically solved via intersecting the unit circle with Bolyai's equilateral hyperbola. An exact solution should solve an equation of the third degree, too. In the following chapter, we present a possibility for a graphic solution using well-known properties of epicycloids.*

5 $p-$sectioning an angle using epicycloids

A generally applicable graphical solution of $p-$sectioning ($p \in \mathbb{Q}$) a given angle can be based on epicycloids, see e.g. [4, p.156] and [5]. Due to a theorem of F.E. Eckhardt (c.f. [4]) the line connecting the points B and P_1, which move along the unit circle u with speed $p\alpha$ resp. α envelops an epicycloid e with the parameter representation

$$e \ldots \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{p+1} \begin{pmatrix} p\cos\alpha + \cos(p\alpha) \\ p\sin\alpha + \sin(p\alpha) \end{pmatrix}. \quad (30)$$

Such a cycloid admits two kinematic generations with circles m and m' rolling on a fixed circle f. The radii r_f, r_m, r'_m of fixed circle f and moving circles m and m' are therewith

$$r_f = \frac{1-p}{1+p}, \quad r_m = \frac{p}{1+p}, \quad r'_m = \frac{1}{1+p}. \quad (31)$$

The following Figures 10, 11 and 12 show examples of such graphical angle p-sections. Thereby some additional properties of cycloids become obvious:
In both cases shown in Figure 10, the angle bisection and trisection, we notice that, besides of two orthogonal tangents t_1, t_2, resp. three tangents t_1, t_2, t_3 (inclosing 120° angles) intersect the (unit) circle u in the solution points P_1, P_2, resp. P_1, P_2, P_3. There occurs an additional tangent t with no meaning for the bisection problem. Figure 11 shows these properties, too. In all cases the touching points E_j of t_j with the epicycloids e are the intersection points of the moving circle m' (center $M' \in OB$) with t_j. The intersection of m' with the additional tangent t is not a point of epicycloid e. The segments $[E_jM']$ are parallel to the solution segments $[OP_j]$. The circle u is an "orthoptic locus" of the cardioid e (Figure 10 (left)) and, as a consequence of the "angle at circumference theorem", u is a "multi-isoptic locus" of the epicycloids e (Figure 10 (right) and Figure 11). The points

E_j form a regular $n-$gon inscribed to e and m', such that this $n-$gon moves along e, when B moves along the circle u. (By the way, this well-known property of epitrochoids has the "Wankel-motor" as a technical application, see e.g. [6].) Figure 12 shows examples of $p-$sectioning an angle for the cases $p = \frac{2}{5}$ and $p = \frac{3}{5}$. Here we find in fact the same properties as described above.
For $p = \frac{2}{5}$ it follows from (31) that $r_f = \frac{3}{7}$, $r_m = \frac{2}{7}$, $r'_m = \frac{5}{7}$, and for $p = \frac{3}{5}$ we get that $r_f = \frac{1}{4}$, $r_m = \frac{3}{8}$, $r'_m = \frac{5}{8}$. The cycloids e have threefold resp. twofold symmetry, and again, a regular pentagon can move in e.

We collect these results in

Theorem 2 *Let $\varphi = \angle AOB$ be the main value of a given angle, (A, B points of the (unit) circle u), and let e be the $p-$epicycloid with fixed circle f (radius $\frac{1-p}{1+p}$) concentric with u and A as vertex. Then one can construct the p^{th} part(s) ($p \in \mathbb{Q}$) of φ by drawing the tangents t_j from B to e and intersect them with u. The intersection points P_j forming a regular polygon define the solution angles $\angle AOP_j$. The circumcircle u of e is a multi-isoptic locus for the epicycloid e.*

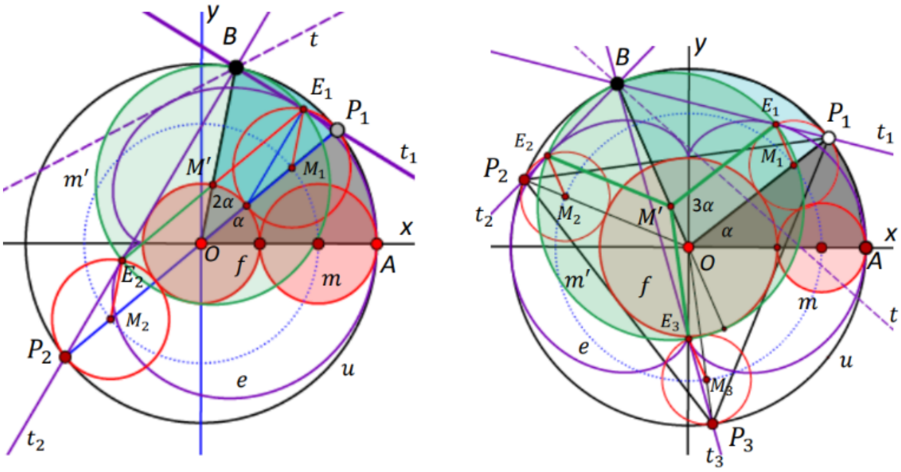


Figure 10: *Angle bisection and trisection with help of a cardioid resp. a nephroid.*

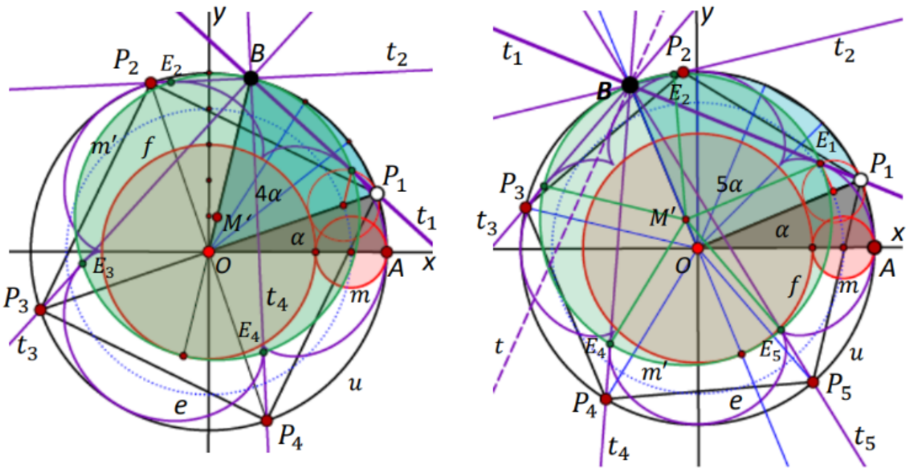


Figure 11: *Angle 4-section and 5-section with help of epicycloids.*

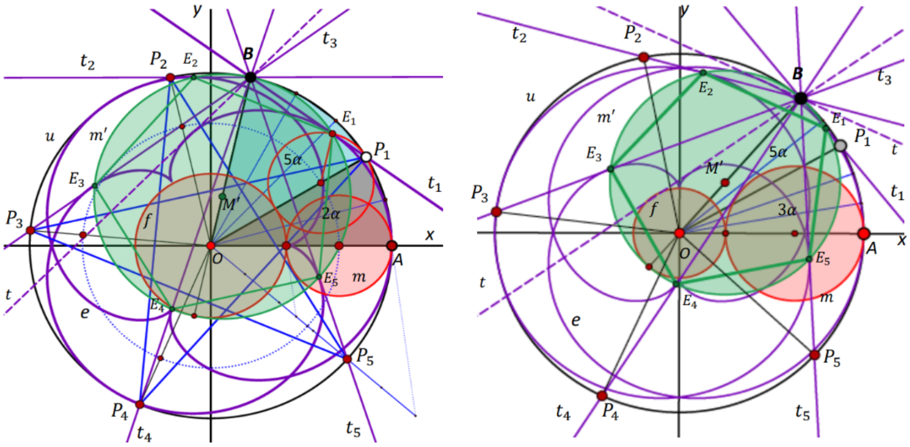


Figure 12: *The $\frac{2}{5}$ and $\frac{3}{5}$ of an angle with help of epicycloids.*

6 Conclusion

In this paper, we tried to "explain" Bolyai's classical method of angle trisection and extend it to $n-$ resp. $p-$sectioning an angle, ($n \in \mathbb{N}$, $p \in \mathbb{Q}$). While the trisection uses an equilateral hyperbola in standard position, the 5−section must use cubics (or curves of higher degree), which have to be calculated individually to each given angle 5α. An equilateral hyperbola c allows a "similarity-motion" of an equilateral triangle, such that its vertices move along c. We could not find a cubic c allowing a similarity- motion to a regular pentagon, such that its vertices move along c. Therefore, this extension of Bolyai's method has no practical application and is finally adapted to replacing c with epicycloids e in the standard position. For $p \in \mathbb{Q}$ these epicycloids e are closed, and they admit a congruence motion of regular $p-$stars, such that their vertices move along e, a well-known property, which is basic for Wankel motors. Obviously, because of the theorem of the angle at circumference, the circumcircle u of e is a "multi-isoptic locus" for e. Finally, one might add that this "epicycloid-method" also works for $p \in \mathbb{R}$, but in such cases, one should restrict the construction to the main value $P_1 \in u$.

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