GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 185 – 201

ON THE MULTIPLICITY IN PILLAI'S PROBLEM WITH FIBONACCI NUMBERS AND POWERS OF A FIXED PRIME

Herbert Batte, Mahadi Ddamulira, Juma Kasozi and Florian Luca

Makerere University, Uganda and University of the Witwatersrand, South Africa

Abstract. Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers and let p be a prime. For an integer c we write $m_{F,p}(c)$ for the number of distinct representations of c as $F_k - p^\ell$ with $k \geq 2$ and $\ell \geq 0$. We prove that $m_{F,p}(c) \leq 4$.

1. Introduction

1.1. *Background.* Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. The first few terms of this sequence are given by

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \ldots.$$

For fixed integers $a > 1, b > 1, c$, the Diophantine equation

$$a^x - b^y = c, \tag{1.1}$$

in nonnegative integers x, y is known as the Pillai equation, see [13]. Pillai was interested if the above equation can have more than one solution (x, y) and proved that if a and b are positive and coprime and $|c| > c_0(a, b)$, then the above equation has at most one solution (x, y). Variants of the Pillai problem have been recently considered in which one takes a to be 2 or 3 but replaces the sequence of powers of b by some other sequence of positive integers of exponential growth such as Fibonacci numbers, Tribonacci numbers, Pell numbers and even k–generalized Fibonacci numbers where k is also unknown. In all of

2020 *Mathematics Subject Classification.* 11B39, 11D61, 11J86.

Key words and phrases. Fibonacci numbers, prime numbers, linear forms in logarithms, Pillai's problem.

these results, it was shown that the conclusion of the original Pillai problem is retained (so, every large integer has at most one such representation) except for some cases where parametric families exist which are completely classified together with the small exceptional cases with multiple such representations which have also been computed. See, for example, [3, 5, 6, 7]. Here, we retain the Fibonacci sequence but replace powers of 2 or 3 by powers of an arbitrary but fixed prime p. Write

$$m_{F,p}(c) := \#\{(k,\ell) : k \geq 2,\ \ell \geq 0,\ c = F_k - p^\ell\}.$$

We imposed the condition $k \geq 2$ above because $F_1 = F_2 = 1$. Our result is the following.

1.2. *Main result.*

THEOREM 1.1. *The inequality $m_{F,p}(c) \leq 4$ holds for all primes p and all integers c.*

We believe that the better result $m_{F,p}(c) \leq 3$ holds, but we did not succeed in proving this. Further, quite possibly $m_{F,p}(c) = 3$ holds only for finitely many pairs (p, c), and maybe only for the following 3 pairs

$$\begin{array}{rrcccccc} (2,-3): & -3 & = & F_7 - 2^4 & = & F_5 - 2^3 & = & F_2 - 2^2, \\ (2,0): & 0 & = & F_6 - 2^3 & = & F_3 - 2^1 & = & F_2 - 2^0, \\ (2,1): & 1 & = & F_5 - 2^2 & = & F_4 - 2^1 & = & F_3 - 2^0. \end{array} \tag{1.2}$$

We leave proving that $m_{F,p}(c) \leq 3$ and classifying the pairs of integers (p, c) with $m_{F,p}(c) = 3$ as a problem to the reader. On the other hand, we believe that $m_{F,p}(c) = 2$ holds for infinitely many pairs (c, p). For that, it suffices to look for c with two representations of the form $c = F_{k_1} - p = F_{k_2} - 1$, so the two representations (k_1, ℓ_1) and (k_2, ℓ_2) have $\ell_1 = 1,\ \ell_2 = 0$. Then $p = F_{k_1} - F_{k_2} + 1$. A calculation revealed 2161 primes p of the above form in the range $2 \leq k_2 < k_1 \leq 1000$.

Before embarking to the proof, let us remark that it is not surprising that $m_{F,p}(c)$ is bounded by an absolute constant. Indeed, letting $m_{F,p}(c) = m$, then the equation

$$c = F_k - p^\ell$$

has m solutions (k, ℓ) with $k \geq 2$ and $\ell \geq 0$. If $c = 0$, we get $F_k = p^\ell$. The only solution of this equation with $\ell \geq 2$ is $F_6 = 2^3$ by the well-known result concerning perfect powers in the Fibonacci sequence [4]. Thus, $m_{F,p}(0) \leq 3$, and $m_{F,p}(0) = 3$ holds only for the prime $p = 2$. When $c \neq 0$, then by using the Binet formula for the Fibonacci sequence

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \tag{1.3}$$

valid for all $k \geq 0$, our equation can be rewritten as

$$1 = \frac{1}{c\sqrt{5}}\alpha^k - \frac{1}{c\sqrt{5}}\beta^k - \frac{1}{c}p^\ell.$$

This is a particular case of the equation $1 = a_1x_1 + \cdots + a_sx_s$ with $s = 3$, $(a_1, a_2, \ldots, a_s) = (1/(c\sqrt{5}),\ \pm 1/(c\sqrt{5}), -1/c)$ and $x_1, x_2, \ldots, x_s$ unknowns in the multiplicative group Γ generated by $\{\alpha, p\}$ inside $\overline{\mathbb{C}}^*$ of rank $r = 2$ (note that $\beta = -\alpha^{-1}$). Furthermore, such a solution is nondegenerate in the sense that no subsum $\sum_{i\in I} a_ix_i$ vanishes for some subset $I \subseteq \{1, 2, \ldots, s\}$ since for us $k \geq 2$. Theorem [1, Theorem 6.1] immediately gives that for a fixed choice of nonzero coefficients $(a_1, a_2, \ldots, a_s)$ the number of such solutions is

$$\leq (8s)^{4s^4(s+r+1)},$$

so for us

$$m \leq 2 \cdot (8 \cdot 3)^{4\cdot 3^4(3+2+1)}$$

and the right–hand side above exceeds 10^{2500}. Thus, while the boundedness of m follows easily from known results on the finiteness of non-degenerate solutions to $\mathcal{S}$-unit equations, the merit of our paper is to give a bound on m which is quite close to the best possible.

2. Methods

We use three times Baker-type lower bounds for nonzero linear forms in two or three logarithms of algebraic numbers. There are many such bounds mentioned in the literature like that of Baker and Wüstholz from [2] or Matveev from [10]. Before we can formulate such inequalities we need the notion of height of an algebraic number recalled below.

Definition 2.1. *Let γ be an algebraic number of degree d with minimal primitive polynomial over the integers*

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \gamma^{(i)}),$$

where the leading coefficient a_0 is positive. Then, the logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d}\log\max\{|\gamma^{(i)}|, 1\}\Big).$$

In particular, if γ is a rational number represented as $\gamma = p/q$ with coprime integers p and $q \geq 1$, then $h(\gamma) = \log\max\{|p|, q\}$.

The following properties of the logarithmic height function $h(\cdot)$ will be used in the rest of the paper without further reference:

$$\begin{aligned} h(\gamma_1 \pm \gamma_2) &\leq h(\gamma_1) + h(\gamma_2) + \log 2; \\ h(\gamma_1 \gamma_2^{\pm 1}) &\leq h(\gamma_1) + h(\gamma_2); \\ h(\gamma^s) &= |s| h(\gamma) \quad \text{valid for} \quad s \in \mathbb{Z}. \end{aligned}$$

A linear form in logarithms is an expression

$$\Lambda := b_1 \log \alpha_1 + \cdots + b_t \log \alpha_t, \tag{2.1}$$

where for us $\alpha_1, \ldots, \alpha_t$ are positive real algebraic numbers and $b_1, \ldots, b_t$ are nonzero integers. We assume, $\Lambda \neq 0$. We need lower bounds for $|\Lambda|$. We write $\mathbb{K} = \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ and D for the degree of $\mathbb{K}$. We start with the general form due to Matveev [10].

Theorem 2.2. *Put* $\Gamma := \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1 = e^{\Lambda} - 1$. *Then*

$$\log |\Gamma| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t,$$

where $B \geq \max\{|b_1|, \ldots, |b_t|\}$ *and* $A_i \geq \max\{D h(\alpha_i), |\log \alpha_i|, 0.16\}$ *for* $i = 1, \ldots, t$.

We continue with $t = 2$. Let $A_1 > 1,\ A_2 > 1$ be real numbers such that

$$\log A_i \geq \max \left\{ h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D} \right\} \quad \text{for} \quad i = 1, 2. \tag{2.2}$$

Put

$$b' := \frac{|b_1|}{D \log A_2} + \frac{|b_2|}{D \log A_1}.$$

The following result is Corollary 2 in [9].

Theorem 2.3. *In case* $t = 2$, *we have*

$$\log |\Lambda| \geq -24.34 D^4 \left(\max \left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 \log A_1 \log A_2.$$

After some calculations with the above theorems, we end up with some upper bounds on our variables which are too large, thus we need to reduce them. We use the following result of Legendre which is related to continued fractions (see [12, Theorem 8.2.4]).

Lemma 2.4. *Let* τ *be an irrational number with continued fraction* $[a_0, a_1, \ldots]$ *and convergents* $p_0/q_0,\ p_1/q_1, \ldots$. *Let* M *be a positive integer. Let* N *be a non-negative integer such that* $q_N > M$. *Then putting* $a(M) := \max\{a_i : 0 \leq i \leq N\}$, *the inequality*

$$\left| \tau - \frac{r}{s} \right| > \frac{1}{(a(M) + 2) s^2},$$

holds for all pairs (r, s) *of positive integers with* $0 < s < M$.

Let $\{L_n\}_{n\geq 0}$ be the Lucas sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. We need the following lemma.

LEMMA 2.5. *If $m \geq n$ and $m \equiv n \pmod 2$, then*

$$F_m - F_n = F_{(m-\delta n)/2} L_{(m+\delta n)/2}, \quad \text{where} \quad \delta = (-1)^{(m-n)/2}.$$

For a prime p let $z(p)$ be *the order of appearance* of the prime p in the Fibonacci sequence (sometimes also called *the entry point* of p) which is the smallest positive integer k such that $p \mid F_k$. This exists for every prime number p. It is well-known that $p \mid F_k$ if and only if $z(p) \mid k$. Further, writing $\nu_p(m)$ for the exponent of p in the factorization of m, and putting $e_p := \nu_p(F_{z(p)})$, then it is well-known that whenever $p \mid F_k$ we have $\nu_p(F_k) \geq e_p$ and further if $f = \nu_p(F_k)$ is positive then $p^{f-e_p} z(p) \mid k$. The following computational result is due to McIntosh and Roettger ([11]).

LEMMA 2.6. *If $p < 10^{14}$, then $p \| F_{z(p)}$.*

Finally, we present an analytic argument which is [8, Lemma 7]. It is useful when obtaining upper bounds on some positive real variable involving powers of the logarithm of the variable itself.

LEMMA 2.7. *If $s \geq 1$, $T > (4s^2)^s$ and $T > \dfrac{x}{(\log x)^s}$, then*

$$x < 2^s T (\log T)^s.$$

In the addition to the above results, we also used computations with Mathematica.

3. Proof of Theorem 1.1

3.1. *Notation.* From now on, we work with pairs (p, c) such that $m_{F,p}(c) \geq 3$. We may assume that $p \geq 5$ since the cases $p = 2, 3$ were treated in [6] and [7], respectively. We write $m := m_{F,p}(c)$ and write

$$c = F_{k_i} - p^{\ell_i} \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{3.1}$$

We assume $\ell_1 > \ell_2 > \cdots > \ell_m \geq 0$. Then,

$$F_{k_i} - F_{k_j} = p^{\ell_i} - p^{\ell_j} > 0, \quad \text{for} \quad 1 \leq i < j \leq m, \tag{3.2}$$

so $k_i > k_j$. Thus, $k_1 > k_2 > \cdots > k_m \geq 2$.

3.2. *$k_1 \leq 1000$.* Suppose that $k_1 \leq 1000$. We considered the Diophantine equation $F_{k_2} - F_{k_3} = p^{\ell_2} - p^{\ell_3}$ for $2 \leq k_3 < k_2 \leq 1000$, $p \geq 5$ and $0 \leq \ell_3 < \ell_2$. Taking $\ell_3 = 0$, we get $F_{k_2} - F_{k_3} + 1 = p^{\ell_2}$. The above equation has 2161 solutions (k_2, k_3, p, ℓ_2) in the range $1000 \geq k_2 > k_3 \geq 2$ with $\ell_2 = 1$ and only one solution with $\ell_2 > 1$ which is $F_{14} - F_{11} + 1 = 17^2$ (other interesting formulas are $F_{12} - F_2 + 1 = 12^2$ and $F_{24} - F_{12} + 1 = 215^2$, but 12 and 215 are not primes). For each one of these 2162 quadruplets (k_2, k_3, p, ℓ_2), we checked whether there exists $k_1 \in [k_2+1, 1000]$ such that $F_{k_1} - (F_{k_3} - 1) = p^{\ell_1}$ for some positive exponent ℓ_1 and did not find any such instance. This code ran for a few hours in Mathematica.

Assume next that $\ell_3 \geq 1$. We fix $2 \leq k_3 < k_1 \leq 1000$. Then

$$(p-1)^{\ell_2} < p^{\ell_2} - p^{\ell_3} = F_{k_2} - F_{k_3} < p^{\ell_2}.$$

So, $\ell_2 \leq (\log(F_{k_2} - F_{k_3}))/\log 4$ and once ℓ_2 is a fixed positive integer in the above range, we have $p = 1 + \lfloor (F_{k_2} - F_{k_3})^{1/\ell_2} \rfloor$. Having found p, we calculate

$$\ell_3 = \lfloor \log(p^{\ell_2} - (F_{k_2} - F_{k_3}))/\log p \rfloor,$$

and check whether $\ell_3 \geq 1$ and $p^{\ell_3} = p^{\ell_2} - (F_{k_2} - F_{k_3})$. This program ran for a day or so in Mathematica and did not find any solutions. The only solutions found for $F_{k_1} - p^{\ell_1} = F_{k_2} - p^{\ell_2}$ where $1000 \geq k_1 > k_2 \geq 2$ and $\ell_1 > \ell_2 \geq 1$ were

$$F_8 - 5^2 = F_2 - 5, \quad F_{10} - 7^2 = F_7 - 7, \quad F_{12} - 11^2 = F_9 - 11.$$

So, our computation shows that there is no integer c having at least three representations as $F_{k_i} - p^{\ell_i}$ with $2 \leq k_3 < k_2 < k_1 \leq 1000$ and some prime $p \geq 5$. So, from now on we assume that $k_1 > 1000$ when $m \geq 3$ and $k_2 > 1000$ when $m \geq 4$.

3.3. *Inequalities for k_i in terms of ℓ_i.* Recall the Binet formula (1.3)

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all} \quad n \geq 0.$$

It is well-known and can be easily checked by induction that the inequalities

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{hold for all} \quad n \geq 1. \tag{3.3}$$

Let $i \in \{1, 2, \ldots, m-1\}$ and $j \in \{i+1, \ldots, m\}$. Then

$$\alpha^{k_i-4} \leq F_{k_i-2} = F_{k_i} - F_{k_i-1} \leq F_{k_i} - F_{k_j} = p^{\ell_i} - p^{\ell_j} < p^{\ell_i}$$
$$\alpha^{k_i-1} \geq F_{k_i} > F_{k_i} - F_{k_j} = p^{\ell_i} - p^{\ell_j} \geq 0.8p^{\ell_i},$$

where for the last inequality we used the fact that $p \geq 5$. So, we get

$$k_i \log\alpha - 4\log\alpha < \ell_i \log p < k_i \log\alpha - \log(\alpha/1.25).$$

Since $4\log\alpha < 2$ and $\log(\alpha/1.25) > 0.25$, we can record the following lemma.

LEMMA 3.1. *For $i = 1, 2, \ldots, m-1$ we have*

$$k_i \log\alpha - \ell_i \log p \in (\log(\alpha/1.25), 4\log\alpha) \subset (c_1, c_2),$$

where $c_1 := 0.25$, $c_2 := 2$.

3.4. *Two small linear forms in logarithms.* We assume that $m \geq 3$. We let $(k, \ell) := (k_i, \ell_i)$ for $i = 1, 2, \ldots, m-2$ and $(k', \ell') := (k_j, \ell_j)$ for some $j = i+1, \ldots, m-1$. Then

$$F_k - p^\ell = F_{k'} - p^{\ell'} \tag{3.4}$$

can be rewritten as

$$\begin{aligned} \left|\frac{\alpha^k}{\sqrt{5}} - p^\ell\right| &= \left|\frac{\alpha^{k'}}{\sqrt{5}} - \frac{\beta^{k'}}{\sqrt{5}} + \frac{\beta^k}{\sqrt{5}} - p^{\ell'}\right| \\ &\leq \frac{\alpha^{k'}}{\sqrt{5}} + p^{\ell'} + \frac{|\beta|^2 + |\beta|^3}{\sqrt{5}} \\ &= \frac{\alpha^{k'}}{\sqrt{5}} + p^{\ell'} + \frac{1}{\sqrt{5}\alpha}. \end{aligned}$$

Thus,

$$\begin{aligned} \left|\alpha^k p^{-\ell}(\sqrt{5})^{-1} - 1\right| &< \frac{\alpha^{k'}/\sqrt{5} + p^{\ell'} + 1/(\sqrt{5}\alpha)}{p^\ell} \\ &< \frac{(\alpha^4/\sqrt{5})p^{\ell'} + p^{\ell'} + 1/(\sqrt{5}\alpha)}{p^\ell} \\ &< \frac{\alpha^4/\sqrt{5} + 1 + 1/(p\alpha\sqrt{5})}{p^{\ell-\ell'}} < \frac{4.2}{p^{\ell-\ell'}}. \end{aligned}$$

In the above, we used that $\alpha^{k'} < \alpha^4 p^{\ell'}$ which follows from Lemma 3.1. So,

$$|\alpha^k p^{-\ell}(\sqrt{5})^{-1} - 1| < \frac{4.2}{p^{\ell-\ell'}}. \tag{3.5}$$

We write

$$\Gamma_{k,\ell} := \alpha^k p^{-\ell}(\sqrt{5})^{-1} - 1. \tag{3.6}$$

We have that $\Gamma_{k,\ell} \neq 0$, since otherwise $\alpha^{2k} = 5p^{2\ell} \in \mathbb{N}$, which is impossible. Inequality (3.5) shows that

$$(\ell - \ell')\log p < -\log|\Gamma_{k,\ell}| + \log(4.2), \tag{3.7}$$

and using Lemma 3.1, we also have

$$(k-k')\log\alpha < (\ell-\ell')\log p + (c_2 - c_1) < -\log|\Gamma_{k,\ell}| + (1.75 + \log(4.2)). \tag{3.8}$$

This is the first small linear form in logarithms. We return to equation (3.4) and use the Binet formula to rewrite it as

$$\left|\frac{\alpha^k(1-\alpha^{k'-k})}{\sqrt{5}} - p^\ell(1 - p^{\ell'-\ell})\right| = \left|\frac{\beta^k}{\sqrt{5}} - \frac{\beta^{k'}}{\sqrt{5}}\right|. \tag{3.9}$$

The above implies that

$$\begin{aligned}\left|\frac{\alpha^k(1-\alpha^{k'-k})}{\sqrt{5}} - p^\ell(1-p^{\ell'-\ell})\right| &\le \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha^k}+\frac{1}{\alpha^{k'}}\right)\\ &= \frac{1}{\sqrt{5}\alpha^{k'}}\left(1+\frac{1}{\alpha}\right)\\ &= \frac{\alpha}{\sqrt{5}\alpha^{k'}}.\end{aligned}$$

Dividing across by $p^\ell(1-p^{\ell'-\ell})$, we get

$$\begin{aligned}\left|\alpha^k p^{-\ell}\left(\frac{\sqrt{5}(1-p^{\ell'-\ell})}{1-\alpha^{k'-k}}\right)^{-1} - 1\right| &< \frac{\alpha}{\sqrt{5}\alpha^{k'}p^\ell(1-p^{\ell'-\ell})}\\ &\le \frac{(5/4)(\alpha^4)\alpha}{\sqrt{5}\alpha^{k'+k}} < \frac{6.2}{\alpha^{k+k'}},\end{aligned} \tag{3.10}$$

where we used the fact that $p \ge 5$ (so $1-p^{\ell'-\ell} \ge 1-1/5$), as well as the fact that $p^\ell > \alpha^k/\alpha^4$, which follows from Lemma 3.1. As before, we put

$$\Gamma'_{k,\ell} := \alpha^k p^{-\ell}\left(\frac{\sqrt{5}(1-p^{\ell'-\ell})}{1-\alpha^{k'-k})}\right)^{-1} - 1;$$

Note that $\Gamma'_{k,\ell} \neq 0$ since otherwise (3.9) gives that $\beta^k = \beta^{k'}$, so $k = k'$ which is impossible. Inequality (3.10) shows that

$$(k+k')\log\alpha < -\log|\Gamma'_{k,\ell}| + \log(6.2),$$

which together with (3.8) gives

$$k < \frac{1}{2\log\alpha}\left(-\log|\Gamma_{k,\ell}| - \log|\Gamma'_{k,\ell}| + (1.75+\log(4.2)+\log(6.2))\right). \tag{3.11}$$

This is the second small linear form in logarithms.

3.5. *Bounds on k and p.*

Lemma 3.2. *If $m \ge 3$, we have:*

(i) $k < 7.2\cdot 10^{24}(1+\log k)^2(\log p)^2$;

(ii) $k < 5\cdot 10^{29}(\log p)^2(\log\log p)^2$.

Proof. We need lower bounds on $\log|\Gamma_{k,\ell}|$ and $\log|\Gamma'_{k,\ell}|$. We get these bounds by Theorem 2.2. In both cases

$$t := 3,\ \alpha_1 := \alpha,\ \alpha_2 := p,\ b_1 := k,\ b_2 := \ell \text{ and } b_3 := -1.$$

Further,

$$\alpha_3 := \sqrt{5} \quad \text{for} \quad \Gamma_{k,\ell} \quad \text{and} \quad \alpha_3 := \frac{\sqrt{5}(1-p^{\ell'-\ell})}{1-\alpha^{k'-k}} \quad \text{for} \quad \Gamma'_{k,\ell}.$$

In both cases $\mathbb{K} := \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{5})$ has $D := 2$. Further, we must take

$$B \geq \max\{|b_1|, |b_2|, |b_3|\} = \max\{k, \ell, 1\},$$

and since

$$\ell \leq \frac{k \log \alpha}{\log p} < \frac{k}{3} < k \quad \text{(because} \quad p \geq 5 > \alpha^3)$$

(see also Lemma 3.1), it follows that we can take $B := k$. Next, we must choose A_j such that

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}$$

for $j = 1, 2, 3$. So, we choose

$$A_1 := Dh(\alpha_1) = \log \alpha, \;\; A_2 := Dh(\alpha_2) = 2 \log p$$

and for $\Gamma_{k,\ell}$ we choose $A_3 := Dh(\alpha_3) = \log 5$. Then, by Theorem 2.2, we get

$$\begin{aligned} \log |\Gamma_{k,\ell}| &> -1.4 \cdot 10^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log k) \\ &\quad \times (\log 5)(\log \alpha)(2 \log p) \\ &> -1.51 \cdot 10^{12} (\log p)(1 + \log k). \end{aligned} \tag{3.12}$$

Inequalities (3.7) and (3.8) give

$$\begin{aligned} \max\{(\ell - \ell') \log p, (k - k') \log \alpha\} &< -\log |\Gamma_{k,\ell}| + (1.75 + \log(4.2)) \\ &< 1.52 \cdot 10^{12} (\log p)(1 + \log k), \end{aligned} \tag{3.13}$$

so, we can pass to estimate a lower bound for $\Gamma'_{k,\ell}$. We only need to estimate the height of α_3:

$$\begin{aligned} h(\alpha_3) &\leq h(1 - \alpha^{\ell' - \ell}) + h(1 - p^{k' - k}) + h(\sqrt{5}) \\ &\leq h(\alpha^{\ell' - \ell}) + h(p^{k' - k}) + (1/2) \log 5 + 2 \log 2 \\ &\leq (1/2)(\ell - \ell') \log \alpha + (k - k') \log p + (1/2) \log 5 + \log 2 \\ &< (1.52/2 + 1.52) \times 10^{12} (1 + \log k) \log p + (1/2) \log 5 + 2 \log 2 \\ &< 2.28 \times 10^{12} (1 + \log k) \log p + (1/2) \log 5 + 2 \log 2 \\ &< 2.29 \times 10^{12} (1 + \log k) \log p, \end{aligned}$$

where we used inequality (3.13). So, we can take

$$A_3 := 4.6 \times 10^{12} (1 + \log k) \log p \quad \text{for} \quad \Gamma'_{k,\ell}.$$

We get

$$\begin{aligned} \log |\Gamma'_{k,\ell}| &> -1.4 \cdot 10^6 \cdot 3^{4.5} 2^2 (1 + \log 2)(1 + \log k)(\log \alpha) \\ &\quad \times (2 \log p)(4.6 \times 10^{12} (1 + \log k) \log p), \end{aligned}$$

or simply

$$\log |\Gamma'_{k,\ell}| > -6.91 \cdot 10^{24} (1 + \log k)^2 (\log p)^2. \tag{3.14}$$

Inserting (3.14) and (3.12) into (3.11), we get

$$k < 7.2 \cdot 10^{24}(1+\log k)^2(\log p)^2.$$

This is (i). Assuming $k > 10^{10}$, we get

$$\begin{aligned} k &< 7.2 \cdot 10^{24}\left(1+\frac{1}{\log(10^{10})}\right)^2 (\log p)^2(\log k)^2 \\ &< 7.9 \cdot 10^{24}(\log p)^2(\log k)^2. \end{aligned}$$

Finally, we apply Lemma 2.7 with $s := 2$ and $T := 7.9 \cdot 10^{24}(\log p)^2$, to get that

$$\begin{aligned} k &< 2^2 T(\log T)^2 < 4 \cdot 7.9 \cdot 10^{24}(\log p)^2 \left(\log(7.9 \cdot 10^{24}) + 2\log\log p\right)^2 \\ &< 31.6 \cdot 10^{24}(\log p)^2(2\log\log p)^2\left(1+\frac{\log(7.9\cdot 10^{24})}{2\log\log 5}\right)^2 \\ &< 5 \cdot 10^{29}(\log p)^2(\log\log p)^2, \end{aligned}$$

which is (ii). ∎

3.6. *An absolute bound on* k_1. We assume that $m \geq 4$. We write inequality (3.6) in logarithmic form. Namely, we put

$$\Lambda_{k,\ell} := k\log\alpha - \ell\log p - \log\sqrt{5}.$$

Note that $\Gamma_{k,\ell} = e^{\Lambda_{k,\ell}} - 1 \neq 0$ so $\Lambda_{k,\ell} \neq 0$. Further, inequality (3.6) shows that

$$|e^{\Lambda_{k,\ell}} - 1| < \frac{4.2}{p^{\ell-\ell'}}. \tag{3.15}$$

If $\Lambda_{k,\ell} > 0$, then

$$|\Lambda_{k,\ell}| < e^{\Lambda_{k,\ell}} - 1 < \frac{4.2}{p^{\ell-\ell'}}.$$

If $\Lambda_{k,\ell} < 0$, then inequality (3.15) together with the fact that $p \geq 5$ implies that

$$e^{|\Lambda_{k,\ell}|} < \frac{1}{1-\frac{4.2}{5}} = 6.25,$$

so

$$|\Lambda_{k,\ell}| < e^{|\Lambda_{k,\ell}|}\left|1 - e^{\Lambda_{k,\ell}}\right| < \frac{6.25 \times 4.2}{p^{\ell-\ell'}} = \frac{26.5}{p^{\ell-\ell'}}.$$

Hence, inequality

$$|\Lambda_{k,\ell}| < \frac{26.5}{p^{\ell-\ell'}} \tag{3.16}$$

holds in all cases. We write inequalities (3.16) for

$$(k,\ell,k',\ell') = (k_i,\ell_i,k_j,\ell_j),\ (k_{i+1},\ell_{i+1},k_j,\ell_j),\ \text{where } j \in [i+2, m-1]$$

getting

$$\left|k_i \log \alpha - \ell_i \log p - \log \sqrt{5}\right| \le \frac{26.5}{p^{\ell_i - \ell_j}},$$

$$\left|k_{i+1} \log \alpha - \ell_{i+1} \log p - \log \sqrt{5}\right| \le \frac{26.5}{p^{\ell_{i+1} - \ell_j}},$$

and take a linear combination of them to get

(3.17)

$$|(k_{i+1}\ell_i - k_i\ell_{i+1}) \log \alpha - (\ell_i - \ell_{i+1}) \log \sqrt{5}| < \frac{26.5(\ell_i + \ell_{i+1})}{p^{\ell_{i+1} - \ell_j}} < \frac{53\ell_i}{p^{\ell_{i+1} - \ell_j}}.$$

If the left–hand side is larger than $1/2$, then

$$\begin{aligned} p \le p^{\ell_{i+1} - \ell_j} &< 106\ell_i \le 106\ell_1 < \frac{106 k_1 \log \alpha}{\log p} \\ &< \frac{106(\log \alpha) \cdot 5 \cdot 10^{29} (\log p)^2 (\log \log p)^2}{\log p} \\ &< 3 \cdot 10^{31} (\log p)(\log \log p)^2, \end{aligned} \tag{3.18}$$

which implies $p < 5 \cdot 10^{34}$ and next

$$k_1 < 5 \cdot 10^{29} (\log p)^2 (\log \log p)^2 < 7 \cdot 10^{34},$$

which is a pretty good bound on k_1. So, assume that the right–hand side of (3.17) is smaller than $1/2$. Then $k_{i+1}\ell_i - k_i\ell_{i+1}$ is positive and

$$\begin{aligned} k_{i+1}\ell_i - k_i\ell_{i+1} &< \frac{(\ell_i - \ell_{i+1}) \log \sqrt{5} + 1/2}{\log \alpha} < \frac{\ell_1 \log \sqrt{5}}{\log \alpha} \\ &< \frac{k_1 \log \alpha \log \sqrt{5}}{\log \alpha \log p} \le \frac{k_1}{2} < k_1, \end{aligned} \tag{3.19}$$

where we used Lemma 3.1 and the fact that $p \ge 5$. Let Λ be the linear form under the absolute value in the left–hand side of (3.17). It is nonzero since α and $\sqrt{5}$ are multiplicatively independent, so if it were zero we would have $\ell_i - \ell_{i+1} = 0$, which is not the case. Thus, we get

$$\log p \le (\ell_{i+1} - \ell_j) \log p < -\log |\Lambda| + \log(53\ell_1).$$

We need upper bounds on the left–hand side above. The second term has already been estimated in (3.18):

$$53\ell_1 < 1.5 \cdot 10^{31} (\log p)(\log \log p)^2.$$

As for the first term, we use Theorem 2.3. We have

$$t := 2,\ \alpha_1 := \alpha,\ \alpha_2 := \sqrt{5}.$$

We have $D := 2$, $\log A_1 := 1/2$, $\log A_2 := (\log 5)/2$. Finally,

$$b' := \frac{k_{i+1}\ell_i - k_i\ell_{i+1}}{D \log A_2} + \frac{\ell_i - \ell_{i+1}}{D \log A_1} < k_1 \left(1 + \frac{1}{\log 5}\right) = 1.7 k_1.$$

Thus,

$$\begin{aligned}-\log|\Lambda| &< 24.34\cdot D^2(1/2)((\log 5)/2)(\max\{\log b'+0.14, 10.5\})^2\\ &< 40(\max\{\log b'+0.14, 10.5\})^2.\end{aligned}$$

If the maximum is 10.5, we get

$$-\log|\Lambda| < 5000.$$

Thus, in this case

$$\log p < 5000+\log(1.5\cdot 10^{31}(\log p)^2(\log\log p)^2).$$

This gives $\log p < 5100$. If the maximum is in

$$\begin{aligned}\log b'+0.14 = \log(e^{0.14}b') < \log(e^{0.14}\cdot 1.7k_1) &< \log(2k_1)\\ &< \log(10^{30}(\log p)^2(\log\log p)^2),\end{aligned}$$

we get

$$-\log|\Lambda| < 40(\log(10^{30}(\log p)^2(\log\log p)^2)^2,$$

so in this case $\log p$ is smaller than

$$40(\log(10^{30}(\log p)^2(\log\log p)^2)^2+\log(1.5\cdot 10^{31}(\log p)^2(\log\log p)^2),$$

which gives $\log p < 4.1\cdot 10^5$. Feeding this into Lemma 3.2, we get

$$k_1 < 5\cdot 10^{29}(\log p)^2(\log\log p)^2 < 1.5\cdot 10^{43}.$$

So, we record what we have.

Lemma 3.3. *If $m\ge 4$, we then have $p < e^{4.1\cdot 10^5}$ and $k_1 < 1.5\cdot 10^{43}$.*

3.7. *There are no solutions with $m=4$ and $p<10^{14}$.* The main scope of this section is to prove the following lemma.

Lemma 3.4. *There are no solutions with $m=4$ and $p<10^{14}$.*

Proof. Well, assume that $p<10^{14}$. Lemma 3.2 gives

$$k_1 < 5\cdot 10^{29}(\log(10^{14}))^2(\log\log 10^{14})^2 < 10^{34}.$$

We return to estimate (3.17) with the aim of bounding $p^{\ell_2-\ell_3}$. If the right–hand side in (3.17) is at least 1/2, then

$$p^{\ell_2-\ell_3}\le 106\ell_1 < 106k_1 < 1.1\cdot 10^{36}.$$

Otherwise, the right–hand side is at most 1/2, so $k_2\ell_1-k_1\ell_2$ is positive and smaller than k_1 as in (3.19). Now $F_{170} > 10^{35} > k_1$. We generate the first 171 convergents of $\tau := \log\alpha/\log\sqrt{5} = [0,1,1,\ldots] = [a_0,a_1,a_2,\ldots]$ and get that $\max\{a_j : 0\le j\le 170\} = 330$. Hence, by Lemma 2.4, we get that the left–hand side of (3.17) is at least

$$\frac{1}{(330+2)k_1}.$$

Thus, we get

$$p^{\ell_2-\ell_3} < 53 \cdot 332 \cdot k_1^2 < 10^{73}.$$

Next p^{ℓ_3} divides $F_{k_i} - F_{k_j}$ for all $i > j \in \{1, 2, 3\}$. There are two indices $k_i,\ k_j$ which are congruent modulo 2; hence,

$$F_{k_i} - F_{k_j} = F_{(k_i \pm k_j)/2} L_{(k_i \mp k_j)/2}$$

by Lemma 2.5. Let $z(p)$ be the order of appearance of p in the Fibonacci sequence. Since $p < 10^{14}$, Lemma 2.6 shows that $p \| F_{z(p)}$. Assume that $p^a \| F_{(k_i \pm k_j)/2}$ and $p^b \| L_{(n_i \mp n_j)/2}$. If $a \geq 1$, then $p^{a-1} \mid (k_i \pm k_j)/2$, so $p^{a-1} \leq k_1 < 10^{34}$. Similarly, $p^b \| L_{(k_i \mp k_j)/2} \mid F_{k_i \mp k_j}$, so $p^{b-1} \mid (k_i \mp k_j)/2$, so $p^{b-1} < 10^{34}$. So,

$$p^{\ell_3} \leq p^{a-1} \cdot p^{b-1} \cdot p^2 < 10^{34} \cdot 10^{34} (10^{14})^2 < 10^{96}.$$

Thus,

$$F_{k_2-2} \leq F_{k_2} - F_{k_3} < p^{\ell_2} = p^{\ell_2-\ell_3} \cdot p^{\ell_3} < 10^{96} \cdot 10^{73} = 10^{169},$$

so $k_2 < 1000$. But we have already shown that in the range $k_2 \leq 1000$, there are no instances of $k_2 > k_3 > k_4 \geq 2$ and $\ell_2 > \ell_3 > \ell_4 \geq 0$ such that

$$F_{k_2} - p^{\ell_2} = F_{k_3} - p^{\ell_3} = F_{k_4} - p^{\ell_4}$$

with some prime $p \geq 5$. This finishes the proof of the current lemma. ∎

3.8. *The conclusion.*

Lemma 3.5. *We have $m \leq 4$.*

Proof. Assume $m \geq 5$. We return to (3.17) and take $i = 1, j = 4$. We have

$$\ell_1 - \ell_4 < \ell_1 < \frac{k_1 \log \alpha}{\log p} < \frac{1.5 \cdot 10^{43} \log \alpha}{\log(10^{14})} < 2.3 \cdot 10^{41}.$$

Now $2.3 \cdot 10^{41} < F_{200}$. We calculated $[a_0, a_1, \ldots, a_{200}]$ for the number $\tau = \log \alpha / \log \sqrt{5}$ obtaining $\max\{a_j : 0 \leq j \leq 200\} = 330$. Hence, the left–hand side of (3.17) is at least

$$\frac{1}{332 k_1},$$

which gives that

$$(\ell_2 - \ell_4) \log p < \log(332 k_1 \ell_1) < \log(332 \cdot (2.3 \cdot 10^{41}) \cdot (1.5 \cdot 10^{43})) < 201. \tag{3.20}$$

By Lemma 3.1, we get

$$k_2 - k_4 < \frac{203}{\log \alpha} < 422.$$

Thus, $k_2 - k_3 =: a < k_2 - k_4 =: b$ are in $[1, 421]$. Fix $1 \leq a < b \in [1, 421]$. Then $k_2 = k_3 + a = k_4 + b$, so $k_3 = k_4 + (b - a) := k_4 + c$. Hence,

$$F_{k_4+b} - F_{k_4} = p^{\ell_2} - p^{\ell_4} \equiv 0 \pmod{p^{\ell_4}};$$
$$F_{k_4+c} - F_{k_4} = p^{\ell_3} - p^{\ell_4} \equiv 0 \pmod{p^{\ell_4}}.$$

Using the Binet formula, we get

$$\alpha^{k_4}(\alpha^b-1)\equiv\beta^{k_4}(\beta^b-1)\pmod{p^{\ell_4}};$$
$$\alpha^{k_4}(\alpha^c-1)\equiv\beta^{k_4}(\beta^c-1)\pmod{p^{\ell_4}}.$$

In the above, for algebraic integers γ,δ,u we write $\gamma\equiv\delta\pmod u$ if $(\gamma-\delta)/u$ is an algebraic integer. Since $\beta=-\alpha^{-1}$ is a unit, we get

$$\alpha^{2k_4}(\alpha^b-1)\equiv(-1)^{k_4}(\beta^b-1)\pmod{p^{\ell_4}};$$
$$\alpha^{2k_4}(\alpha^c-1)\equiv(-1)^{k_4}(\beta^c-1)\pmod{p^{\ell_4}}.$$

Multiplying both sides of the second congruence above by α^b-1 and using also the first congruence, we get

$$(-1)^{k_4}(\beta^b-1)(\alpha^c-1)\equiv(-1)^{k_4}(\alpha^b-1)(\beta^c-1)\pmod{p^{\ell_4}}.$$

Hence, p^{ℓ_4} divides

$$\begin{aligned}&\left|(-1)^c\alpha^{b-c}-\alpha^b-\beta^c+1-((-1)^c\beta^{b-c}-\beta^b-\alpha^c+1)\right|\\&=\left|(\alpha^b-\beta^b)-(\alpha^c-\beta^c)\pm(\alpha^{b-c}-\beta^{b-c})\right|.\end{aligned}$$

In particular,

$$p^{\ell_4}\mid F_b-F_c\pm F_{b-c}. \tag{3.21}$$

We show that $F_b-F_c\pm F_{b-c}$ is nonzero. This is clear if the sign of F_{b-c} is positive since $b>c$. It is also clear if $\max\{c,b-c\}\le b-2$ since then

$$F_b-F_c-F_{b-c}=F_{b-1}+F_{b-2}-F_c-F_{b-c}>0.$$

Thus, either $c=b-1$ or $b-c=b-1$. If $c=b-1$, we get

$$F_b-F_c-F_{b-c}=F_{b-2}-F_1$$

and this is positive unless $b\in\{2,3,4\}$. Similarly, if $b-c=b-1$, so $c=1$, we get that $F_b-F_c-F_{b-c}=F_{b-2}-1$ and again this is positive unless $b\in\{2,3,4\}$.

If $b=2$, then $a=1$, $k_2=k_3+1=k_4+2$, so p^{ℓ_4} divides

$$F_{k_2}-F_{k_3}=F_{k_3-1}=F_{k_4}\quad\text{and also}\quad F_{k_2}-F_{k_4}=F_{k_4+1},$$

and this is false since $\gcd(F_{k_4},F_{k_4+1})=1$.

If $b=3$, then either $a=1$, or $a=2$. When $a=1$, we have $k_2=k_3+1=k_4+3$. So,

$$p^{\ell_3}\|F_{k_2}-F_{k_3}=F_{k_3-1}=F_{k_4+1},$$

and also

$$p^{\ell_4}\|F_{k_2}-F_{k_4}=F_{k_4+3}-F_{k_4}=F_{k_4+2}+F_{k_4+1}-F_{k_4}=2F_{k_4+1},$$

which implies that $\ell_3=\ell_4$, and this is impossible. If $a=2$, then $k_2=k_3+2=k_4+3$. Thus,

$$p^{\ell_4}\|F_{k_4+3}-F_{k_4}=2F_{k_4+1}\quad\text{and}\quad p^{\ell_4}\|F_{k_4+1}-F_{k_4}=F_{k_4-1},$$

and this is impossible since $\gcd(F_{k_4+1},F_{k_4-1})=1$.

If $b=4$, then $a \in \{1,2,3\}$. If $a=1$, then $k_2=k_3+1=k_4+4$. Then

$$p^{\ell_3} \| F_{k_2}-F_{k_3}=F_{k_3-1}=F_{k_4+2}$$

and

$$p^{\ell_4} \| F_{k_3}-F_{k_4}=F_{k_4+3}-F_{k_4}=2F_{k_4+1},$$

so again $p \mid (\gcd(F_{k_4+2},F_{k_4+1})=1$, a contradiction. If $a=2$, then $k_2=k_3+2=k_4+4$, so p divides $F_{k_2}-F_{k_3}=F_{k_3+1}=F_{k_4+3}$ and also $F_{k_3}-F_{k_4}=F_{k_4+1}$. Thus, p divides $\gcd(F_{k_4+1},F_{k_4+3}) \mid F_2=1$, a contradiction.

Finally, if $a=3$, then $k_2=k_3+3=k_4+4$, so p divides

$$F_{k_2}-F_{k_3}=F_{k_2+3}-F_{k_3}=2F_{k_3+1}=2F_{k_4+2}$$

and

$$F_{k_3}-F_{k_4}=F_{k_4-1}$$

and since $\gcd(F_{k_4+2},F_{k_4-1}) \mid F_3=2$, we get a contradiction.

The above argument shows that the integer which appears in the right–hand side of (3.21) is nonzero. Its size is at most

$$F_b+F_{b-c} \leq F_{b+1} < \alpha^{421}.$$

Thus, $p^{\ell_4} < \alpha^{421}$. Since also $p^{\ell_2-\ell_4} < e^{201}$ (see (3.20)), we get that

$$\alpha^{k_2-4} < F_{k_2-2} \leq F_{k_2}-F_{k_3} < p^{\ell_2} = (p^{\ell_2-\ell_4})(p^{\ell_4}) < e^{201} \cdot \alpha^{421},$$

so

$$k_2 < 4 + \frac{201+421\log\alpha}{\log\alpha} < 850,$$

but again due to the computation that we did at the beginning, we saw that there do not exist $k_2>k_3>k_4$ in $[1,1000]$ such that $F_{k_2}-p^{\ell_2}=F_{k_3}-p^{\ell_3}=F_{k_4}-p^{\ell_4}$ for some prime p and integers ℓ_2,ℓ_3,ℓ_4. This finishes the proof. $\square$

Acknowledgements.

We thank the referees for careful reading of the first version of this manuscript and for helpful comments and suggestions. This work has been supported by the Mathematics for Sustainable Development (MATH4SDG) project, which is a research and development project running in the period 2021-2026 at Makerere University, Uganda, University of Dar es Salaam, Tanzania, and the University of Bergen, Norway, funded through the NORHED II program under the Norwegian Agency for Development Cooperation (NORAD, project no 68105).

References

[1] F. Amoroso and E. Viada, *Small points on subvarieties of a torus*, Duke Math. J. **150** (2009), 407–442.

[2] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. **442** (1993), 19–62.

[3] J. J. Bravo, F. Luca and K. Yazán, *On Pillai's problem with Tribonacci numbers and Powers of 2*, Bull. Korean Math. Soc. **54** (2017), 1069–1080.

[4] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers*, Ann. of Math. (2) **163** (2006), 969–1018.

[5] K. C. Chim, I. Pink and V. Ziegler, *On a variant of Pillai's problem II*, J. Number Theory **183** (2018), 269–290.

[6] M. Ddamulira, F. Luca and M. Rakotomalala, *On a problem of Pillai with Fibonacci numbers and powers of 2*, Proc. Indian Acad. Sci. Math. Sci. **127** (2017), 411–421.

[7] M. Ddamulira, *On a problem of Pillai with Fibonacci numbers and powers of 3.* Bol. Soc. Mat. Mex. **26** (2020), 263–277.

[8] S. Gúzman and F. Luca, *Linear combinations of factorials and s-units in a binary recurrence sequence*, Ann. Math. Qué. **38** (2014), 169–188.

[9] M. Laurent, M. Mignotte and Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory **55** (1995), 285–321.

[10] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), 125–180; translation in Izv. Math. **64** (2000), 1217–1269.

[11] R. J. McIntosh and E. L. Roettger, *A search for Fibonacci-Wieferich and Wolstenholme primes*, Math. Comp. **76** (2007), 2087–2094.

[12] M. R. Murty and J. Esmonde, *Problems in algebraic number theory*, Second edition. Graduate Texts in Mathematics, **190**, Springer Verlag, New York, 2005.

[13] S. S. Pillai, *On* $a^x - b^y = c$, J. Indian Math. Soc. N. Ser. **2** (1936), 119–122.

H. Batte
Department of Mathematics, Makerere University
Kampala, Uganda
E-mail: hbatte91@gmail.com

M. Ddamulira
Department of Mathematics, Makerere University
Kampala, Uganda
E-mail: mahadi.ddamulira@mak.ac.ug

J. Kasozi
Department of Mathematics, Makerere University
Kampala, Uganda
E-mail: juma.kasozi@mak.ac.ug

F. Luca
School of Mathematics, University of the Witwatersrand
Johannesburg, South Africa
&
Research Group in Algebraic Structures and Applications, King Abdulaziz University
Jeddah, Saudi Arabia
&
Max Planck Institute for Software Systems
Saarbrücken, Germany
&
Centro de Ciencias Matemáticas UNAM
Morelia, Mexico
E-mail: Florian.Luca@wits.ac.za

Received: 15.5.2022.
Revised: 20.7.2022.