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ON THE EXISTENCE OF $D(-3)$-QUADRUPLES OVER $\mathbb{Z}$

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Abstract. In this paper we prove that there does not exist a set of four non-zero polynomials from $\mathbb{Z}[X]$, not all constant, such that the product of any two of its distinct elements decreased by 3 is a square of a polynomial from $\mathbb{Z}[X]$.

1. Introduction

Since Diophantus ([3]) noted that the product of any two elements of the set $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ increased by 1 is a square of rational number, many generalizations of his original problem were also studied. The following definition describes a more general problem.

Definition 1.1. *Let $m \geq 2$ and let R be a commutative ring with unity. Let $n \in R$ be a non-zero element and let $\{a_1, \ldots, a_m\}$ be a set of m distinct non-zero elements from R such that $a_i a_j + n$ is a square of an element of R for $1 \leq i < j \leq m$. The set $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or simply a $D(n)$-m-tuple in R.*

There are many results concerning the upper bounds for the number of elements of such sets (see for example [1, 4, 6, 7, 8, 9, 12, 17]). Brown ([1]) proved that if n is an integer and $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple of integers. Furthermore, Dujella ([4]) proved that if n is an integer $n \not\equiv 2 \pmod 4$, and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple of integers, and moreover if $n \notin S \cup T$, where $T = \{-15, -12, -7, 7, 13, 15, 21, 24, 28, 32, 48, 60, 84\}$, then there exist at least two distinct $D(n)$-quadruples of integers. For some integers the

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question of the existence of such a quadruple is still open, as it is stated in Dujella's conjecture ([5]).

CONJECTURE 1.2. *For* $n \in S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$ *there does not exist a* $D(n)$*-quadruple of integers.*

The question of whether there exists a $D(n)$-quadruple of integers can be reduced to elements of the set $S' = \{-3, -1, 3, 5, 8, 20\}$ (see [4, Remark 3]). Bonciocat, Cipu and Mignotte ([2]) recently proved that there are no $D(-1)$-quadruples (and $D(-4)$-quadruples).

A polynomial variant of the problem of Diophantus was firstly studied by Jones ([18, 19]) for the case $R = \mathbb{Z}[X]$ and $n = 1$. Since then, a lot of other variants of such a polynomial problem were also considered (for example [9, 10, 11, 12, 13, 14, 15]). In case where R is a polynomial ring with coefficients in a ring-extension of $\mathbb{Z}$, and n is a non-zero integer, which will be the subject of our interest, it is usually assumed that not all polynomials in such a $D(n)$-tuple are constant. We also assume that R has a characteristic 0. We call a $D(n)$-tuple a polynomial $D(n)$-tuple in this case and we specify, when necessary, which polynomial ring R we are referring to. Let $\{a, b, c\}$ be a polynomial $D(n)$-triple. Then, there exist polynomials r, s and t from R such that

$$ab + n = r^2, \quad ac + n = s^2, \quad bc + n = t^2. \tag{1.1}$$

Also, there is the following well-known definition:

DEFINITION 1.3. *A polynomial* $D(n)$*-triple* $\{a, b, c\}$*, where* $n \in \mathbb{Z} \setminus \{0\}$*, is called regular if*

$$(c - b - a)^2 = 4(ab + n). \tag{1.2}$$

Equation (1.2) is symmetric under the permutations of a, b, and c, and from (1.2), using (1.1), we get

$$c = c_{\pm} = a + b \pm 2r, \tag{1.3}$$

$$ac_{\pm} + n = (a \pm r)^2, \; bc_{\pm} + n = (b \pm r)^2. \tag{1.4}$$

For a polynomial $D(n)$-quadruple $\{a, b, c, d\}$, with $n \in \mathbb{Z} \setminus \{0\}$, there also exist polynomials x, y and z from R such that, beside (1.1), there hold

$$ad + n = x^2, \quad bd + n = y^2, \quad cd + n = z^2.$$

We can also distinguish regular and irregular polynomial $D(n)$-quadruples (see [16]).

DEFINITION 1.4. *A polynomial* $D(n)$*-quadruple* $\{a, b, c, d\}$*, with* $n \in \mathbb{Z} \setminus \{0\}$*, is called regular if*

$$n(d + c - a - b)^2 = 4(ab + n)(cd + n) \tag{1.5}$$

or, equivalently, if

$$d = d_{\pm} = a + b + c + \frac{2}{n}(abc \pm rst).$$

Equation (1.5) is also symmetric under permutations of a, b, c and d. In $\mathbb{R}[X]$, for example, a regular polynomial $D(n)$-quadruple exists for every positive integer n.

A multi-set $\{a_1, \ldots, a_m\}$ of elements of R such that $a_i a_j + n$ is a square of an element in R for $1 \leq i < j \leq m$ is called an improper $D(n)$-m-tuple in R. Thus compared with Definition 1.1 the elements need not be distinct and need not be non-zero. If this is the case we say that the improper $D(n)$-m-tuple in R is proper. As above we consider polynomial $D(n)$-m-tuples. In an (improper) polynomial $D(n)$-triple we can not have two equal non-constant polynomials because this would imply for example $(b-t)(b+t) = -n$, which is not possible since not both factors on the left hand side of this equation can be constant.

Now we state a polynomial variant of Conjecture 1.2.

Conjecture 1.5. *For $n \in S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$ there does not exist a polynomial $D(n)$-quadruple in $\mathbb{Z}[X]$.*

Let us mention that in a polynomial case we can not reduce the set S to the set S'.

The authors ([15]) proved that there does not exist a polynomial $D(n)$-quadruple in $\mathbb{Z}[X]$ for any positive integer n which is not a perfect square. Dujella and Fuchs ([8]) proved that there does not exist a polynomial $D(-1)$-quadruple in $\mathbb{Z}[X]$. As a consequence of the result from [2], there does not exist a polynomial $D(-4)$-quadruple in $\mathbb{Z}[X]$. In this paper, we consider the case where $R = \mathbb{Z}[X]$ and $n = -3$ and we prove the following theorem.

Theorem 1.6. *There does not exist a polynomial $D(-3)$-quadruple in $\mathbb{Z}[X]$.*

Therefore, by proving Theorem 1.6, we complete the proof of Conjecture 1.5.

Let $\mathbb{Z}^+[X]$ denote the set of all polynomials with integer coefficients with positive leading coefficient. For $a', b' \in \mathbb{Z}[X]$, $a' < b'$ means that $b' - a' \in \mathbb{Z}^+[X]$. Let $\{a', b', c'\}$, such that $0 < a' < b' < c'$, be a polynomial $D(-3)$-triple in $\mathbb{Z}[X]$ and

$$a'b' - 3 = (r')^2,\ a'c' - 3 = (s')^2,\ b'c' - 3 = (t')^2, \tag{1.6}$$

where $r', s', t' \in \mathbb{Z}^+[X]$. Such sets are, for example, the regular polynomial $D(-3)$-triple

$$\{1, X^2 + 3, X^2 + 2X + 4\}$$

and the irregular polynomial $D(-3)$-triple

$$\{1, 9X^2 + 3, 1296X^6 + 864X^5 + 792X^4 + 288X^3 + 105X^2 + 18X + 4\}.$$

The proof that a polynomial $D(-3)$-triple in $\mathbb{Z}[X]$ can not be extended to a polynomial $D(-3)$-quadruple in $\mathbb{Z}[X]$ will be presented through the following sections. In Section 2 we transform the problem of extending a polynomial $D(-3)$-triple $\{a', b', c'\}$ to a polynomial $D(-3)$-quadruple $\{a', b', c', d'\}$ in $\mathbb{Z}[X]$ into solving a system of simultaneous Pellian equations. Solutions of that system appear from the intersections of the obtained binary recurrent sequences of polynomials. It is a well-known method, but while solving our problem we have to seek solutions in $\mathbb{R}[X]$ instead of solutions in $\mathbb{Z}[X]$, firstly. In Section 3 we completely determine possible initial terms of the observed recurring sequences. It is not difficult to see that if $\{a', b', c', d'\}$ is a polynomial $D(-3)$-quadruple in $\mathbb{Z}[X]$, then the set $\left\{\frac{a'}{\sqrt{3}}, \frac{b'}{\sqrt{3}}, \frac{c'}{\sqrt{3}}, \frac{d'}{\sqrt{3}}\right\}$ is a polynomial $D(-1)$-quadruple in $\mathbb{R}[X]$ and also the set $\left\{\frac{a'}{\sqrt{3}}i, \frac{b'}{\sqrt{3}}i, \frac{c'}{\sqrt{3}}i, \frac{d'}{\sqrt{3}}i\right\}$ is a polynomial $D(1)$-quadruple in $\mathbb{C}[X]$. We will use this parallelism of the original problem with the problems in $\mathbb{R}[X]$ and $\mathbb{C}[X]$ to be able to draw conclusions in a polynomial ring R in which they are most apparent. That is the main difference and novelty from the methods and results from [8] where it was possible to consider everything in $\mathbb{Z}[X]$. In Section 4 we prove Theorem 1.6, using the results from the previous sections.

2. Binary recursive sequences of polynomials

Assume that a $D(-3)$-triple $\{a', b', c'\}$, where $0 < a' < b' < c'$, in $\mathbb{Z}[X]$ can be extended to a $D(-3)$-quadruple $\{a', b', c', d'\}$ in $\mathbb{Z}[X]$, where $d' > c'$. Let the equations (1.6) hold and let

$$a'd' - 3 = (x')^2, \quad b'd' - 3 = (y')^2, \quad c'd' - 3 = (z')^2, \tag{2.1}$$

where $x', y', z' \in \mathbb{Z}^+[X]$. Let a, b, c and d be obtained by dividing a', b', c' and d' with $\sqrt{3}$. The polynomial $D(-1)$-triple $\{a, b, c\}$ in $\mathbb{R}[X]$ can in turn be extended to a polynomial $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{R}[X]$, where all suitable elements are obtained from (1.6) and (2.1) by dividing with 3, i.e. we get equations

$$ab - 1 = r^2, \quad ac - 1 = s^2, \quad bc - 1 = t^2, \tag{2.2}$$

$$ad - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2, \tag{2.3}$$

where $r, s, t, x, y, z \in \mathbb{R}^+[X]$ and $\mathbb{R}^+[X]$ is defined analogously as $\mathbb{Z}^+[X]$. Since, by [13, Lemma 1], in a polynimial $D(1)$-quadruple $\{ai, bi, ci, di\}$ in $\mathbb{C}[X]$ there is at most one constant, it follows that a polynomial $D(-1)$-quadruple in $\mathbb{R}[X]$ and also a polynomial $D(-3)$-quadruple in $\mathbb{Z}[X]$ can not contain two constants.

Remark 2.1. We will firstly consider our problem in $\mathbb{R}[X]$ since, as we will explain later, it will be the most suitable polynomial ring for our conclusions. When we study polynomial $D(-1)$-tuples in $\mathbb{R}[X]$, we consider only polynomials of the form $p = \frac{p'}{\sqrt{3}}$, where $p' \in \mathbb{Z}[X]$. In other words, we are interested only in polynomial $D(-1)$-tuples in $\mathbb{R}[X]$ whose existence follows from the existence of polynomial $D(-3)$-tuples in $\mathbb{Z}[X]$. Analogously, in $\mathbb{C}[X]$ we are interested only in polynomial $D(1)$-tuples which follow from the existence of polynomial $D(-3)$-tuples in $\mathbb{Z}[X]$.

We will need the following construction for the elements of a polynomial $D(-1)$-triple in $\mathbb{R}[X]$. The following lemma is a modification of the analogue statement for the integer case in [6, Lemma 3], but also an analogue version of [8, Lemma 7] for $\mathbb{Z}[X]$.

LEMMA 2.2. *Let $\{a, b, c\}$, where $0 < a < b < c$, be a polynomial $D(-1)$-triple in $\mathbb{R}[X]$ such that (2.2) holds. Then, there exist polynomials $e, u, v, w \in \mathbb{R}[X]$ such that*

$$ae + 1 = u^2,\ be + 1 = v^2,\ ce + 1 = w^2 \tag{2.4}$$

and

$$c = a + b - e + 2(abe + ruv). \tag{2.5}$$

PROOF. The proof is analogous to the proof of [8, Lemma 7]. ∎

REMARK 2.3. As in the proof of [8, Lemma 7], we define

$$e = -(a + b + c) + 2(abc - rst) \tag{2.6}$$

and we have

$$u = at - rs,\ v = bs - rt,\ w = cr - st.$$

LEMMA 2.4. *Let $\{a, b, c\}$ be a polynomial $D(-1)$-triple in $\mathbb{R}[X]$, where $0 < a < b < c$, and let e be defined by (2.6). Let us denote by α, β, γ the degrees of polynomials a, b, c, respectively. Then $e = 0$ and $\beta = \gamma$ or $e > 0$ and $\beta < \gamma$.*

PROOF. From (2.4), we conclude that $e \geq 0$. For $e = 0$, by (2.5), we get $c = c_{\pm}$ and then, from (1.3), $\gamma \leq \beta$ which implies $\beta = \gamma$.

Let $e > 0$. Let us define

$$\overline{e} = -(a + b + c) + 2(abc + rst). \tag{2.7}$$

By (2.6) and (2.7), we have

$$e\overline{e} = -2ab - 2ac - 2bc + a^2 + b^2 + c^2 + 4. \tag{2.8}$$

Also, by (2.7),

$$\deg(\overline{e}) = \alpha + \beta + \gamma.$$

From (2.8), $\deg(e) + \deg(\overline{e}) \leq 2\gamma$, where the inequality can occur only if $\beta = \gamma$. But, then we would also have $\deg(e) + \alpha < 0$, which is not possible. Therefore, we have $\deg(e) + \deg(\overline{e}) = 2\gamma$, from which we conclude

$$\deg(e) = \gamma - \alpha - \beta.$$

If in this case we have $\beta = \gamma$, then it would be $\deg(e) = -\alpha = 0$ and $\{ai, bi, ci, -ei\}$ would be a polynomial $D(1)$-quadruple in $\mathbb{C}[X]$ with two constants. Hence, $ai = -ei$, so $a = -e < 0$, which is not possible. We conclude that $\beta < \gamma$. ∎

We also need an analogue of [13, Lemma 4] so that we can use it in finding all possible polynomials d. Let us denote by δ the degree of the polynomial d. Therefore, $0 \leq \alpha \leq \beta \leq \gamma \leq \delta$ and $\beta, \gamma, \delta > 0$. Eliminating d from (2.3), we obtain the system of simultaneous Pellian equations

$$az^2 - cx^2 = c - a, \tag{2.9}$$

$$bz^2 - cy^2 = c - b. \tag{2.10}$$

If we could find solutions (z, x) and (z, y) of (2.9) and (2.10), respectively, we would have a polynomial d by following the classical arguments from [13, Lemma 4] and also the results from [8].

We consider our problem in $\mathbb{R}[X]$, since in $\mathbb{Z}[X]$ we can not generate all solutions of the equations (2.9) and (2.10) such that all of the solutions in one class would be in $\mathbb{Z}[X]$. This is the main difference between considering polynomial $D(-3)$-tuples and polynomial $D(-1)$-tuples, for which in [8] all solutions of the equations (2.9) and (2.10) are generated in the way which is standard by now. On the other hand, in $\mathbb{R}[X]$ we can not pick the minimal element of the observed set as we can in $\mathbb{Z}[X]$, just the element with minimal degree. This makes the proofs in $\mathbb{R}[X]$ more complex. In the following lemma we firstly obtain bounds for the degrees of the fundamental solutions of the equations (2.9) and (2.10). Later, we will completely determine the form of these fundamental solutions so that we can prove Theorem 1.6.

Lemma 2.5. *Let (z, x) and (z, y) be solutions, with $x, y, z \in \mathbb{R}^+[X]$, of (2.9) and (2.10), respectively. Then, there exist solutions (z_0, x_0) and (z_1, y_1), with $z_0, x_0, z_1, y_1 \in \mathbb{R}[X]$, of (2.9) and (2.10), respectively, such that:*

$$\frac{\gamma}{2} \leq \deg(z_0) \leq \gamma, \qquad \frac{\alpha}{2} \leq \deg(x_0) \leq \frac{\alpha + \gamma}{2}, \tag{2.11}$$

$$\frac{\gamma}{2} \leq \deg(z_1) \leq \gamma, \qquad \frac{\beta}{2} \leq \deg(y_1) \leq \frac{\beta + \gamma}{2}. \tag{2.12}$$

More precisely, if $\deg(z_0) = \gamma$, then $z_0 = c \mp s$ and if $\deg(z_1) = \gamma$, then $z_1 = c \mp t$.

Moreover, there also exist non-negative integers m and n such that

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^{2m}, \tag{2.13}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^{2n}. \tag{2.14}$$

Proof. We prove the statements of the lemma for the equation (2.9). Statements for the equation (2.10) can be obtained analogously.

From (2.3), we conclude that $z \neq 0$, since otherwise we would have $cd = 1$, i.e. $\gamma + \delta = 0$, which is not possible. By (2.9), we have

$$a(z^2+1) = c(x^2+1), \tag{2.15}$$

so if z is a constant, then x is a constant. Moreover, $z^2+1 \neq 0$ and $x^2+1 \neq 0$, since the case with $z = \pm i$ and $x = \pm i$ is not possible in $\mathbb{R}[X]$. Hence, in that case, $c = c_1 a$ with $c_1 \in \mathbb{R} \setminus \{0\}$. Also, $\alpha = \beta = \gamma$ and, by (2.2), we have $a_1 c^2 - 1 = s^2$. This leads to $(\sqrt{a_1}c - s)(\sqrt{a_1}c + s) = 1$, which is not possible since at least one of the factors on the left hand side of this equation has to be non-constant. It follows that $\deg(z) \geq 1$.

Notice that

$$(s \pm \sqrt{ac})^{2m} = (s^2 \pm 2s\sqrt{ac} + ac)^m = (2ac - 1 \pm 2s\sqrt{ac})^m$$

and

$$(s+\sqrt{ac})^{2m}(s-\sqrt{ac})^{2m} = (s^2 - ac)^{2m} = (-1)^{2m} = 1. \tag{2.16}$$

Let (z, x) be a solution of the equation (2.9) in polynomials from $\mathbb{R}^+[X]$. Let us consider all pairs (z^*, x^*) of polynomials from $\mathbb{R}[X]$ for which

$$z^*\sqrt{a} \pm x^*\sqrt{c} = (z\sqrt{a} \pm x\sqrt{c})(s \pm \sqrt{ac})^{2m}, \tag{2.17}$$

$m \in \mathbb{Z}$. By (2.16), (z^*, x^*) is a solution of (2.9), so $z^* \neq 0$ and $\deg(z^*) \geq 1$. Analogously as in [8], we can prove that $z^* > 0$.

Among all pairs (z^*, x^*), we choose one with the minimal $\deg(z^*)$. We denote that pair by (z_0, x_0). By (2.17),

$$z\sqrt{a} + \varepsilon_1 x\sqrt{c} = (z_0\sqrt{a} + \varepsilon_2 x_0\sqrt{c})(s + \sqrt{ac})^{2m},$$

for $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and some $m \in \mathbb{Z}$. We follow the arguments from [13, Lemma 4] for $n = 1$. By choosing appropriate signs of x and x_0, which does not change the degree of z_0, we may assume that

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^{2m} \tag{2.18}$$

holds for $m \geq 0$ and a pair (z_0, x_0) for which $\deg(z_0)$ is minimal. For $m < 0$, by (2.16), it holds

$$z\sqrt{a} - x\sqrt{c} = (z_0\sqrt{a} - x_0\sqrt{c})(s - \sqrt{ac})^{-2m}.$$

Therefore, by changing the signs of x and x_0, which does not have effect on the degree of x_0 neither the degree of z_0, we may assume that (2.18) holds for $m \geq 0$ and for $z_0 > 0$ with minimal degree, up to the change of the signs of x and x_0.

Let us now find bounds on $\deg(z_0)$ and $\deg(x_0)$. Let $(v_m)_{m\in\mathbb{Z}}$ and $(x_m)_{m\in\mathbb{Z}}$ be the sequences of polynomials given by

$$v_m\sqrt{a} + x_m\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^{2m}. \tag{2.19}$$

Then, $v_0 = z_0$,

$$v_1\sqrt{a} + x_1\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^2$$

and

$$v_{-1}\sqrt{a}+x_{-1}\sqrt{c}=(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^{-2}=(z_0\sqrt{a}+x_0\sqrt{c})(s-\sqrt{ac})^2.$$

More precisely,

$$v_1 = (2ac-1)z_0+2scx_0=2c(az_0+sx_0)-z_0, \tag{2.20}$$

$$v_{-1} = (2ac-1)z_0-2scx_0=2c(az_0-sx_0)-z_0. \tag{2.21}$$

By the minimality of $\deg(z_0)$ and by using the fact that $z^*\neq 0$, we get

$$\deg(v_1) \geq \deg(v_0)=\deg(z_0), \tag{2.22}$$

$$\deg(v_{-1}) \geq \deg(v_0)=\deg(z_0). \tag{2.23}$$

From (2.15), we can conclude that if $c|z^2+1$, then $a|x^2+1$. For the pair (z_0,x_0), we prove the similar statement by following the arguments from [13, Lemma 4 v.]. We assume that $c|z^2+1$ and, by induction on $m\geq 0$ from (2.18), we have to show that $c|z_0^2+1$. For $m=0$, from (2.18), we have $z=z_0$, so the statement follows trivially. For $m=1$, in (2.18) it holds $z\sqrt{a}+x\sqrt{c}=(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^2$. But, since $z_0\sqrt{a}+x_0\sqrt{c}=(z\sqrt{a}+x\sqrt{c})(s-\sqrt{ac})^2$ also holds, we conclude that $z_0=(2ac-1)z-2scx$. Finally, $z_0\equiv -z \pmod{c}$, so $(z_0)^2\equiv z^2 \pmod{c}$. Since $z^2\equiv -1 \pmod{c}$, then $(z_0)^2\equiv -1 \pmod{c}$. By induction on m from (2.18), we can conclude that if $c|z^2+1$, then $c|z_0^2+1$. Hence, there exists

$$d_0=\frac{z_0^2+1}{c}\in\mathbb{R}^+[X]. \tag{2.24}$$

By (2.15), it furthermore follows that $a|x_0^2+1$, i.e. we have

$$x_0^2=ad_0-1 \quad\text{and}\quad z_0^2=cd_0-1. \tag{2.25}$$

If x_0 is a constant, then a and d_0 are constants and $d_0=a$, since otherwise we would have two different constants in a polynomial $D(-1)$-triple $\{a,d_0,c\}$, which is not possible. Also, by (2.2) and (2.25), $x_0^2=a^2-1$, $z_0=s$. Hence, $\deg(v_0)=\deg(z_0)=\frac{\gamma}{2}$. Since $az_0\pm sx_0=s(a\pm x_0)$, by (2.20) and (2.21), $\deg(v_{\pm 1})=\frac{3\gamma}{2}$. If x_0 is not a constant, since z_0^2+1 is a multiple of c and $2\deg(z_0)=\deg(z_0^2+1)\geq\gamma$, it follows that $\deg(z_0)\geq\frac{\gamma}{2}$. Also, from (2.15), we get

$$\deg(x_0)=\frac{\alpha-\gamma}{2}+\deg(z_0), \tag{2.26}$$

thus $\deg(x_0)\geq\frac{\alpha}{2}$. By (2.25), the leading coefficients of the polynomials az_0 and sx_0 are equal to $A\sqrt{CD_0}$, where A, C and D_0 are the leading coefficients of the polynomials a, c and d_0, respectively. Also, by (2.26), $\deg(az_0)=\deg(sx_0)=\alpha+\deg(z_0)$. Hence, one of the polynomials $az_0\pm sx_0$ obviously has degree equal to $\alpha+\deg(z_0)$. By (2.20) and (2.21), one of the polynomials

v_1 and v_{-1} has degree equal to $\alpha+\gamma+\deg(z_0)$, which is strictly greater than $\deg(v_0)=\deg(z_0)$. Therefore,

$$\max\{\deg(v_1),\deg(v_{-1})\}=\alpha+\gamma+\deg(z_0)>\deg(v_0). \tag{2.27}$$

Using (2.9), (2.22), (2.23) and (2.27), we obtain

$$\deg(z_0^2)<\deg(v_1v_{-1})=\deg(4c(c-a)s^2+z_0^2). \tag{2.28}$$

From (2.28), it follows that $\deg(v_{-1})+\deg(v_1)\leq\alpha+3\gamma$. Furthermore, $\deg(z_0)+\alpha+\gamma+\deg(z_0)\leq\alpha+3\gamma$, so $\deg(z_0)\leq\gamma$. More precisely, if $\min\{\deg(v_1),\deg(v_{-1})\}>\deg(v_0)$, then we have

$$\deg(z_0)<\gamma.$$

In that case, by (2.26),

$$\deg(x_0)<\frac{\alpha+\gamma}{2}.$$

If $\min\{\deg(v_1),\deg(v_{-1})\}=\deg(v_0)=\deg(z_0)$, then we have

$$\deg(z_0)=\gamma.$$

By (2.26), $\deg(x_0)=\frac{\alpha+\gamma}{2}$. Also, by (2.25), $\deg(d_0)=\gamma$. In this case, by Lemma 2.4, a polynomial $D(-1)$-triple $\{a,c,d_0\}$ in $\mathbb{R}[X]$ is regular. Depending whether $c>d_0$ or $c<d_0$, in (1.3) we have $c=a+d_0\pm2x_0$, respectively, where we take $x_0>0$. Then, by (1.4), $c=\pm s+z_0$, respectively. Hence, $z_0=c\mp s$. For a $D(-1)$-triple $\{b,c,d_1\}$, similarly we conclude that $z_1=c\mp t$. □

Now, we investigate the sequence $(v_m)_{m\geq0}$, given by (2.19) (obtained from (2.13)) for some initial values (z_0,x_0) for which estimates (2.11) hold, looking for $d_m\in\mathbb{R}^+[X]$ such that $v_m^2=cd_m-1$, for some $m\geq0$. We also study the sequence $(w_n)_{n\geq0}$, obtained from (2.14) for some initial values (z_1,y_1) for which estimates (2.12) hold, trying to find $d_n\in\mathbb{R}^+[X]$ such that $w_n^2=cd_n-1$, for some $n\geq0$. Since we want to extend a polynomial $D(-1)$-triple $\{a,b,c\}$ in $\mathbb{R}[X]$ to a polynomial $D(-1)$-quadruple with an element $d\in\mathbb{R}^+[X]$, we need to solve the equation

$$v_m=w_n \tag{2.29}$$

with $m,n\geq0$. Then, for some non-negative integers m and n we would have $z=v_m=w_n$ where, by (2.13) and (2.14), the sequences $(v_m)_{m\geq0}$ and $(w_n)_{n\geq0}$ are given by

$$\begin{aligned}v_0&=z_0,\\ v_1&=(2ac-1)z_0+2scx_0,\\ v_{m+2}&=(4ac-2)v_{m+1}-v_m,\end{aligned} \tag{2.30}$$

$$
\begin{aligned}
w_0 &= z_1, \\
w_1 &= (2bc-1)z_1 + 2tcy_1, \qquad (2.31)\\
w_{n+2} &= (4bc-2)w_{n+1} - w_n.
\end{aligned}
$$

REMARK 2.6. Note that in (2.30) and (2.31) we take $z_0, z_1 > 0$ and we have to consider both possibilities $\pm x_0$, $\pm y_1$. Note also that elements of the sequences $(v_m)_{m\geq 0}$ and $(w_n)_{n\geq 0}$, multiplied by $\pm i$ correspond to elements of the analogous sequences, with even indices, obtained for a polynomial $D(1)$-triple $\{ai, bi, ci\}$ in $\mathbb{C}[X]$. In other words, by multiplying (2.13) and (2.14) with $\pm i$, we obtain a subcase (described in [12, Lemma 3]) of the case with $n = 1$ in $\mathbb{C}[X]$ which corresponds to the case with $n = -1$ in $\mathbb{R}[X]$.

The following lemma implies that if the equation (2.29) has a solution, then it has a solution for $m = n = 0$. In the part a) of its proof we consider congruences in $\mathbb{R}[X]$, but in the rest of the proof we have to consider congruences in $\mathbb{Z}[X]$, which appear after taking into consideration that polynomials in $\mathbb{R}[X]$ we are dealing with in our problem have the form $\frac{p'}{\sqrt{3}}$, where $p' \in \mathbb{Z}[X]$.

LEMMA 2.7. *Let $\{a, b, c\}$, where $0 < a < b < c$ and $a = \frac{a'}{\sqrt{3}}$, $b = \frac{b'}{\sqrt{3}}$ and $c = \frac{c'}{\sqrt{3}}$ with $a', b', c' \in \mathbb{Z}[X]$, be a polynomial $D(-1)$-triple in $\mathbb{R}[X]$. If that triple can be exstended with an element $d = \frac{d'}{\sqrt{3}}$, where $d' \in \mathbb{Z}[X]$ and $d > c$, to a polynomial $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{R}[X]$ i.e. if the equation (2.29) has a solution, then $z_0 = z_1$ and $\deg(z_0) = \deg(z_1) < \gamma$.*

PROOF. Let $v_m = w_n$, for some integers $m, n \geq 0$. From (2.30) and (2.31), by induction, it easily follows that

$$v_m \equiv (-1)^m z_0 \pmod{c}, \quad w_n \equiv (-1)^n z_1 \pmod{c},$$

where we consider congruences in $\mathbb{R}[X]$. Since $v_m = w_n$, we have

$$z_0 \equiv \pm z_1 \pmod{c}. \qquad (2.32)$$

By Lemma 2.5, $\deg(z_0), \deg(z_1) \leq \gamma$. We have to consider four possible combinations of degrees:

a) If $\deg(z_0), \deg(z_1) < \gamma$, then $z_0 = \pm z_1$. Moreover, since $z_0, z_1 > 0$, we conclude that $z_0 = z_1$. We can also conclude that m and n have the same parity.

b) If $\deg(z_0) < \gamma$ and $\deg(z_1) = \gamma$ then, from the proof of Lemma 2.5, we have that the triple $\{b, c, d_1\}$ is regular and $z_1 = c \mp t$ (the sign $-$ appears if $c > d_1$ and the sign $+$ if $c < d_1$). In this case $y_1 = b \mp t$.

c) If $\deg(z_0) = \gamma$ and $\deg(z_1) < \gamma$ then, similarly as in **b)**, the triple $\{a, c, d_0\}$ is regular and $z_0 = c \mp s$ (the sign $-$ appears if $c > d_0$ and the sign $+$ if $c < d_0$). In this case $x_0 = a \mp s$.

d) Let $\deg(z_0) = \gamma$ and $\deg(z_1) = \gamma$. Then the triples $\{a, c, d_0\}$ and $\{b, c, d_1\}$ are both regular, so $z_0 = c \mp s$ ($x_0 = a \mp s$) and $z_1 = c \mp t$ ($y_1 = b \mp t$).

In cases b) - d) polynomials x_0, z_0, y_1 and z_1 have the form $\frac{x_0'}{\sqrt{3}}$, $\frac{z_0'}{\sqrt{3}}$, $\frac{y_1'}{\sqrt{3}}$ and $\frac{z_1'}{\sqrt{3}}$, respectively, where $x_0', z_0', y_1', z_1' \in \mathbb{Z}[X]$. After taking this into consideration, from (2.30) we can conclude that $v_m = \frac{v_m'}{3^m\sqrt{3}}$, where $v_m' \in \mathbb{Z}[X]$, for $m \geq 0$. Also, from (2.31), $w_n = \frac{w_n'}{3^n\sqrt{3}}$, where $w_n' \in \mathbb{Z}[X]$, for $n \geq 0$. Therefore, by induction, we get congruences in $\mathbb{Z}[X]$

$$v_m' \equiv 3^m(-1)^m z_0' \pmod{2c'}, \quad w_n' \equiv 3^n(-1)^n z_1' \pmod{2c'}.$$

Since $v_m = w_n$, we have $3^n v_m' = 3^m w_n'$ and then $3^{m+n} z_0' \equiv \pm 3^{m+n} z_1' \pmod{2c'}$. If $3 \mid c'$ then, by (1.6), $3 \mid s'$ and $3 \mid t'$ so $3 \mid z_0'$ and $3 \mid z_1'$. If $3^2 \mid c'$ then, since $3^2 \mid (t')^2$, we are led to the contradiction $3^2 \mid 3$. Therefore, if $3 \mid c'$ then we may take $c' = 3c''$, where $c'' \in \mathbb{Z}[X]$ and $\gcd(3, c'') = 1$. We also take $s' = 3s''$, $t' = 3t''$, $z_0' = 3z_0''$ and $z_1' = 3z_1''$, where $s'', t'', z_0'', z_1'' \in \mathbb{Z}[X]$. If $\gcd(3, c') = 1$ then we simply take $c'' := c'$, $s'' := s'$, $z_0'' := z_0'$ and $z_1'' := z_1'$. Finally, we can consider the congruence

$$z_0'' \equiv \pm z_1'' \pmod{2c''} \tag{2.33}$$

in $\mathbb{Z}[X]$, instead of the congruence (2.32) in $\mathbb{R}[X]$.

In the case b), from (2.33), we get $z_0'' \equiv \pm(c'' \mp t'') \pmod{2c''}$. By (1.6), $0 < t'' < c''$, therefore, $z_0'' \equiv \pm t'' \pmod{c''}$, which leads to $z_0'' = \pm t''$. But, then in (2.33) we would have $\pm t'' \equiv \pm(c'' \mp t'') \pmod{2c''}$. Finally, either $2c'' \mid c''$, which is not possible, or $c'' \mid 2t''$ which is also not possible because of (1.6).

In the case c), we conclude analogously as in b).

In the case d), from (2.33), we get $c'' \mp s'' \equiv \pm(c'' \mp t'') \pmod{2c''}$. From that, we get

$$\pm s'' \pm t'' \equiv 0 \pmod{2c''}. \tag{2.34}$$

By (1.6), we conclude that $0 < s'' < c''$ and $0 < t'' < c''$. Hence $| \pm s'' \pm t'' | < 2c''$ and we obtain a contradiction with (2.34). With this we showed that cases b) - d) are not possible. ∎

In the proof of Lemma 2.5, we proved the existence of $d_0 \in \mathbb{R}^+[X]$ such that (2.25) holds, where $x_0, z_0 \in \mathbb{R}[X]$ and $z_0 > 0$. Analogously, there exists $d_1 \in \mathbb{R}^+[X]$ such that

$$bd_1 - 1 = y_1^2 \quad \text{and} \quad cd_1 - 1 = z_1^2, \tag{2.35}$$

with $y_1, z_1 \in \mathbb{R}[X]$ and $z_1 > 0$. In the following lemma, we will prove that $d_0 = d_1$.

Lemma 2.8. *Let $\{a, b, c, d\}$, where $0 < a < b < c < d$ and $a = \frac{a'}{\sqrt{3}}$, $b = \frac{b'}{\sqrt{3}}$, $c = \frac{c'}{\sqrt{3}}$ and $d = \frac{d'}{\sqrt{3}}$, with $a', b', c', d' \in \mathbb{Z}[X]$, be a $D(-1)$-quadruple* *in* $\mathbb{R}[X]$. *Then, there exists* $d_0 \in \mathbb{R}^+[X]$ *such that* $\deg(d_0) < \gamma$ *and* $ad_0 - 1$, $bd_0 - 1$ *and* $cd_0 - 1$ *are perfect squares.*

PROOF. By the proof of Lemma 2.5, there exists $d_0 \in \mathbb{R}^+[X]$ defined by (2.24), such that (2.25) holds, where $x_0, z_0 \in \mathbb{R}[X]$ and $z_0 > 0$. By Lemma 2.5, Lemma 2.7, (2.10) and (2.24),

$$bd_0 - 1 = b\frac{z_1^2 + 1}{c} - 1 = y_1^2.$$

By (2.35), $d_0 = d_1$. Also, since $\deg(z_0) < \gamma$, from (2.24) we conclude that $\deg(d_0) < \gamma$. ∎

REMARK 2.9. Generally, $d_0, x_0, z_0 \in \mathbb{R}[X]$. But, by Remark 2.1, we consider a problem of extending a polynomial $D(-1)$-triple to a polynomial $D(-1)$-quadruple in $\mathbb{R}[X]$ which follows from a problem of extending a polynomial $D(-3)$-triple to a polynomial $D(-3)$-quadruple in $\mathbb{Z}[X]$, so we only consider the situation which is a consequence of that problem and we have to assume that $d_0 = \frac{d_0'}{\sqrt{3}}$, $x_0 = \frac{x_0'}{\sqrt{3}}$, $z_0 = \frac{z_0'}{\sqrt{3}}$ with $d_0', x_0', z_0' \in \mathbb{Z}[X]$, respectively.

3. Determination of the initial terms

We assume that $\{a, b, c, d\}$, with $0 < a < b < c < d$ and $a = \frac{a'}{\sqrt{3}}$, $b = \frac{b'}{\sqrt{3}}$, $c = \frac{c'}{\sqrt{3}}$ and $d = \frac{d'}{\sqrt{3}}$ with $a', b', c', d' \in \mathbb{Z}[X]$, is a polynomial $D(-1)$-quadruple in $\mathbb{R}[X]$ with minimal δ among all such polynomial $D(-1)$-quadruples. By Lemma 2.8, there also exists a polynomial $D(-1)$-quadruple $\{a, b, c, d_0\}$ in $\mathbb{R}[X]$. By Remark 2.9, $d_0 = \frac{d_0'}{\sqrt{3}}$, $x_0 = \frac{x_0'}{\sqrt{3}}$, $z_0 = \frac{z_0'}{\sqrt{3}}$ with $d_0', x_0', z_0' \in \mathbb{Z}[X]$, respectively. Since $\deg(d_0) < \gamma$, it follows that $\{a, b, c, d_0\}$ is an improper polynomial $D(-1)$-quadruple. From our considerations in the Section 1, we conclude that $\{a, b, c, d_0\} = \{a, a, b, c\}$, i.e. $d_0 = a$ and $\alpha = 0$. By (2.25), $a^2 - 1 = x_0^2$, where $a = \frac{a'}{\sqrt{3}}$ and $x_0 = \frac{x_0'}{\sqrt{3}}$, with $a', x_0' \in \mathbb{Z}$. Therefore, we solve the equation $(a')^2 - 3 = (x_0')^2$ in integers and we get $(a', x_0') = (\pm 2, \pm 1)$. Since $a' > 0$, we conclude that it suffices to consider a polynomial $D(-1)$-quadruples in $\mathbb{R}[X]$ with $a = \frac{2}{\sqrt{3}}$, i.e. a polynomial $D(-1)$-quadruples $\{\frac{2}{\sqrt{3}}, b, c, d\}$, with $\frac{2}{\sqrt{3}} < b < c < d$ and $b = \frac{b'}{\sqrt{3}}$, $c = \frac{c'}{\sqrt{3}}$, $d = \frac{d'}{\sqrt{3}}$, with $b', c', d' \in \mathbb{Z}[X]$. Hence, we also have a polynomial $D(-1)$-quadruple $\{\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, b, c\}$ in $\mathbb{R}[X]$ and a polynomial $D(1)$-quadruple $\{\frac{2}{\sqrt{3}}i, \frac{2}{\sqrt{3}}i, bi, ci\}$ in $\mathbb{C}[X]$, so we can draw the following conclusion.

LEMMA 3.1. *For a polynomial* $D(-1)$*-quadruple* $\{\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, b, c\}$ *in* $\mathbb{R}[X]$, *it holds* $\beta < \gamma$.

PROOF. Assume that $\beta = \gamma$. Then, by [12, Lemma 5], a polynomial $D(1)$-quadruple $\{\frac{2}{\sqrt{3}}i, \frac{2}{\sqrt{3}}i, bi, ci\}$ in $\mathbb{C}[X]$ is regular. Applying (1.5) for that quadruple, since $a = \frac{2}{\sqrt{3}}$, we obtain

$$(ci + bi - 2ai)^2 = 4\Big(\frac{4i^2}{3} + 1\Big)(bci^2 + 1).$$

But, then it also holds

$$-(c + b - 2a)^2 = \frac{4}{3}(bc - 1) = \frac{4}{3}t^2,$$

which is not possible in $\mathbb{R}[X]$. Since $\beta \leq \gamma$, it follows that $\beta < \gamma$. ∎

For the polynomial $D(-1)$-triple $\{\frac{2}{\sqrt{3}}, b, c\}$ in $\mathbb{R}[X]$, with $\frac{2}{\sqrt{3}} < b < c$, we have $d_0 = d_1 = a = \frac{2}{\sqrt{3}}$, hence, by (2.25) and (2.35), $x_0 = \pm\frac{1}{\sqrt{3}}$, $z_0 = z_1 = s$ (since $z_0, z_1 > 0$) and $y_1 = \pm r$. With that initial terms, from (2.30) and (2.31), the following four sequences arise. Namely, the sequences $(v_m^{\pm})_{m\geq 0}$ are given by

$$\begin{aligned} v_0^+ &= s, \\ v_1^+ &= \frac{s}{\sqrt{3}}(6c - \sqrt{3}), \\ v_{m+2}^+ &= \frac{2}{\sqrt{3}}(4c - \sqrt{3})v_{m+1}^+ - v_m^+, \end{aligned} \tag{3.1}$$

$$\begin{aligned} v_0^- &= s, \\ v_1^- &= \frac{s}{\sqrt{3}}(2c - \sqrt{3}), \\ v_{m+2}^- &= \frac{2}{\sqrt{3}}(4c - \sqrt{3})v_{m+1}^- - v_m^- \end{aligned} \tag{3.2}$$

and the sequences $(w_n^{\pm})_{n\geq 0}$ are given as follows

$$\begin{aligned} w_0^+ &= s, \\ w_1^+ &= (2bc - 1)s + 2crt, \\ w_{n+2}^+ &= (4bc - 2)w_{n+1}^+ - w_n^+, \end{aligned} \tag{3.3}$$

$$\begin{aligned} w_0^- &= s, \\ w_1^- &= (2bc - 1)s - 2crt, \\ w_{n+2}^- &= (4bc - 2)w_{n+1}^- - w_n^-. \end{aligned} \tag{3.4}$$

4. The final part of the proof of Theorem 1.6

In order to extend a polynomial $D(-1)$-triple $\{\frac{2}{\sqrt{3}}, b, c\}$ to a polynomial $D(-1)$-quadruple, we are trying to find a suitable solution of equation (2.29), where for $(v_m)_{m\geq 0}$ we take $(v_m^{\pm})_{m\geq 0}$, defined by (3.1) and (3.2), and for $(w_n)_{n\geq 0}$ we take $(w_n^{\pm})_{n\geq 0}$, defined by (3.3) and (3.4). The trivial solution

$v_0 = w_0 = s$ leads to an improper polynomial $D(-1)$-quadruple $\{\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, b, c\}$ in $\mathbb{R}[X]$. But, we will prove that neither of the above mentioned four sequences leads to a nontrivial solution. Therefore, the proof of Theorem 1.6 follows directly from the following lemma.

LEMMA 4.1. *Let the sequences $(v_m^{\pm})_{m\geq 0}$ and $(w_n^{\pm})_{n\geq 0}$ be defined by (3.1) - (3.4). Then, the equations $v_m^{\pm} = w_n^{\pm}$ have no nontrivial solutions for $m \geq 0$ and $n \geq 0$.*

PROOF. From (3.1) and (3.2), by induction, it follows that

$$\deg(v_m^{\pm}) = m\gamma + \frac{\gamma}{2}, \tag{4.1}$$

for $m \geq 0$. Hence, $\deg(v_m^{\pm}) < \deg(v_{m+1}^{\pm})$, for $m = 0, 1, 2, \ldots$. Similarly, from (3.3), it follows that

$$\deg(w_n^{+}) = n(\beta + \gamma) + \frac{\gamma}{2}, \tag{4.2}$$

for $n \geq 0$. Hence, $\deg(w_n^{+}) < \deg(w_{n+1}^{+})$, for $n = 0, 1, 2, \ldots$.

Also, from (3.1), $V_0^{+} = \sqrt{\frac{2C'}{3}}$ is the leading coefficient of the polynomial v_0^{+} and, by induction, we conclude that the leading coefficients of the polynomials v_m^{+} are

$$V_m^{+} = \frac{2^{3m-2}}{3^{m-1}}(C')^m\sqrt{\frac{2C'}{3}}, \tag{4.3}$$

for $m \geq 1$, where C' is the leading coefficient of the polynomial c'. Analogously, from (3.2), $V_0^{-} = \sqrt{\frac{2C'}{3}}$ is the leading coefficient of the polynomial v_0^{-} and, by induction, we conclude that the leading coefficients of the polynomials v_m^{-} are

$$V_m^{-} = \frac{2^{3m-2}}{3^{m}}(C')^m\sqrt{\frac{2C'}{3}}, \tag{4.4}$$

for $m \geq 1$, where C' is the leading coefficient of the polynomial c'. Furthermore, from (3.3), we find that the leading coefficients of the polynomials w_n^{+} are

$$W_n^{+} = \frac{2^{2n}}{3^{n}}(B')^n(C')^n\sqrt{\frac{2C'}{3}}, \tag{4.5}$$

for $n \geq 0$, where B' and C' are the leading coefficients of the polynomials b' and c', respectively.

If the equations $v_m^{\pm} = w_n^{+}$ have a solution, by comparing the degrees on both sides of these equations and using (4.1) and (4.2), we obtain a homogeneous linear Diophantine equation

$$(m - n)\gamma - n\beta = 0.$$

For fixed β and γ, there holds $m-n=q\beta$ and $n=q\gamma$, with $q\in\mathbb{Z}$. Since for the triple $\{\frac{2}{\sqrt{3}},b,c\}$ we have $\alpha=0$, by (2.2), we can conclude that β and γ are even positive integers. Therefore, n is even and m is even. Especially, for $m=0$, we have $n=0$, which leads to the trivial solution.

If the equation $v_m^+=w_n^+$ has a nontrivial solution then, from (4.3) and (4.5), assuming that $m=2k$ and $n=2l$ with $k,l\in\mathbb{N}$, we also conclude that

$$\Big(\frac{3^{k-l}(B')^l}{2^{3k-2l-1}(C')^{k-l}}\Big)^2=3.$$

But, since $\sqrt{3}\notin\mathbb{Q}$, this is not possible.

By comparing the leading coefficients in equations (1.6), obtained for the triple $\{\frac{2}{\sqrt{3}},b,c\}$, we conclude that $B'=2(B'')^2$ and $C'=2(C'')^2$, where B' and C' are the leading coefficients of the polynomials b' and c', respectively, and $B'',C''\in\mathbb{N}$. If the equation $v_m^-=w_n^+$ has a nontrivial solution then, from (4.4) and (4.5), assuming that $m=2k$ and $n=2l$ with $k,l\in\mathbb{N}$, we conclude that

$$\Big(\frac{2^{2k}(C'')^k}{2^{2l}(B'')^l(C'')^l}\Big)^2=2\cdot 3^{k-l}.$$

But, then we would obtain $\sqrt{2}\in\mathbb{Q}$, if $k-l$ is even, or $\sqrt{6}\in\mathbb{Q}$, if $k-l$ is odd, a contradiction in each case.

Now we have to consider the equations $v_m^\pm=w_n^-$. From (3.3) and (3.4), using (2.2), we get

$$\begin{aligned} w_1^+w_1^- &= (2bc-1)^2s^2-4c^2r^2t^2 \\ &= 4bc^3-4c^2-4b^2c^2+4bc+ac-1. \end{aligned} \tag{4.6}$$

By Lemma 3.1, $\beta<\gamma$ so, by comparing the leading coefficients in (4.6), we obtain

$$W_1^+W_1^-=4BC^3=\frac{4B'(C')^3}{3^2}, \tag{4.7}$$

where $W_1^\pm$ are the leading coefficients of the polynomials $w_1^\pm$, and, by comparing the degrees in (4.6), we obtain

$$\deg(w_1^+w_1^-)=\beta+3\gamma. \tag{4.8}$$

By (4.2),

$$\deg(w_1^+)=\beta+\frac{3\gamma}{2} \tag{4.9}$$

and, by (4.5), we get

$$W_1^+=\frac{2^2B'C'}{3}\sqrt{\frac{2C'}{3}}. \tag{4.10}$$

Hence, by (4.7) and (4.10),

$$W_1^- = \frac{C'\sqrt{2C'}}{2\sqrt{3}} \tag{4.11}$$

and, by (4.8) and (4.9), we conclude $\deg(w_1^-) = \frac{3\gamma}{2}$. From (3.4), it follows that $\deg(w_0^-) = \frac{\gamma}{2}$ and

$$\deg(w_n^-) = (n-1)\beta + n\gamma + \frac{\gamma}{2}, \tag{4.12}$$

for $n \geq 1$. Hence, $\deg(w_n^-) < \deg(w_{n+1}^-)$, for $n = 0, 1, 2, \ldots.$

Also, from (3.4), the leading coefficient of the polynomial w_0^- is $W_0^- = \sqrt{\frac{2C'}{3}}$ and, by induction, we conclude that the leading coefficients of the polynomials w_n^- are

$$W_n^- = \frac{2^{2n-3}(B')^{n-1}(C')^n}{3^{n-1}}\sqrt{\frac{2C'}{3}}, \tag{4.13}$$

for $n \geq 1$, where B' and C' are the leading coefficients of the polynomials b' and c', respectively.

If the equation $v_m^{\pm} = w_n^-$ has a solution, by comparing the degrees on both sides of this equation and using (4.1) and (4.12), we obtain a homogeneous linear Diophantine equation

$$(m-n)\gamma - (n-1)\beta = 0.$$

For fixed β and γ, there holds $m - n = q\beta$ and $n - 1 = q\gamma$, with $q \in \mathbb{Z}$. Since for the triple $\{\frac{2}{\sqrt{3}}, b, c\}$ we already concluded that β and γ are even positive integers, then n is odd and m is odd.

If the equation $v_m^+ = w_n^-$ has a nontrivial solution then, from (4.3) and (4.13), assuming that $m = 2k+1$ and $n = 2l+1$ with $k, l \in \mathbb{N}_0$, we conclude that

$$\Big(\frac{(B'')^l}{2^{2(k-l)}(C'')^{k-l}}\Big)^2 = \frac{2}{3^{k-l}}.$$

If $k-l$ is even, then we would have $\sqrt{2} \in \mathbb{Q}$, which is a contradiction. If $k-l$ is odd, we get $\sqrt{\frac{2}{3}} \in \mathbb{Q}$, which is also not possible.

If the equation $v_m^- = w_n^-$ has a nontrivial solution then, from (4.4) and (4.13), assuming that $m = 2k+1$ and $n = 2l+1$ with $k, l \in \mathbb{N}_0$, we conclude that

$$\Big(\frac{2^{3k-2l+1}(C')^{k-l}}{3^{k-l}(B')^l}\Big)^2 = 3.$$

But, then we would obtain $\sqrt{3} \in \mathbb{Q}$, which is again a contradiction. ∎

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