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THE HAUSDORFF DIMENSION OF DIRECTIONAL EDGE ESCAPING POINTS SET

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Abstract. In this paper, we define the directional edge escaping points set of function iteration under a given plane partition and then prove that the upper bound of Hausdorff dimension of the directional edge escaping points set of $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, is no more than 1.

1. Introduction

The Julia sets of transcendental entire functions always have very complicated fractal structures (see [11]). We often use the Hausdorff dimension to describe them. Many profound results about the Hausdorff dimension of Julia sets of transcendental entire functions have been obtained. For example, Stallard and Bishop proved that there is a transcendental entire function such that the Hausdorff dimension of its Julia set is equal to any pre-specified number in the closed interval [1,2] (see [2, 17, 18]).

In addition to Julia set, the closely related escaping set (see [4]) is also the subject of increasing interest. In particular, there are many studies on the escaping sets of specific transcendental entire functions. Take the escaping set of the exponential function for example. Schleicher and Zimmer proved that the escaping points set of λe^z with $\lambda \neq 0$ is the Cantor set of curves and has a peculiar phenomenon of ”dimension paradox”, which was first found by Karpińska (see [8, 9]); that is, the Hausdorff dimension of the hairs without endpoints is 1, while, the Hausdorff dimension of the set of endpoints is 2 (see [16]). Furthermore, it is not only the escaping points set of the exponential function that has been intensively studied, but also the escaping parameters

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set of a family of the exponential functions. For example, Schleiher, Forster, Rempe, Bailesteanu and Balan proved that the escaping parameters set of a family of exponential functions also has the properties of Cantor bundle structure and "dimension paradox" (see [12, 14, 1, 6]). Of course, there are many other entire functions that have been studied deeply, such as the cosine function $ae^z + be^{-z}$, where $ab \neq 0$ (see [7, 10, 12, 13, 15, 19]).

In this paper, we will combine exponential and cosine functions to study the function $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. Moreover, we will also study escaping points set of a special kind, which we call directional edge escaping points set. For a function $S(z)$, its directional edge escaping points set under a given plane partition is defined below.

First, we divide the complex plane into squares. Denote by $S^n(z)$ the n-fold iterate of $S(z)$, where $n \in \mathbb{N}$. Take one of the squares arbitrarily, denote it by B_0. A point z in it is called directional edge escaping point if it satisfies

- $S^n(z) \to \infty$ as $n \to \infty$,
- $|\operatorname{Im} S^n(z)| \leq \lambda |S^n(z)|$ for all $n \in \mathbb{N}$,
- $B_{n+1}(z) \cap \partial S(B_n(z)) \neq \emptyset$ for all $n \in \mathbb{N}$,

where $\lambda \in (0, 1)$ is a constant, $B_{n+1}(z)$ is the square $S^{n+1}(z)$ belongs to, $\partial S(B_n(z))$ is the boundary of the image of $B_n(z)$ under function $S(z)$. See Figure 1.

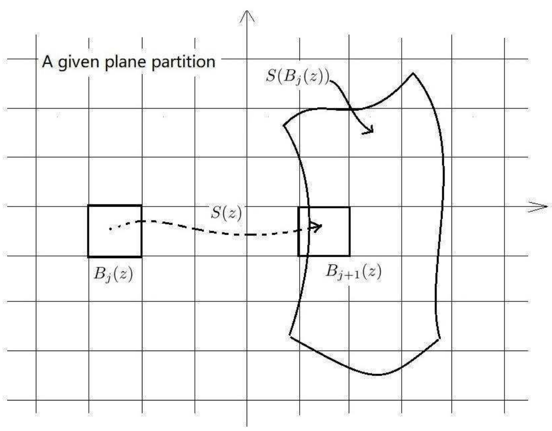

FIGURE 1. directional edge escaping point

As we all know, a very important method is to study the transcendental dynamics by dividing the plane (see [3]). If we imagine a series of objects connected by one rope, see Figure 2, the concept of directional edge escaping point can emerge.

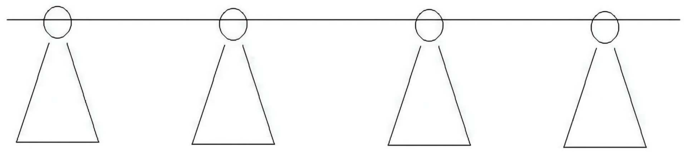

FIGURE 2. objects connected by one rope

It should be pointed out that the above directional edge escaping points set is very likely complicated and interesting. In order to be more intuitive, we limit the observation to the real axis and a simple linear function. Divide the real axis by partitioning it with integer points as endpoints and consider the directional edge escaping points set of the map $y = 3x$, but only in the interval $[0, 1]$. According to the concept of directional edge escaping points set, we can infer that the directional edge escaping points set of map $y = 3x$ in the interval $[0, 1]$ is a classic Cantor set without $\{0\}$, whose Hausdorff dimension is $\log_3 2$. See Figure 3.

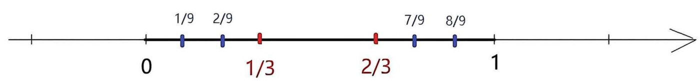

FIGURE 3. the directional edge escaping points set of $y = 3x$

In this paper, We will prove that the Hausdorff dimension of directional edge escaping points set of $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, is no more than 1 under one kind of complex plane partition. In order to state our conclusion, we turn to briefly introduce the concept of Hausdorff dimension (see [5]) and some notation.

For any set $U \subseteq \mathbb{C}$, denote the diameter of U by $|U| := \sup\{|z - w| : z, w \in U\}$. Let F be a set in $\mathbb{C}$, and s a positive number. Define s-dimensional measure $H^s(F)$ of F by

$$H^s(F) := \lim_{\delta \to 0} \inf \Big\{ \sum_{i=1}^{\infty} |U_i|^s : |U_i| < \delta, F \subseteq \bigcup_i U_i \Big\}$$

and define the Hausdorff dimension $\dim(F)$ of F by

$$\dim(F) := \inf \big\{ s \geq 0 : H^s(F) = 0 \big\} = \sup \big\{ s \geq 0 : H^s(F) = \infty \big\}.$$

For convenience, we might as well denote function $ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, by $S(z)$.

Define $I := \{z \in \mathbb{C} : \ S^n(z) \to \infty$, as $\ n \to \infty$ and $|\operatorname{Im} S^n(z)| \leq \lambda |S^n(z)|$ for all $n \in \mathbb{N}\}$. Denote by E_∞ the directional edge escaping points set, that is $E_\infty := \{z \in I : \ B_{n+1}(z) \cap \partial S(B_n(z)) \neq \emptyset$ for all $n \in \mathbb{N}\}$ and divide the complex plane as follows.

$$
\begin{aligned}
\mathbb{C} &:= \cup_{k=-\infty}^{\infty} P_k \\
&:= \cup_{k=-\infty}^{\infty} \{z \in \mathbb{C} : -\frac{\pi}{2} + k\pi \leq \operatorname{Im} z < \frac{\pi}{2} + k\pi\} \\
&:= \cup_{k=-\infty}^{\infty} \cup_{j=-\infty}^{\infty} B_{j,k} \\
&:= \cup_{k=-\infty}^{\infty} \cup_{j=-\infty}^{\infty} \{z \in P_k : j\pi \leq \operatorname{Re} z < (j+1)\pi\},
\end{aligned}
$$

see Figure 4.

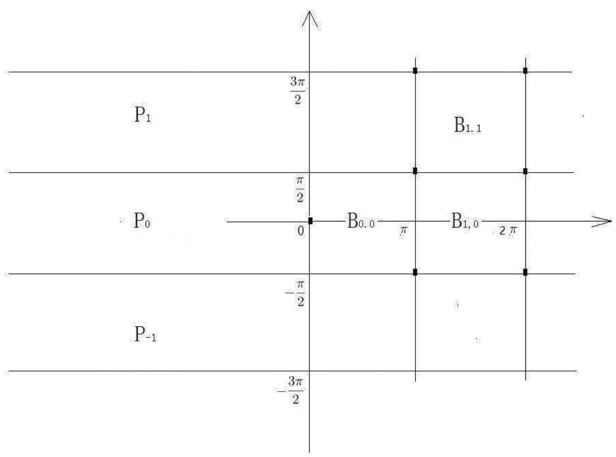

FIGURE 4. plane division

THEOREM 1.1. *If E_∞ is the directional edge escaping points of $S(z)$ under the aforementioned division of the plane, then* $\dim(E_\infty) \leq 1$.

2. Preliminaries

LEMMA 2.1. *Let F be a subset of $\mathbb{C}$, $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. Then* $\dim(F) = \dim(S(F))$.

PROOF. If $a = 0$ or $b = 0$, $S'(z) \neq 0$.
If $ab \neq 0$, let $S'(z) = ae^z - be^{-z} = 0$, then $z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}\operatorname{Arg}(\frac{b}{a})$. So

$S'(z) \neq 0$ on $\mathbb{C} \setminus \{z : z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}\operatorname{Arg}(\frac{b}{a})\}$, which means that $S(z)$ is locally univalent except for a countable set. By noting that ignoring a countable subset has no effect on the Hausdorff dimension of the original set, we get $\dim(F) = \dim(S(F))$. ∎

For $z \in I$, according to

$$|S^n(z)| \leq |a|\exp(\operatorname{Re} S^{n-1}(z)) + |b|\exp(-\operatorname{Re} S^{n-1}(z)),$$

we have $\{z \in I : |\operatorname{Re}(S^n(z))| \to \infty \text{ as } n \to \infty\}$. So we can limit our discussion to the points in

$$H^q := \{z \in \mathbb{C} : |\operatorname{Re} z| \geq q\},$$

where q is large enough. Otherwise, by Lemma 2.1, we consider the set $S^n(E_\infty \cap B_{j,k})$.

Lemma 2.2. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$, m, n be nonnegative integers, $q > 0$ be sufficiently large and $z \in H^q$. Then*

(a) $S^{(m)}(z) \neq 0$ *, where $S^{(m)}(z)$ is the m-order derivative, $S^{(0)}(z) = S(z)$;*

(b) the horizontal strip domain with width smaller than 2π and the real part no less than q (or no more than $-q$) is the univalent domain of $S^{(m)}(z)$;

(c) $e < \frac{2}{3}\min\{|a|, |b|\}e^{|\operatorname{Re} z|} < |S^{(m)}(z)| < \frac{3}{2}\max\{|a|, |b|\}e^{|\operatorname{Re} z|}$*;*

(d) $\frac{1}{2e^\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} < 2e^\pi$*, where $|\operatorname{Re} z_1 - \operatorname{Re} z_2| < \pi$ and $z_i \in H^q$, $i = 1, 2$.*

Proof. (a) If $S^{(m)}(z) = ae^z \pm be^{-z} = 0$, then $z = \frac{1}{2}\log|\frac{b}{a}| + \frac{i}{2}\operatorname{Arg}(\pm\frac{b}{a})$. Because $|\operatorname{Re} z| \geq q > |\frac{1}{2}\log|\frac{b}{a}||$, then $S^{(m)}(z) \neq 0$.

(b) Note that $S^{(m)}(z) = ae^z \pm be^{-z} = \sqrt{ab}(\sqrt{\frac{a}{b}}e^z \pm \sqrt{\frac{b}{a}}e^{-z})$. If $S^{(m)}(z_1) = S^{(m)}(z_2)$, then

$$\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2} \text{ or } |\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| = 1.$$

Since q is large enough such that $|\sqrt{\frac{a}{b}}e^{z_1} \cdot \sqrt{\frac{a}{b}}e^{z_2}| \neq 1$, we have $\sqrt{\frac{a}{b}}e^{z_1} = \sqrt{\frac{a}{b}}e^{z_2}$ and then $z_1 = z_2$ (width of strip $< 2\pi$).

(c) Suppose $\operatorname{Re} z \geq q > 0$, as q is large enough. Then

$$\begin{aligned}
|S^{(m)}(z)| &\geq ||a|e^{\operatorname{Re} z} - |b|e^{-\operatorname{Re} z}| > |a|e^{\operatorname{Re} z} - \frac{1}{3}|a|e^{\operatorname{Re} z} \\
&= \frac{2}{3}|a|e^{\operatorname{Re} z} \geq \frac{2}{3}\min\{|a|, |b|\}e^{|\operatorname{Re} z|} > e, \\
|S^{(m)}(z)| &\leq |a|e^{\operatorname{Re} z} + |b|e^{-\operatorname{Re} z} < |a|e^{\operatorname{Re} z} + \frac{1}{2}|a|e^{\operatorname{Re} z} \\
&= \frac{3}{2}|a|e^{|\operatorname{Re} z|} \leq \frac{3}{2}\max\{|a|, |b|\}e^{|\operatorname{Re} z|}.
\end{aligned}$$

The proof is very similar when $\operatorname{Re} z \leq -q < 0$.

(d) Without losing generality, suppose $\operatorname{Re} z \geq q > 0$. The claim can be proved similarly when $\operatorname{Re} z \leq -q < 0$.

$$\frac{||a|e^{\operatorname{Re} z_1} - |b|e^{-\operatorname{Re} z_1}|}{|a|e^{\operatorname{Re} z_2} + |b|e^{-\operatorname{Re} z_2}} \leq \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \leq \frac{|a|e^{\operatorname{Re} z_1} + |b|e^{-\operatorname{Re} z_1}}{||a|e^{\operatorname{Re} z_2} - |b|e^{-\operatorname{Re} z_2}|}.$$

If $q > 0$ is large enough and $|\operatorname{Re} z_1 - \operatorname{Re} z_2| < \pi$, then $\operatorname{Re} z_1$ and $\operatorname{Re} z_2$ are positive and large enough, so

$$\frac{1}{2}e^{-\pi} < \frac{|S^{(m)}(z_1)|}{|S^{(n)}(z_2)|} \approx e^{\operatorname{Re} z_1 - \operatorname{Re} z_2} < 2e^{\pi}.$$

□

According to Lemma 2.2, we can further observe $S(z)$. For any given small positive number θ, as long as $q > 0$ is large enough, we have that

$$\max\{|a|, |b|\}e^{-|\operatorname{Re} z|} < \theta. \tag{2.1}$$

Thus, $S(z) \approx ae^z$ or $S(z) \approx be^{-z}$ in H^q.

Take $B := B_{j,k}$ and $j > 0$ for example; $S(B)$ contains a half-annulus with inner radius $|a|e^{j\pi} + \theta$ and outer radius $|a|e^{(j+1)\pi} - \theta$. At the same time, $S(B)$ is contained in a half-annulus with inner radius $|a|e^{j\pi} - \theta$ and outer radius $|a|e^{(j+1)\pi} + \theta$. As the positive number θ is very small, $S(B)$ can be viewed as 'approximate-half-annulus'.

Let $R(S(B)) := \sup |S(B)|$, $r(S(B)) := \inf |S(B)|$, and

$$\widetilde{A}(r(S(B)), R(S(B))) := S(B) \cap H^q \cap \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \lambda |z|, \lambda \in (0,1)\},$$

which is a partial approximate-annulus.

Denote $\widetilde{A}(a_0 r(S(B)) + a_1, b_0 R(S(B)) + b_1)$ as the 'approximate-half-annulus' in H^q, which is enclosed by the image of inner and outer boundary of $S(B)$ under linear transformation $a_0 z + a_1$ and $b_0 z + b_1$, respectively, along radial direction, where a_0, a_1, b_0, b_1 are real numbers.

Since $\{z, S^1(z), S^2(z), \ldots\}$ stay in H^q, for every $n \geq 0$ there exists a unique square $B_n(z) \subseteq H^q$ such that

$$S^n(z) \in B_n(z).$$

If necessary, we can ask q to be sufficiently large that the above Lemma 2.2 holds when $|\operatorname{Re} z| > \frac{q}{2}$. It follows immediately from Lemma 2.2 (b) and (c) that there exists a unique holomorphic inverse branch $S_z^{-n} : B_n(z) \to H^{q-\pi} := \{z \in \mathbb{C} : |\operatorname{Re} z| \geq q - \pi\}$ of S^n sending $S^n(z)$ to z. Denote

$$K_n(z) = S_z^{-n}(B_n(z)).$$

In addition, denote $R(S(B_{n-1}(z)))$ and $r(S(B_{n-1}(z)))$, i.e. $R(S^n(K_{n-1}(z)))$ and $r(S^n(K_{n-1}(z)))$, respectively, by $R_n(z)$ and $r_n(z)$. See Figure 5 to become familiar with the above symbols.

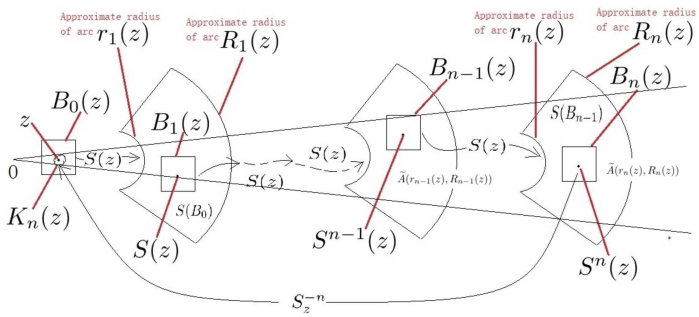

FIGURE 5. diagrammatic sketch of symbols

LEMMA 2.3. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. If q is large enough, then there exist constants K_1 and K_2 independent of n and z such that*

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \leq K_1$$

for all $x, y \in B_n(z)$, and

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} \leq K_2$$

for all $x, y \in K_{n-1}(z)$, i.e. $S^{n-1}(x), S^{n-1}(y) \in B_{n-1}(z)$.

PROOF. Denote by $\widetilde{B_i(z)} \supset B_i(z)$ the open square of side length 2π with sides parallel to $B_i(z)$ and center coincident with $B_i(z)$. By Lemma 2.2 (b), we know that $S(z)$ is univalent on $\widetilde{B_i(z)}$ and $S(\widetilde{B_i(z)})$ contains $\widetilde{B_{i+1}(z)}$ for $i = 0, 1, 2, \ldots$. See Figure 6.

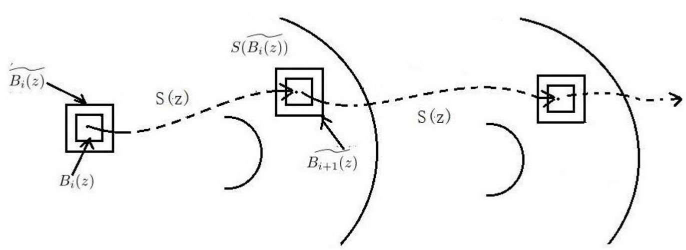

FIGURE 6. deviation property of $S(z)$

The module of $\widetilde{B_i(z)} \setminus B_i(z)$ is constant and, by distortion theorem, for all $x, y \in B_n(z)$

$$\frac{|(S_z^{-n})'(x)|}{|(S_z^{-n})'(y)|} \leq K_1.$$

By Lemma 2.2 (d)

$$\frac{|(S^n)'(x)|}{|(S^n)'(y)|} = \frac{|S'(S^{n-1}(x))|}{|S'(S^{n-1}(y))|} \cdot \frac{|(S^{n-1})'(x)|}{|(S^{n-1})'(y)|} \leq 2e^{\pi} K_1 = K_2.$$

If we let $K = \max\{K_1, K_2\}$, then K_1, K_2 can both be replaced with K. ∎

LEMMA 2.4. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. If $z \in I$ and q is large enough, then $|\operatorname{Re}(S^n(z))|$ tends to infinity uniformly.*

PROOF. For any given $z \in I$, we have

$$\begin{aligned} |\operatorname{Re}(S^n(z))| &= \sqrt{|S^n(z)|^2 - (|\operatorname{Im} S^n(z)|)^2} \\ &\geq \sqrt{|S^n(z)|^2 - (\lambda |S^n(z)|)^2} \\ &= (1-\lambda^2)^{\frac{1}{2}} |S^n(z)|. \end{aligned} \tag{2.2}$$

According to Lemma 2.2 (c), if q is large enough, we get

$$\begin{aligned} |S^{n+1}(z)| &= |ae^{S^n(z)} + be^{-S^n(z)}| \\ &\geq \frac{2}{3} \min\{|a|, |b|\} e^{|\operatorname{Re} S^n(z)|} \\ &\geq \frac{2}{3} \min\{|a|, |b|\} \exp((1-\lambda^2)^{\frac{1}{2}} |S^n(z)|) \\ &\geq \frac{2}{(1-\lambda^2)^{\frac{1}{2}}} |S^n(z)|. \end{aligned}$$

Hence,

$$\begin{aligned} |\operatorname{Re}(S^{n+1}(z))| &\geq (1-\lambda^2)^{\frac{1}{2}} |S^{n+1}(z)| \geq 2|S^n(z)| \\ &\geq 2|\operatorname{Re}(S^n(z))| \geq \cdots \geq 2^{n+1} q. \end{aligned}$$

∎

LEMMA 2.5. *Let $S(z) = ae^z + be^{-z}$, where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 \neq 0$. For any given $\alpha > 0$ and $T > 0$, there exist $K_3 > 0$ and $n_0 \geq 0$ such that for every $n \geq n_0$,*

$$|(S^{n+1})'(z)| \geq K_3 |(S^n)'(z)|^{\alpha}$$

for all $z \in I \cap B(0, T)$.

PROOF. By Lemma 2.4, for any given $\alpha > 0$, there is $n_0 \geq 0$ such that

$$\frac{1}{2e^{\pi}} \frac{2}{3} \min\{|a|, |b|\} e^{(1-\lambda^2)^{\frac{1}{2}} |S^{n+1}(z)|} \geq (2e^{\pi})^{\alpha} |S^{n+1}(z)|^{\alpha}$$

for all $z \in I$ when $n \geq n_0$.

We claim that

$$\inf_{z\in\overline{I\cap B(0,T)}} \frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha} \neq 0.$$

If there exist no $j \in \{0, 1, \ldots, n_0\}$ and $z_0 \in \overline{I \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$, $\frac{|(S^{n_0+1})'(z)|}{|(S^{n_0})'(z)|^\alpha}$ is a positive continuous function on bounded closed sets, the claim holds. Suppose that there exist $j \in \{0, 1, \ldots, n_0\}$ and $z_0 \in \overline{I \cap B(0,T)}$ such that $S'(S^j(z_0)) = 0$. Then there exists $\{z_n\} \subseteq I \cap B(0,T)$ such that $z_n \to z_0$ or $z_n \equiv z_0$. By Lemma 2.2 (d)

$$|S'(S^j(z_0))| \leftarrow |S'(S^j(z_n))| \geq \frac{1}{2e^\pi}|S^{j+1}(z_n)| \geq \frac{1}{2e^\pi}q,$$

which contradicts $S'(S^j(z_0)) = 0$.

Let K_3 be the infimum of the function $z \mapsto |(S^{n_0+1})'(z)||(S^{n_0})'(z)|^{-\alpha}$ in $I \cap B(0,T)$. Then K_3 is a positive number. Proof by induction. According to the definition of K_3, the lemma holds when $n = n_0$. Suppose it is true for $n \geq n_0$, so

$$\begin{aligned} |(S^{n+2})'(z)| &= |(S'(S^{n+1}(z))| \cdot |(S^{n+1})'(z)| \\ &\geq K_3|(S'(S^{n+1}(z))| \cdot |(S^n)'(z)|^\alpha. \end{aligned}$$

By Lemma 2.2(d), Lemma 2.2(c) and (2.2)

$$\begin{aligned} |(S'(S^{n+1}(z))| &\geq \frac{1}{2e^\pi}|S^{n+2}(z)| \geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{|\operatorname{Re} S^{n+1}(z)|} \\ &\geq \frac{1}{2e^\pi}\frac{2}{3}\min\{|a|,|b|\}e^{(1-\lambda^2)^{\frac{1}{2}}|S^{n+1}(z)|} \geq (2e^\pi)^\alpha|S^{n+1}(z)|^\alpha \\ &\geq (2e^\pi)^\alpha \cdot (\frac{1}{2e^\pi})^\alpha|S'(S^n(z))|^\alpha = |S'(S^n(z))|^\alpha. \end{aligned}$$

Therefore

$$|(S^{n+2})'(z)| \geq K_3|S'(S^n(z))|^\alpha \cdot |(S^n)'(z)|^\alpha = K_3|(S^{n+1})'(z)|^\alpha.$$

□

3. The proof of the theorem

Based on the above preliminaries, we can begin proving the main result of this paper.

Proof. Let $E_n := \cup_{z\in I} S_z^{-n}(\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z)))$.

Then E_∞ can be covered by the set $\cup_{n\geq k} E_n$ for every $k \geq 0$ and the approximate-half-annuli $\widetilde{A}(r_n(z), r_n(z)+2\pi) \cup \widetilde{A}(R_n(z)-2\pi, R_n(z))$ can be covered by $M_1 r_n(z)$ squares with diameters less than 1, where M_1 is a constant. Therefore, according to Lemma 2.3, $K_{n-1}(z) \cap E_n$ can be covered with no more than $M_1 r_n(z)$ sets $J_{i,n}(z)$ of diameters less than $K|(S^n)'(z)|^{-1}$.

Let $T \geq 2q$. Note that any two sets $K_{n-1}(z)$ and $K_{n-1}(z')$ are either disjoint or equal, so we can find a set $Z_n \subset I$ such that $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint for $z, z' \in Z_n, z \neq z'$ and

$$E_n \cap B(0,T) \subset \cup_{z\in Z_n} K_{n-1}(z) \subset B(0,2T).$$

For the given $\epsilon > 0$, let n be large enough such that Lemma 2.5 is satisfied for $\alpha = 2/\epsilon$ and $2T$. Using Lemma 2.2 (d), Lemma 2.5 and (2.1), we get

$$\begin{aligned}
\sum_{z\in Z_n}\sum_{J_{i,n}}(\operatorname{diam} J_{i,n}(z))^{1+\epsilon} &\leq \sum_{z\in Z_n} M_1K^{1+\epsilon} r_n(z)|(S^n)'(z)|^{-(1+\epsilon)}\\
&\leq 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||(S^n)'(z)|^{-(1+\epsilon)}\\
&\leq 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))||S'(S^{n-1}(z))|^{-(1+\epsilon)}|(S^{n-1})'(z)|^{-(1+\epsilon)}\\
&\leq 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|S'(S^{n-1}(z))|^{-\epsilon}|(S^{n-1})'(z)|^{-\epsilon}|(S^{n-1})'(z)|^{-1}\\
&\leq 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}|(S^{n})'(z)|^{-\epsilon}|(S^{n-1})'(z))|^{-1}\\
&\leq 2e^{\pi}M_1K^{1+\epsilon}\sum_{z\in Z_n}K_3^{-\epsilon}|(S^{n-1})'(z)|^{-2}|(S^{n-1})'(z))|^{-1}\\
&\leq 2e^{\pi}K_3^{-\epsilon}M_1K^{1+\epsilon}e^{-(n-1)}\sum_{z\in Z_n}|(S^{n-1})'(z)|^{-2}.
\end{aligned}$$

Because $K_{n-1}(z)$ and $K_{n-1}(z')$ are disjoint and the Lebesgue measure of each set of the form $K_{n-1}(z)$ is proportional to $|(S^{n-1})'(z)|^{-2}$ by Lemma 2.3, we get that there exists a constant $M_2 > 0$ such that the last term in the above inequality is no more than $M_2e^{-(n-1)} \cdot \operatorname{area}(B(0,2T))$.

Hence,

$$\begin{aligned}
\sum_{n=k}^{\infty}\sum_{z\in Z_n}\sum_{J_{i,n}}(\operatorname{diam} J_{i,n}(z))^{1+\epsilon} &\leq M_2\cdot \operatorname{area}(B(0,2T))\sum_{n=k}^{\infty}e^{-(n-1)}\\
&= 4\pi T^2M_2\frac{e^{-k+2}}{e-1}.
\end{aligned}$$

If we let $k \to \infty$, then $4\pi T^2 M_2\frac{e^{-k+2}}{e-1} \to 0$. That is, for any given $\epsilon > 0$, the $(1+\epsilon)$-dimensional Hausdorff measure of $E_\infty \cap B(0,T)$ is equal to zero. Hence,

$$\dim(E_\infty) \leq 1.$$

$\square$

Question: Does the same result hold for more general analytic functions?

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REFERENCES

[1] M. Bailesteanu, H. V. Balan and D. Schleicher, *Hausdorff dimension of exponential parameter rays and their endpoints*, Nonlinearity **21** (2008), 113–120.
[2] C. J. Bishop, *A transcendental Julia set of dimension 1,* Invent. Math. **212** (2018), 407–460.
[3] R. L. Devaney and M. Krych, *Dynamics of* e^z*,* Ergodic Theory Dynam. Systems **4** (1984), 35–52.
[4] A. E. Eremenko, *On the iteration of entire functions*, in Dynamical systems and ergodic theory, PWN, Warsaw, 1989, 339–345.
[5] K. Falconer, Fractal geometry. Mathematical foundations and applications, Wiley Publishing, Chichester, 1990.
[6] M. Förster and D. Schleiher, *Parameter rays in the space of exponential maps*, Ergodic Theory Dynam. Systems **29** (2009), 515–544.
[7] X. J. Huang and W. Y. Qiu, *The dimension paradox in parameter space of cosine family*, Chinese Ann. Math. Ser. B **41** (2020), 645–656.
[8] B. Karpińska, *Hausdorff dimension of the hairs without endpoints for* $\lambda \exp(z)$, C. R. Acad. Sci. Paris Sér. I Math **328** (1999), 1039–1044.
[9] B. Karpińska, *Area and Hausdorff dimension of the set of accessible points of the Julia sets of* λe^z *and* $\lambda \sin(z)$, Fund. Math. **159** (1999), 269–287.
[10] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), 329–342.
[11] F. C. Moon, Chaotic and fractal dynamics, Wiley Publishing, New York, 1992.
[12] W. Y. Qiu, *Hausdorff Dimension of the M-Set of* $\lambda \exp(z)$, Acta Math. Sin. (N.S.) **10** (1994), 362–368.
[13] G. Rottenfusser and D. Schleicher, *Escaping points of the cosine family*, in Transcendental dynamics and complex analysis, Cambridge Univ. Press, Cambridge, 2008, 396–424.
[14] L. Rempe, D. Schleicher and M. Förster, *Classification of escaping exponential maps*, Proc. Amer. Math. Soc. **136** (2008), 651–663.
[15] D. Schleicher, *The dynamical fine structure of iterated cosine maps and a dimension paradox*, Duke Math. J. **136** (2007), 343–356.
[16] D. Schleicher and J. Zimmer, *Escaping points of exponential maps*, J. London Math. Soc. (2) **67** (2003), 380–400.
[17] G. M. Stallard, *The Hausdorff dimension of Julia sets of entire functions. III*, Math. Proc. Cambridge Philos. Soc. **122** (1997), 223–244.
[18] G. M. Stallard, *The Hausdorff dimension of Julia sets of entire functions. IV*, J. London Math. Soc. (2) **61** (2000), 471–488.
[19] T. Tian, *Parameter rays for the cosine family*, J. Fudan Univ. Nat. Sci. **50** (2011), 10–22.

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