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ON A GENERALIZATION OF SOME INSTABILITY RESULTS FOR RICCATI EQUATIONS VIA NONASSOCIATIVE ALGEBRAS

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Abstract. In [28], for any real non associative algebra of dimension $m \geq 2$, having k linearly independent nilpotent elements $n_1, n_2, \ldots, n_k$, $1 \leq k \leq m-1$, Mencinger and Zalar defined near idempotents and near nilpotents associated to $n_1, n_2, \ldots, n_k$. Assuming $\mathcal{N}_k\mathcal{N}_k = \{0\}$, where $\mathcal{N}_k = \operatorname{span}\{n_1, n_2, \ldots, n_k\}$, they showed that if there exists a near idempotent or a near nilpotent, called u, associated to $n_1, n_2, \ldots, n_k$ verifying $n_i u \in \mathbb{R}n_i$, for $1 \leq i \leq k$, then any nilpotent element in $\mathcal{N}_k$ is unstable. They also raised the question of extending their results to cases where $\mathcal{N}_k\mathcal{N}_k \neq \{0\}$ with $\mathcal{N}_k\mathcal{N}_k \subset \mathcal{N}_k$ and to cases where $\mathcal{N}_k\mathcal{N}_k \not\subset \mathcal{N}_k$.

In this paper, positive answers are emphasized and in some cases under the weaker conditions $n_i u \in \mathcal{N}_k$. In addition, we characterize all such algebras in dimension 3.

1. Introduction

An autonomous homogeneous polynomial systems of ODEs of degree k is defined by

$$x' = \frac{dx}{dt} = H(x), \tag{1.1}$$

where the vector function $x : I \subset \mathbb{R} \to \mathbb{R}^m$ is defined on some open interval I and $H : \mathbb{R}^m \to \mathbb{R}^m$ is a homogeneous form of degree k

$$H(\alpha x) = \alpha^k H(x) \quad \forall \alpha \in \mathbb{R}, \quad \forall x \in \mathbb{R}^m. \tag{1.2}$$

If H is homogeneous of degree two, system (1.1) is called a *homogeneous quadratic system*. In this case, it is common to write $x' = Q(x)$.

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Lawrence Markus in [17] was the first one who associated system (1.1) for $k = 2$ with nonassociative algebra $\mathcal{A} = (\mathbb{R}^m, \cdot)$, where the algebra multiplication $\cdot$ is defined by

$$x \cdot y = \frac{1}{2} \left(Q\left(x+y\right) - Q\left(x\right) - Q\left(y\right)\right). \tag{1.3}$$

Conversely, given a nonassociative algebra of dimension m, we associate the homogeneous quadratic differential equation in $\mathcal{A}$:

$$x' = x \cdot x.$$

Therefore, there is a one-to-one correspondence between real non-associative algebras in dimension n and homogeneous quadratic dynamical systems in $\mathbb{R}^m$.

After his classification of planar systems, many other authors considered various classifications, mostly limited to $k = 2$, see [4, 5, 6, 7, 8, 18, 22]. For systems (1.1), the question of (in)stability of singular point(s) is nontrivial, see [3, 19, 23, 24, 25, 28], since the origin is a nonelementary nonhyperbolic singular point in any dimension and for any degree of homogeneity. There are also some papers about (algebraic) structure(s) and dynamics in systems (1.1), e.g. [1, 4, 5, 8] and applications beyond ODEs (see e.g. [12, 13, 14, 15, 26, 16, 21]). Finally, let us mention some review papers [10, 11, 12, 20] and a monograph [27]. The list of the references below is far from exhaustive, but the references in [27] are quite exhaustive until year 1991.

Concerning the solutions of a quadratic system (1.1) and special algebraic elements in the corresponding algebra $\mathcal{A}$, it is well-known that any nonzero *idempotent* (for which $p \cdot p = p$ holds) in $\mathcal{A}$ implies the existence of a ray solution which yields instability of the origin. The ray solutions on the line $\mathbb{R}p$ are examples of so called *blow-up solutions* (e.g. [10, 11, 20, 27]). On the other hand, nilpotent elements, defined as nonzero elements n verifying $n^2 = n \cdot n = 0$, lead to a line of nilpotents $\mathbb{R}n$ and to a line of stationary points for the associated dynamical system. As a consequence, a nilpotent element is never asymptotically stable.

We recall the definition of critical point stability in the sense of Lyapunov.

Definition 1.1. *Consider a dynamical system in $\mathbb{R}^m$ and $M \in \mathbb{R}^m$ a stationary point. M is a stable critical point if for any neighbourhood $V_M \subset \mathbb{R}^m$ of M, there exist a neighbourhood $W_M \subset V_M$ of M such that, for any point $P_0 \in W_M$, the trajectory $P_0(t)$ via P_0 remains in V_M for $t > 0$ as long as solution is defined.*

In [28], the authors studied the stability of non-zero singular points of a quadratic system (1.1) using the so called $\lambda-$space of a nonzero element u defined by

$$\mathcal{A}_\lambda\left(u\right) = \left\{x \in \mathcal{A};\, u \cdot x = \lambda x\right\}. \tag{1.4}$$

If $\mathcal{A}_\lambda(u) \neq \{0\}$, it is called an eigenspace of u. Obviously, all such λ are eigenvalues of the linear map $x \longmapsto u \cdot x$ and the maximal number of $\lambda-$eigenspaces is smaller than $\dim(\mathcal{A})$. In [3], the authors proved that the nilpotent line $\mathbb{R}n$ consists of unstable singular points, if n is included in $\mathcal{A}_\lambda(p)$ for some idempotent $p \in \mathcal{A}$ and that this result remains true even when p is not necessarily an idempotent but an element satisfying a weaker algebraic condition. In [28], the authors defined a more general algebraic framework in which the first main result of [3] was reinterpreted in sense of the following two definitions.

Definition 1.2 ([28]). *Let $\mathcal{A}$ be a commutative real algebra of dimension m and $\mathcal{N}_k \subset \mathcal{A}$ the subspace spanned by a set of linearly independent nilpotent elements $n_1, \dots, n_k$ in $\mathcal{A}$, $1 \leq k \leq m-1$. An element $u \in \mathcal{A} \setminus \mathcal{N}_k$ will be called a near-nilpotent associated to $\mathcal{N}_k$ if*

$$u^2 = \sum_{i=1}^{k} \lambda_i n_i, \tag{1.5}$$

where all $\lambda_i \in \mathbb{R}$ are nonzero.

Definition 1.3 ([28]). *An element $u \in \mathcal{A} \setminus \mathcal{N}_k$ will be called a near-idempotent associated to $\mathcal{N}_k$ if*

$$u^2 - u = \sum_{i=1}^{k} \lambda_i n_i, \tag{1.6}$$

where all $\lambda_i \in \mathbb{R}$ are nonzero. The largest possible number k from equation (1.5) or (1.6) will be called the rank of u. Note that the above definitions imply that u^2 is always a nonzero element.

The authors noted that near-idempotents and near-nilpotents exist even in algebras which do not contain idempotents and proved that under suitable conditions (see [28, Th.1]) the existence of such special algebraic elements affects the (in)stability of (all) singular points and implies that the origin of the corresponding system of ODEs cannot be stable.

To make the paper self-contained, we summarize the following three results from [28].

Proposition 1.4 ([28]). *Let $\mathcal{A}$ be a real nonassociative algebra of finite dimension m and $u \in \mathcal{A}$ either a near-idempotent or near-nilpotent of rank 1, associated to the subspace $\mathcal{N}_1 = \mathbb{R}n_1$, where n_1 is a nonzero nilpotent. If $\mathcal{N}_1$ is included in one of the eigenspaces of u, then every $n \in \mathcal{N}_1$ is an unstable singular point of the Riccati equation $x' = x^2$ associated with $\mathcal{A}$.*

Corollary 1.5 ([28]). *The additional assumption, about $\mathcal{N}_1$ being an eigenspace of u, cannot be removed from Proposition 1.4.*

Theorem 1.6 ([28]). *Let $\mathcal{A}$ be a real nonassociative algebra of finite dimension $m \geq 2$, $n_1, \dots, n_k$ k nonzero linearly independent nilpotents of rank* *two, for* $1 \leq k \leq m-1$, $\mathcal{N}_k = \operatorname{span}\{n_1, \ldots, n_k\}$, $u \in \mathcal{A} \backslash \mathcal{N}_k$ *be a near-nilpotent or a near-idempotent associated with* $\mathcal{N}_k$. *If* $n_i \cdot n_j = 0$ *for all* $1 \leq i, j \leq k$ *and* $n_i \in A_{\lambda_i}(u)$, *for all* $1 \leq i \leq k$ *and some scalars* $\lambda_1, \ldots, \lambda_k$, *then any* $n \in \mathcal{N}_k$ *is a unstable singular point of the Riccati equation* $x' = x^2$ *in* $\mathcal{A}$.

The authors in [28] also raised the following question: Is it possible to generalize the instability results from the case where $\mathcal{N}_k\mathcal{N}_k = \{0\}$, which makes $\mathcal{N}_k$ trivial as subalgebra, to the case where $\mathcal{N}_k$ is no longer trivial and, even more, to the case when $\mathcal{N}_k$ is not a subalgebra, that is $\mathcal{N}_k\mathcal{N}_k \not\subset \mathcal{N}_k$.

In the present paper, we continue to eliminate some classes of algebras corresponding to systems (1.1) with unstable origin. We consider the stability of singular points in the classical sense of Lyapunov. As already mentioned, the origin $x = 0 \in \mathbb{R}^m$ is a totally degenerated nonhyperbolic singular point of (1.1) for every $m \in \mathbb{N}$, $m \geq 2$ and non-zero singular points of a homogeneous system (1.1) clearly correspond to *nilpotents* of rank two defined by $n \cdot n = 0$.

In the following section, when there exists a near nilpotent u, we state an instability theorem for $0 \in \mathcal{A}$ and also for any nilpotent belonging to $\mathbb{R}n_1, \ldots,$ or $\mathbb{R}n_k$ under the weaker condition $n_i u \in \mathcal{N}_k$, $1 \leq i \leq k$, which generalizes $n_i u \in \mathbb{R}n_i$, $1 \leq i \leq k$, by assuming a restriction in the case $\mathcal{N}_k\mathcal{N}_k \subset \mathcal{N}_k$, $\mathcal{N}_k\mathcal{N}_k \neq \{0\}$.

For the near idempotent case, the instability remains for $0 \in \mathcal{A}$ and for any nilpotent element belonging to $\mathbb{R}n_1, \ldots,$ or $\mathbb{R}n_k$ under the weaker condition $\mathcal{N}_k\mathcal{N}_k \subset \mathcal{N}_k$ but with $n_i u \in \mathbb{R}n_i$, for $1 \leq i \leq k$.

When $\mathcal{N}_k$ is no longer a subalgebra and in case there is a near nilpotent with $k = 2$, we prove the instability of $0 \in \mathcal{A}$ and of any nilpotent belonging to $\mathbb{R}n_1, \ldots,$ or $\mathbb{R}n_k$ under the conditions $n_i u \in \mathbb{R}n_i$, for $1 \leq i \leq k$ and justify that some additional restrictions are needed to extend the result to the cases $3 \leq k \leq m - 1$. Also, some remarks are underlined for the near idempotent case.

For each theorem, we give applications by characterizing, in dimension three, all corresponding algebras and we justify that they represent totally new families of algebras not treated in [28].

Finally, when the n_i's are no longer necessarily nilpotents, we obtain an extension of classical instability result stated by Sagle and Kinyon.

2. Main Results

Before starting and proving our main theorems, we recall an efficient tool for proving the instability of some given stationary point. Let $\mathcal{A}$ be a nonassociative algebra, $\mathcal{B} \subset \mathcal{A}$ a subalgebra and $n \in \mathcal{B} \setminus \{0\}$ a nilpotent element. Obviously, considered in $\mathcal{A}$, n is also a nilpotent element. Since $\mathcal{B}$ is a subalgebra, it makes sense to study the quadratic dynamical system associated to $\mathcal{A}$, but this time restricted to the subspace $\mathcal{B}$. If $n \in \mathcal{B}$ is unstable for the restricted system of ODE, it is also unstable for the original system of ODE in $\mathcal{A}$. This remark justifies that we will often deal with restrictions.

Let $\mathcal{A}$ be a real nonassociative algebra of dimension $m = \dim \mathcal{A} \geq 2$, k an integer with $1 \leq k \leq m-1$ and $n_1, \ldots, n_k$ k linearly independent nilpotents. We denote $\mathcal{N}_k = \operatorname{span}\{n_1, \ldots, n_k\}$. An element $u \in \mathcal{A} \setminus \mathcal{N}_k$ verifying $u^2 = \delta u + \sum_{i=1}^k \gamma_i n_i$, where the γ_i's are all nonzero, is a near-nilpotent if and only if $\delta = 0$ and u is a near-idempotent if and only if $\delta = 1$.

2.1. *The case $\mathcal{N}_k\mathcal{N}_k \subset \mathcal{N}_k$ and $\mathcal{N}_k\mathcal{N}_k \neq \{0\}$.*

2.1.1. *The case of a near nilpotent.* We start with the case $\delta = 0$ and for $i, j \in \{1, \ldots, k\}$, $i \neq j$, we let $\mathcal{N}_{ij}^k = \operatorname{span}\{n_i, n_j\}$.

Theorem 2.1. *Let $\mathcal{A}$ be a real nonassociative algebra of dimension $m \geq 3$, $n_1, \ldots, n_k$ k linearly independent nilpotents, $2 \leq k \leq m-1$, $\mathcal{N}_k = \operatorname{span}\{n_1, \ldots, n_k\}$ and $u \in \mathcal{A} \setminus \mathcal{N}_k$ a near-nilpotent associated to $\mathcal{N}_k$. If $\mathcal{N}_k\mathcal{N}_k \subset \mathcal{N}_k$, $n_i \cdot u$ belongs to $\mathcal{N}_k$ for $i = 1, \ldots, k$ and there exist $i_0, j_0 \in \{1, \ldots, k\}$, $i_0 \neq j_0$ with $\mathcal{N}_{i_0j_0}^k\mathcal{N}_{i_0j_0}^k \subset \mathcal{N}_{i_0j_0}^k$ and $\mathcal{N}_{i_0j_0}^k$ not trivial, then $0 \in \mathcal{A}$ and any nilpotent in $\mathbb{R}n_{i_0}$ or $\mathbb{R}n_{j_0}$ are unstable.*

Proof. According to assumptions, we have the following tables:

$$n_i \cdot n_i = 0, \quad \text{for } 1 \leq i \leq k,$$
$$n_i \cdot u = \sum_{\ell=1}^{k} \lambda_i^\ell n_i \quad \text{for } 1 \leq i \leq k \text{ with } \lambda_i^\ell \text{ scalars},$$
$$u \cdot u = \sum_{i=1}^{k} \gamma_i n_i, \quad \gamma_i \neq 0 \text{ for } 1 \leq i \leq k,$$
$$n_i \cdot n_j = \sum_{\ell=1}^{k} A_{ij}^\ell n_\ell \quad \text{for } i, j \in \{1, \ldots, k\} \text{ with } i \neq j, A_{ij}^\ell = A_{ji}^\ell \text{ and } A_{i_0j_0}^m = 0$$
$$\text{if } m \notin \{i_0, j_0\}.$$

Let $\mathcal{P}_k = \operatorname{span}\{n_1, \ldots, n_k, u\}$. In the basis $\{n_1, \ldots, n_k, u\}$ of $\mathcal{P}_k$, let $(x_1, \ldots, x_k, z)$ denote the coordinates. Obviously, $\mathcal{P}_k$ is a subalgebra and, as noted before, it makes sense to consider the Riccati equation restricted to $\mathcal{P}_k$. More precisely, the corresponding ODE's restricted to the subspace $\mathcal{P}_k$ become

$$\dot{x}_1 = x_1 \left(2 \sum_{i \neq 1} A_{1i}^1 x_i \right) + 2 \sum_{i=1}^{k} \lambda_i^1 x_i z + \gamma_1 z^2,$$
$$\dot{x}_2 = x_2 \left(2 \sum_{i \neq 2} A_{2i}^2 x_i \right) + 2 \sum_{i=1}^{k} \lambda_i^2 x_i z + \gamma_2 z^2,$$
$$\vdots$$

$$\dot{x}_k = x_k\left(2\sum_{i\neq k} A_{ki}^k x_i\right) + 2\sum_{i=1}^{k} \lambda_i^k x_i z + \gamma_k z^2,$$
$$\dot{z} = 0.$$

Since $\mathcal{N}_{i_0 j_0}^k$ is also a subalgebra, the system restricted to $\mathcal{N}_{i_0 j_0}^k$ is

$$\dot{x}_{i_0} = x_{i_0}\left(2A_{i_0 j_0}^{i_0} x_{j_0}\right),$$
$$\dot{x}_{j_0} = x_{j_0}\left(2A_{i_0 j_0}^{j_0} x_{i_0}\right),$$

with $(A_{i_0 j_0}^{i_0}, A_{i_0 j_0}^{j_0}) \neq (0,0)$. We deduce: $A_{i_0 j_0}^{j_0}\dot{x}_{i_0} - A_{i_0 j_0}^{i_0}\dot{x}_{j_0} = 0$ which leads to

$$A_{i_0 j_0}^{j_0} x_{i_0}(t) - A_{i_0 j_0}^{i_0} x_{j_0}(t) = K,$$

for some constant K depending on the initial condition.

Therefore, trajectories in $\mathcal{N}_{i_0 j_0}^k$ are either segments or half-lines according to the initial condition. In any given neighbourhood of any critical point, trajectories will enter this neighbourhood from one side and leave it from the other side. Thus, any nilpotent in $\mathbb{R}n_{i_0}$ or $\mathbb{R}n_{j_0}$ is unstable and $0 \in \mathcal{N}_{i_0 j_0}^k$, too. ∎

Remark 2.2. This theorem remains true for any scalars γ_i, $1 \leq i \leq k$.

Applications in dimension 3.

Consider the general algebra $\mathcal{A}$ having two linearly independent nilpotent elements n_1, and n_2 with the conditions $\mathcal{N}_2\mathcal{N}_2 \subset \mathcal{N}_2$, where $\mathcal{N}_2 = \operatorname{span}\{n_1, n_2\}$.

Given a basis $\{n_1, n_2, e_3\}$, we have the general tables:

$$\begin{aligned} n_1^2 &= n_2^2 = 0, \\ n_1 n_2 &= A_{12}^1 n_1 + A_{12}^2 n_2, \\ n_1 e_3 &= \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 e_3, \\ n_2 e_3 &= \beta_1 n_1 + \beta_2 n_2 + \beta_3 e_3, \\ e_3^2 &= \mu_1 n_1 + \mu_2 n_2 + \mu_3 e_3. \end{aligned} \tag{2.1}$$

Such an algebra admits a near-nilpotent verifying hypothesis of Theorem 2.1 if and only if $\alpha_3 = \beta_3 = \mu_3 = 0$. Certainly, if we let $u = an_1 + bn_2 + ce_3 \notin \mathcal{N}_2$, which means $c \neq 0$, it is not difficult to obtain that the two products $n_1 \cdot u$ and $n_2 \cdot u$ belong to $\mathcal{N}_2$ if and only if $\alpha_3 = \beta_3 = 0$ and that u^2 belongs to $\mathcal{N}_2$ if and only if $\mu_3 = 0$. Thus, we have $u^2 = \gamma_1 n_1 + \gamma_2 n_2$, for some convenient scalars γ_1 and γ_2.

If $\gamma_1\gamma_2 = 0$, u is not a near-nilpotent. However, due to the previous remark, the conclusion of the theorem remains true in this case.

Therefore, we characterized all homogeneous quadratic systems in $\mathbb{R}^3$ having two distinct lines of critical points crossing the origin and verifying

Theorem 2.1 assumptions:

$$\begin{aligned}
\dot{x}_1 &= 2A_{12}^1 x_1 x_2 + 2\alpha_1 x_1 x_3 + 2\beta_1 x_2 x_3 + \mu_1 x_3^2,\\
\dot{x}_2 &= 2A_{12}^2 x_1 x_2 + 2\alpha_2 x_1 x_3 + 2\beta_2 x_2 x_3 + \mu_2 x_3^2,\\
\dot{x}_3 &= 0,
\end{aligned}$$

with no condition on scalars, except $\left(A_{12}^1, A_{12}^2\right) \neq (0,0)$. We notice that the condition $\left(A_{12}^1, A_{12}^2\right) = (0,0)$ corresponds to the case studied in [28] for which instability of $(0,0,0)$ and any nilpotent in $\mathcal{N}_2$ holds.

In the following subsection, we study the near idempotent case.

2.1.2. *The case of a near idempotent.* Before stating our second result, we recall a well known result that will be used in the sequel (see [9, Theorem on p. 171]).

Theorem 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{C}^1$, $\dot{x} = f(x)$ be the associated autonomous dynamical system defined on $\mathbb{R}^n$ and let $(0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ be an initial condition. If $x(t)$ is the maximal solution with $x(0) = x_0$, then either $x(t)$ is defined for $t \in [0, \infty)$ or the trajectory blows up.*

When we consider the case of a near-idempotent ($\delta = 1$), the conditions can be made weaker.

Theorem 2.4. *Let $\mathcal{A}$ be a real nonassociative algebra of dimension $m \geq 3$, $n_1, \ldots, n_k$ k linearly independent nilpotents, $1 \leq k \leq m-1$, $\mathcal{N}_k = \operatorname{span}\{n_1, \ldots, n_k\}$ and $u \in \mathcal{A} \setminus \mathcal{N}_k$ a near-idempotent associated to $\mathcal{N}_k$. If $n_i \cdot u \in \mathcal{N}_k$ for $i = 1, \ldots, k$ and $\mathcal{N}_k \mathcal{N}_k \subset \mathcal{N}_k$ then $0 \in \mathcal{A}$ and all nilpotents in $\mathbb{R}n_1$, or in $\mathbb{R}n_2$ $\ldots$ or in $\mathbb{R}n_k$ are unstable.*

Proof. According to hypothesis, we have the following tables:

$$\begin{aligned}
n_i^2 &= 0 \quad \text{for } 1 \leq i \leq k,\\
n_i \cdot n_j &= \sum_{\ell=1}^{k} A_{ij}^{\ell} n_\ell,\\
n_i \cdot u &= \sum_{\ell=1}^{k} \lambda_i^{\ell} n_\ell, \quad \lambda_i^{\ell} \text{ scalars for } i, \ell \in \{1, \ldots, k\},\\
u^2 &= u + \sum_{i=1}^{k} \gamma_i n_i, \quad \gamma_i \neq 0 \text{ for } 1 \leq i \leq k,
\end{aligned}$$

with $A_{ii}^{\ell} = 0$ for $i, \ell \in \{1, \ldots, k\}$, and $A_{ij}^{\ell} = A_{ji}^{\ell}$ for $i, j \in \{1, \ldots, k\}$.

As $\mathcal{P}_k = \operatorname{span}\{n_1, \ldots, n_k, u\}$ is a subalgebra, if we choose the basis $\{n_1, \ldots, n_k, u\}$ and if the corresponding coordinates are denoted by $(x_1, \ldots,$

$x_k, z)$, the corresponding ODEs restricted to $\mathcal{P}_k$ become:

$$\begin{aligned}
\dot{x}_1 &= 2\sum_{m,p=1}^{k} A^1_{mp} x_m x_p + 2\sum_{i=1}^{k} \lambda^1_i x_i z + \gamma_1 z^2,\\
\dot{x}_2 &= 2\sum_{m,p=1}^{k} A^2_{mp} x_m x_p + 2\sum_{i=1}^{k} \lambda^2_i x_i z + \gamma_2 z^2,\\
&\vdots\\
\dot{x}_k &= 2\sum_{m,p=1}^{k} A^k_{mp} x_m x_p + 2\sum_{i=1}^{k} \lambda^k_i x_i z + \gamma_k z^2,\\
\dot{z} &= z^2.
\end{aligned}$$

In any neighborhood of $0 \in \mathcal{P}_k$, we consider the point $M_0 = (0, \ldots, 0, \varepsilon_0)$, with $\varepsilon_0 > 0$, small enough and the initial condition $(t_0, M_0) = (0, M_0)$.

Suppose the trajectory $\varphi_{M_0}(\cdot)$ via M_0 is bounded. Then, according to the result above, $\varphi_{M_0}(t)$ is defined for $t \in [0, \infty)$ which contradicts the equation $\dot{z} = z^2$ as $z(t) = \frac{\varepsilon_0}{1-\varepsilon_0 t}$. Thus, the trajectory via M_0 blows up and $0 \in \mathcal{P}_k$ is unstable.

Using the same idea, we prove that any nilpotent in $\mathbb{R}n_1$ or $\mathbb{R}n_2$ ... or in $\mathbb{R}n_k$ is unstable by selecting a similar initial condition in any given neighbourhood of the nilpotent. ◻

Remark 2.5. Here again we notice that Theorem 2.4 remains true for any scalars γ_i.

If we suppose $\gamma_i = 0$ for $1 \leq i \leq k$ then u is an idempotent and Theorem 2.4 gives not only the instability of $0 \in \mathcal{A}$, which is an obvious conclusion, but also the instability of any other nilpotent in $\mathbb{R}n_1$ or $\mathbb{R}n_2$ ... or in $\mathbb{R}n_k$.

Applications in dimension 3.

We consider the general algebra $\mathcal{A}$ having two linearly independent nilpotents n_1, n_2 with $\mathcal{N}_2\mathcal{N}_2 \subset \mathcal{N}_2$ given by the tables (2.1) and we look for necessary and sufficient conditions ensuring the existence of a near-idempotent as required in Theorem 2.4. As obtained in the previous application, there exist a vector $u = an_1 + bn_2 + ce_3 \notin \mathcal{N}_2$, i.e. $c \neq 0$, verifying $u \cdot n_1 \in \mathcal{N}_2$ and $u \cdot n_2 \in \mathcal{N}_2$ if and only if $\alpha_3 = \beta_3 = 0$. Moreover, u^2 has a nonzero third component if and only if $\mu_3 \neq 0$. Therefore, the conditions $\alpha_3 = \beta_3 = 0$ and $\mu_3 \neq 0$ are equivalent to the existence of a vector u verifying $u \cdot n_1 \in \mathcal{N}_2$ and $u \cdot n_2 \in \mathcal{N}_2$ and $u^2 = a'n_1 + b'n_2 + c'e_3$ with some convenient scalars a', b' and c' with the condition $c' \neq 0$. By replacing, if necessary, u by the vector $v = \alpha u$, where α is an appropriate nonzero scalar, we obtain

$$v^2 = v + a''n_1 + b''n_2,$$

with a'' and b'' scalars. Certainly, if $a''b'' = 0$, v is not a near-idempotent. But even in this case and due to the remark above, the conclusion still holds.

Therefore, the general algebra $\mathcal{A}$ verifies assumptions of Theorem 2.4 if and only if $\alpha_3 = \beta_3 = 0$ and $\mu_3 \neq 0$. This leads to the family of homogeneous quadratic dynamical systems

$$\begin{aligned}\dot{x}_1 &= 2A_{12}^1 x_1 x_2 + 2\alpha_1 x_1 x_3 + 2\beta_1 x_2 x_3 + \mu_1 x_3^2,\\ \dot{x}_2 &= 2A_{12}^2 x_1 x_2 + 2\alpha_2 x_1 x_3 + 2\beta_2 x_2 x_3 + \mu_2 x_3^2,\\ \dot{x}_3 &= \mu_3 x_3^2,\end{aligned}$$

with no conditions on scalars except $\mu_3 \neq 0$.

According to the result in [28], the case $\left(A_{12}^1, A_{12}^2\right) = (0,0)$ can be included.

Remark 2.6. In dimension three under hypothesis of Theorem 2.4, it is easy to show that the only nilpotents are the nonzero elements of $\mathbb{R}n_1$ or $\mathbb{R}n_2$. On the other hand, it is known that the image of a nilpotent under an isomorphism is also a nilpotent. Consequently, all considered algebras are nonisomorphic to any algebra involved in [28] since there it is supposed that $\mathcal{N}_2\mathcal{N}_2 = \{0\}$ and $\mathcal{N}_2$ is a plane of nilpotents.

If we consider two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ verifying hypothesis of theorem 3 with basis $\{n_1, n_2, e_3\}$ and $\{n_1', n_2', e_3'\}$, respectively and if $f : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism, we have either

$$f(n_1) = d_1 n_1' \text{ and } f(n_2) = d_2 n_2', \quad \text{or} \quad f(n_1) = d_1 n_2' \text{ and } f(n_2) = d_2 n_1'$$

with d_1, d_2 nonzero scalars. In addition, the conditions

$$\begin{aligned} f(n_i \cdot e_3) &= f(n_i)\, f(e_3) \qquad \text{for } i = 1, 2,\\ f(n_1 \cdot n_2) &= f(n_1)\, f(n_2),\\ f\left(e_3^2\right) &= \left(f(e_3)\right)^2\end{aligned}$$

lead to many other restrictions. Therefore, one can expect that, up to an isomorphism, it is difficult to reduce significantly the family.

2.2. *The case $\mathcal{N}_k\mathcal{N}_k \not\subset \mathcal{N}_k$.* In this subsection, we suppose $k = 2$, $m \geq 3$ and $\delta = 0$.

Theorem 2.7. *Let $\mathcal{A}$ be a real nonassociative algebra of dimension $m \geq 3$, n_1, n_2 two linearly independent nilpotents, $\mathcal{N}_2 = \operatorname{span}\{n_1, n_2\}$, $u \in \mathcal{A} \setminus \mathcal{N}_2$ a near-nilpotent associated to $\mathcal{N}_2$ and $\mathcal{P}_2 = \operatorname{span}\{n_1, n_2, u\}$. If $\mathcal{N}_2\mathcal{N}_2 \subset \mathcal{P}_2$ and $n_i \in \mathcal{A}_{\lambda_i}(u)$ for $i = 1, 2$ then $0 \in \mathcal{A}$ and any nilpotent in $\mathbb{R}n_1$ or in $\mathbb{R}n_2$ are unstable.*

Proof. As $\mathcal{P}_2$ is a subalgebra, we will prove that $0 \in \mathcal{P}_2$ is unstable. Restricted to $\mathcal{P}_2$, the corresponding ODE's become

$$\dot{x}_1 = x_1\left(2A_{12}^1 x_2 + 2\lambda_1 z\right) + \gamma_1 z^2,$$

$$\dot{x}_2 = x_2\left(2A_{12}^2 x_2 + 2\lambda_2 z\right) + \gamma_2 z^2,$$
$$\dot{z} = 2C_2^1 x_1 x_2,$$

if we let $n_1 n_2 = A_{12}^1 n_1 + A_{12}^2 n_2 + C_2^1 u$ with $C_2^1 \neq 0$.

With no loss of generality, we can suppose $\gamma_1 = \gamma_2 = 1$ by considering the new basis $\left\{\frac{1}{\gamma_1} n_1, \frac{1}{\gamma_2} n_2, u\right\}$. Also, if $C_2^1 < 0$ we can replace u by $-u$, then the new constant, still denoted by C_2^1, is strictly positive.

Let $N_0 = (0, 0, \varepsilon_0) \in \mathcal{P}_2$, with $\varepsilon_0 > 0$ small, be an element of any given neighbourhood of $0 \in \mathcal{P}_k$. We will justify that trajectory via N_0 blows up. Suppose the contrary, it is then defined for any $t > 0$. For $t > 0$ small enough, since $\dot{x}_1(0) > 0$ and $\dot{x}_2(0) > 0$, $x_1(t)$ and $x_2(t)$ increase. Thus, there exist $t_0 > 0$ with $x_1(t_0) > 0$ and $x_2(t_0) > 0$.

If $z \neq 0$, at each point $(0, x_2, z)$ belonging to the plane "$x_1 = 0$", we have $\dot{x}_1 = z^2 > 0$. Consequently, this plane is repellent and the same conclusion holds for any point $(x_1, 0, z)$ in the plane "$x_2 = 0$".

Thus, if $x_1(t_0) > 0$ and $x_2(t_0) > 0$, $x_1(t)$ and $x_2(t)$ remain positive for $t \geq t_0$. As a consequence, we have $z(t) \geq \varepsilon_0$, for all $t \geq 0$.

In addition, there exist some scalars $\mu > 0$ and $t_1 > 0$ with $x_1(t) \geq \mu$ and $x_2(t) \geq \mu$ for all $t \geq t_1$. We will prove it for $x_1(t)$ and the same idea gives the result for $x_2(t)$. Therefore, suppose there exists a sequence $t_n \to +\infty$ for which $x_1(t_n) \to 0$. As the sequences $(x_2(t_n))_n$ and $(z(t_n))_n$ are bounded, up to subsequences still denoted by $(x_1(t_n))_n$, $(x_2(t_n))_n$ and $(z(t_n))_n$, we have:

$$\lim_{n\to\infty} \left(x_1(t_n), x_2(t_n), z(t_n)\right) = (0, \alpha, \beta),$$

where $\alpha \geq 0$ and $\beta > 0$ are some convenient scalars. As the point $(0, \alpha, \beta)$ is repellent, there exist no trajectory that enters any given neighbourhood of $(0, \alpha, \beta)$ infinitely often.

As a consequence, we obtain

$$\dot{z}(t) \geq 2C_2^1 \mu^2 > 0 \quad \text{for} \quad t \geq t_1$$

and $z(t)$ blows up when $t \to \infty$, which is a contradiction. Thus $0 \in \mathcal{A}$ is unstable. Using the same ideas, we can prove that any nilpotent in $\mathbb{R}n_1$ or $\mathbb{R}n_1$ is unstable by setting a convenient initial condition in any given neighbourhood of the considered nilpotent. ∎

Remark 2.8. The conditions $\gamma_i \neq 0$, $1 \leq i \leq k$ are necessary for our proof.

If $k \geq 3$, we need additional restrictions like: $n_i n_j \in \mathcal{N}_k$ for $i \neq j$ with exactly one exception (i_0, j_0), $i_0 < j_0$ for which

$$n_{i_0} n_{j_0} = A_{i_0 j_0}^{i_0} n_{i_0} + A_{i_0 j_0}^{i_0} n_{j_0} + C_{j_0}^{i_0} u,$$

with $C_{j_0}^{i_0} \neq 0$. This assumptions will keep the repellency of the two hyperplanes "$x_{i_0} = 0$" and "$x_{j_0} = 0$" in $\mathcal{P}_k$.

When u is a near-idempotent, $k=2$ and $\mathcal{N}_2\mathcal{N}_2 \not\subset \mathcal{N}_2$ with $\mathcal{N}_2\mathcal{N}_2 \subset \mathcal{P}_2$. The system of ODE's restricted to $\mathcal{P}_2$ writes

$$\begin{aligned} \dot{x}_1 &= x_1\left(2A_{12}^1 x_2 + 2\lambda_1 z\right) + z^2, \\ \dot{x}_2 &= x_2\left(2A_{12}^2 x_2 + 2\lambda_2 z\right) + z^2, \\ \dot{z} &= z^2 + 2C_2^1 x_1 x_2, \end{aligned}$$

with $C_2^1 \neq 0$. If $C_2^1 > 0$, one can prove that $0 \in \mathcal{A}$ and any nilpotent in $\mathbb{R}n_1$ or $\mathbb{R}n_1$ are unstable. However, we believe that the case $C_2^1 < 0$ may include some cases where stability of the origin occurs though we have no available example.

Applications in dimension 3.

We consider the general family of algebras $\mathcal{A}'$ defined by (2.1) with, this time, $n_1n_2 = A_{12}^1 n_1 + A_{12}^2 n_2 + A_{12}^3 e_3$, where $A_{12}^3 \neq 0$ since $\mathcal{N}_2\mathcal{N}_2 \not\subset \mathcal{N}_2$.

Let $u = an_1 + bn_2 + ce_3$ be a vector not in $\mathcal{N}_2$ (i.e. $c \neq 0$). The condition $n_1 \in \mathcal{A}_{\lambda_1}(u)$ is equivalent to

$$\begin{aligned} bA_{12}^2 + c\alpha_2 &= 0, \\ bA_{12}^3 + c\alpha_3 &= 0, \end{aligned}$$

and this system admits nontrivial solutions if and only if

$$\alpha_3 A_{12}^2 = \alpha_2 A_{12}^3. \tag{2.2}$$

On the other hand, the condition $n_2 \in \mathcal{A}_{\lambda_2}(u)$ is equivalent to

$$\begin{aligned} aA_{12}^1 + c\beta_1 &= 0, \\ aA_{12}^3 + c\beta_3 &= 0, \end{aligned}$$

and we obtain nontrivial solutions of and only if

$$\beta_3 A_{12}^1 = \beta_1 A_{12}^3. \tag{2.3}$$

Under conditions (2.2) and (2.3), we obtain

$$u = -c\left(\frac{\beta_3}{A_{12}^3} n_1 + \frac{\alpha_3}{A_{12}^3} n_2 - e_3\right); \quad c \neq 0.$$

Then we compute u^2 and the third component should be zero. A direct computation gives the necessary and sufficient condition

$$\mu_3 = \frac{2\alpha_3\beta_2}{A_{12}^3}.$$

According to the proof, we absolutely need the conditions $\gamma_1\gamma_2 \neq 0$, where $u^2 = \sum_{i=1}^2 \gamma_i n_i$. This leads to

$$\begin{aligned} &2\alpha_3\beta_3 A_{12}^1 - 2A_{12}^3\left(\alpha_1\beta_3 + \alpha_3\beta_1\right) + \mu_1\left(A_{12}^3\right)^2 \neq 0, \\ &2\alpha_3\beta_3 A_{12}^2 - 2A_{12}^3\left(\alpha_2\beta_3 + \alpha_3\beta_2\right) + \mu_2\left(A_{12}^3\right)^2 \neq 0. \end{aligned} \tag{2.4}$$

Consequently, we did characterize the elements of the family that verify hypothesis of Theorem 2.7. They correspond to the quadratic dynamical systems:

$$\dot{x}_1 = 2A_{12}^1 x_1 x_2 + 2\alpha_1 x_1 x_3 + \frac{2\beta_3 A_{12}^1}{A_{12}^3} x_2 x_3 + \mu_1 x_3^2,$$
$$\dot{x}_2 = 2A_{12}^2 x_1 x_2 + \frac{2\alpha_3 A_{12}^2}{A_{12}^3} x_1 x_3 + 2\beta_2 x_2 x_3 + \mu_2 x_3^2,$$
$$\dot{x}_3 = 2A_{12}^3 x_1 x_2 + 2\alpha_3 x_1 x_3 + 2\beta_3 x_2 x_3 + \frac{2\alpha_3\beta_3}{A_{12}^3} x_3^2,$$

with conditions (2.4).

Remark 2.9. Using above theorems, we can give an extension of a classical result of Kinyon and Sagle.

In [10], Kinyon and Sagle stated that if there exist an idempotent in a nonassociative algebra $\mathcal{A}$, the origin is unstable.

When there exist no idempotent but we have a vector u verifying

$$u^2 = u + \sum_{i=1}^{k} \lambda_i e_i,$$

where $e_1, \ldots, e_k$ are linearly independent vectors of $\mathcal{A}$ (not necessarily nilpotents) and all scalars λ_i are not zero, we can derive the following result.

Theorem 2.10. *Let $\mathcal{A}$ be a real nonassociative algebra of dimension m, $e_1, \ldots, e_k$ k linearly independent elements of $\mathcal{A}$, $1 \le k \le m-1$ and $u \in \mathcal{A} \setminus \operatorname{span}\{e_1, \ldots, e_k\}$ verifying*

$$u^2 = u + \sum_{i=1}^{k} \gamma_i e_i,$$

where not all γ_i's are zero. If $\mathcal{M}_k = \operatorname{span}\{e_1, \ldots, e_k\}$ is a subalgebra and $e_i \cdot u \in \mathcal{M}_k$ for $i = 1, \ldots, k$, then $0 \in \mathcal{A}$ is unstable.

Proof. According to hypotheses, $\mathcal{P}_k = span\{e_1, \ldots, e_k, u\}$ is a subalgebra and the corresponding ODE's restricted to $\mathcal{P}_k$ become:

$$\dot{x}_1 = 2\sum_{i,j} A_{ij}^1 x_i x_j + 2\sum_{i=1}^{k} \lambda_i^1 x_i z + \gamma_1 z^2,$$
$$\dot{x}_2 = 2\sum_{i,j} A_{ij}^2 x_i x_j + 2\sum_{i=1}^{k} \lambda_i^2 x_i z + \gamma_2 z^2,$$
$$\vdots$$

$$\dot{x}_k = 2\sum_{i,j} A_{ij}^k x_i x_j + 2\sum_{i=1}^{k} \lambda_i^k x_i z + \gamma_k z^2,$$
$$\dot{z} = z^2,$$

with suitable constants.

For any neighbourhood of $0 \in \mathcal{P}_k$, let $P_0 = (0, \dots, 0, \varepsilon_0) \in \mathcal{P}_k$ with $\varepsilon_0 > 0$ small, be an element of this neighbourhood. Necessarily, the trajectory via P_0 with blows up otherwise it would be defined for $t \in [0, \infty)$ but this is in contradiction with

$$\dot{z} = z^2.$$

□

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