SOME DATKO AND BARBASHIN TYPE CHARACTERIZATIONS FOR THE UNIFORM h-INSTABILITY OF EVOLUTION FAMILIES

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ABSTRACT. The aim of this paper is to give some Datko and Barbashin type characterizations for the uniform h-instability of evolution families in Banach spaces, by using some important sets of growth rates. We prove four characterization theorems of Datko type and two characterization theorems of Barbashin type for uniform h-instability. Variants for uniform h-instability of some well-known results in stability theory (Barbashin (1967), Datko (1972)) are obtained.

1. INTRODUCTION

The topic of exponential stability and instability of dynamical systems on Banach spaces have been intensively investigated for many years. In the last decades, various results concerning this subject have witnessed considerable development. There are two momentous and fundamental results in the exponential stability theory which were obtained by Barbashin in 1967 ([1]) and Datko in 1972 ([10]).

THEOREM 1.1 (Barbashin). *An evolution family* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *with uniform exponential growth is uniformly exponentially stable if and only if*

$$\sup_{t \geq 0} \int_0^t \|U(t,s)x\| \, ds < \infty, \ \forall x \in X.$$

THEOREM 1.2 (Datko). *An evolution family* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *with uniform exponential growth is uniformly exponentially stable if and only if*

there is $p \in [1, \infty)$ such that

$$\sup_{s \geq 0} \int_s^{\infty} \|U(t,s)x\|^p \, dt < \infty, \ \forall x \in X.$$

Note that the Theorem 1.2 was originally proved by Datko in 1970 ([11]) for C_0-semigroups acting on Hilbert spaces and $p = 2$. Later, Pazy in [28] showed the case $p \in [1, \infty)$ on Banach spaces.

After the path-breaking research of Barbashin and Datko, through the usage of Datko-Pazy theorem, admissibility, Banach function spaces, Lyapunov functions, discrete-time methods and other ways, there has been numerous extensions and generalizations devoted to this area (see [2, 6, 9, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 27, 30, 31] and the references therein). For instance, in [12] Dragičević obtained some continuous and discrete versions of Datko type theorem for the concept of the exponential stability in average. In [6] and [23], the Datko and Barbashin type integral characterizations for uniform exponential instability of evolution operators were studied by Boruga and Megan. Also, a significant contribution in this direction was made by Bento et al. in [5] who obtained an interesting Datko type characterization for nonuniform μ-dichotomy. They observed that the uniform Datko's condition (Theorem 1.2) can be generalized to a more enhanced level, by means of differentiable growth rates. The notion of nonuniform μ-dichotomy is very general to cover some particular notions, such as uniform μ-dichotomy, (non)uniform exponential dichotomy, (non)uniform exponential stability, etc.

Since many differential equations arising in nature or engineering involve the discussion of nonexponential asymptotic behaviors and the notion of exponential stability is too restrictive for dynamical systems, there has been an increasing trend to look for more general types of stable behaviors. In recent years, many researcher focused on the polynomial stability and instability of solutions of evolution equations in Banach spaces. The concept of polynomial behaviors was proposed respectively by Bento and Silva in [4] and Barreira and Valls in [3] with slight differences, in the case of discrete and continuous time systems. After the two notable papers appeared, some remarkable and various results of polynomial stability and instability were investigated by Megan et al. in [6, 7, 23, 25, 26, 32] and Hai in [19, 20, 21]. It is worth noting that in [8] the authors proposed a more general concept, the so-called uniform h-stability. This concept includes the classical concepts of uniform exponential stability and uniform polynomial stability as particular cases. In addition, some Datko and Barbashin type conditions for uniform h-stability of evolution operators were obtained in [8]. Naturally, the question arises whether Datko and Barbashin type theorems can be generalized to the case of a uniform h-instability. This paper will give an affirmative answer.

Motivated by the recent work of Boruga, Megan and Toth ([8]), in this paper, we introduce the concept of uniform h-instability for evolution families

which is an extension of classical concepts of uniform exponential instability and uniform polynomial instability. Our main objective is to give some necessary and sufficient conditions of Datko type and Barbashin type for the uniform h-instability concept of evolution families on Banach spaces, and variants for uniform h-instability of some well-known results in stability theory (Barbashin ([1]), Datko ([10]), Boruga et al. ([8])) and instability theory (Boruga and Megan, [6, 23]) are obtained. We emphasize that the set of growth rates considered in this paper are different from that used in [5]. Our approach is based on the properties of Definition 2.12, but the growth functions utilized in [5] only require differentiability. Moreover, although the concept of uniform h-instability can be considered as a particular case of the notion of nonuniform μ-dichotomy, we point out that all Datko type characterizations in this paper do not imply those in [5], and vice versa.

2. Notations and preliminaries

In this section, we give some notations, definitions and preliminary facts which will be used in the sequel. Let $\mathbb{R}$ be the set of all real numbers. We denote by $\mathbb{R}_+ = [0, +\infty)$, by $\Delta = \left\{ (t,s) \in \mathbb{R}_+^2 : t \geq s \geq 0 \right\}$ and by $T = \left\{ (t,r,s) \in \mathbb{R}_+^3 : t \geq r \geq s \geq 0 \right\}$. We assume that X is a Banach space, X^* its dual space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators from X into itself. The norm on X, X^* and $\mathcal{B}(X)$ will be denoted by $||\cdot||$.

DEFINITION 2.1. *A family* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0} \subseteq \mathcal{B}(X)$ *is called an evolution family if the following three conditions are satisfied:*

(i) $U(t,t) = I$ *(where I is the identity operator on X) for all $t \in \mathbb{R}_+$;*
(ii) $U(t,s) = U(t,r)U(r,s)$ *for all $(t,r,s) \in T$;*
(iii) $U(\cdot, s)x$ *is continuous on $[s, \infty)$, for all $(s,x) \in \mathbb{R}_+ \times X$, $U(t, \cdot)x$ is continuous on $[0,t]$, for all $(t,x) \in \mathbb{R}_+ \times X$.*

DEFINITION 2.2. *An evolution family $\mathcal{U}$ is said to be injective if $\|U(t,s)x\| > 0$ for all $(t,s) \in \Delta$ and all $x \in X \setminus \{0\}$.*

DEFINITION 2.3 ([29]). *A nondecreasing function $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ is said to be a growth rate if it is bijective.*

In what follows, we suppose that $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ is a growth rate.

DEFINITION 2.4. *We say that an evolution family $\mathcal{U}$ has uniform h-decay (u.h.d.) if there are $M > 1$ and $\omega > 0$ such that*

$$(2.1) \qquad Mh(t)^\omega \|U(t,s)x\| \geq h(s)^\omega \|x\|, \ \forall (t,s,x) \in \Delta \times X.$$

Obviously, if an evolution family $\mathcal{U}$ has uniform h-decay, then it is injective.

REMARK 2.5. As particular cases of Definition 2.4, we have the following.

(i) If $h(t) = e^t$, then we say that an evolution family $\mathcal{U}$ has uniform exponential decay (u.e.d.).
(ii) If $h(t) = t + 1$, then we say that an evolution family $\mathcal{U}$ has uniform polynomial decay (u.p.d.).

REMARK 2.6. An evolution family $\mathcal{U}$ has uniform h-decay if and only if there are $M > 1$ and $\omega > 0$ such that

$$(2.2) \qquad Mh(t)^\omega \, \|U(t,s)x\| \geq h(r)^\omega \, \|U(r,s)x\| \,, \; \forall (t,r,s,x) \in T \times X.$$

DEFINITION 2.7. *An evolution family $\mathcal{U}$ is said to be uniformly h-unstable (u.h.us.) if there are $N > 1$ and $v \in (0,1)$ such that*

$$(2.3) \qquad Nh(s)^v \, \|U(t,s)x\| \geq h(t)^v \, \|x\| \,, \; \forall (t,s,x) \in \Delta \times X.$$

REMARK 2.8. As particular cases of Definition 2.7, we give the following.
(i) If $h(t) = e^t$, then we say that an evolution family $\mathcal{U}$ is uniformly exponentially unstable (u.e.us.).
(ii) If $h(t) = t + 1$, then we say that an evolution family $\mathcal{U}$ is uniformly polynomially unstable (u.p.us.).

REMARK 2.9. An evolution family $\mathcal{U}$ is uniformly h-unstable if and only if there are $N > 1$ and $v \in (0,1)$ such that

$$(2.4) \qquad Nh(r)^v \, \|U(t,s)x\| \geq h(t)^v \, \|U(r,s)x\| \,, \; \forall (t,r,s,x) \in T \times X.$$

REMARK 2.10. If an evolution family $\mathcal{U}$ is uniformly h-unstable, then it has uniform h-decay. The converse is not necessarily valid. To show this, we consider the following example.

EXAMPLE 2.11. Let $X = \mathbb{R}$. An evolution family $\mathcal{U}$ defined by

$$U(t,s)x = \frac{h(s)}{h(t)}x,$$

for all $(t, s, x) \in \Delta \times X$. It is easy to check that $\mathcal{U}$ satisfies Definition 2.4 for $\omega = 1$ and for all $M > 1$. It results that $\mathcal{U}$ has u.h.d.

If we suppose that $\mathcal{U}$ is u.h.us., then there are $N > 1$ and $v \in (0,1)$ such that

$$Nh(s)^v \cdot \frac{h(s)}{h(t)} \geq h(t)^v,$$

for all $(t, s) \in \Delta$. In particular, for $s = 0$, we obtain $\left(\frac{h(t)}{h(0)}\right)^{v+1} \leq N$, which is absurd for $t \to \infty$. Hence, $\mathcal{U}$ is not u.h.us.

It is worth noting that Definition 2.4 is an important tool in proving our main results. It is clear that the uniform h-decay is the necessary condition for the uniform h-instability. In other words, if an evolution family $\mathcal{U}$ does not have uniform h-decay, then it is not uniformly h-unstable. This is the reason why all the main results in this paper need to assume that an evolution family exhibits uniform h-decay.

DEFINITION 2.12. *We introduce the following sets of growth rates, which are very useful to our study:*

(i) $\mathcal{H}_0$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with*

$$(2.5) \qquad\qquad h(t) \geq t + 1, \ \forall t \geq 0;$$

(ii) $\mathcal{H}$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H > 1$ *such that*

$$(2.6) \qquad\qquad h(t+1) \leq H h(t), \ \forall t \geq 0;$$

(iii) $\mathcal{H}_1$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_1 > 1$ *such that*

$$(2.7) \qquad\qquad \int_t^\infty h(s)^{\alpha-1} ds \leq H_1 h(t)^\alpha, \ \forall \alpha < 0, \ t \geq 0;$$

(iv) $\mathcal{H}_2$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_2 > 1$ *such that*

$$(2.8) \qquad\qquad \int_t^\infty h(s)^\alpha ds \leq H_2 h(t)^\alpha, \ \forall \alpha < 0, \ t \geq 0;$$

(v) $\mathcal{H}_3$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_3 > 1$ *such that*

$$(2.9) \qquad\qquad \int_0^t h(s)^\alpha ds \leq H_3 h(t)^\alpha, \ \forall \alpha > 0, \ t \geq 0;$$

(vi) $\mathcal{H}_4$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_4 \geq 2$ *such that*

$$(2.10) \qquad\qquad h(H_4 h(t)) \leq (H_4)^2 h(t), \ \forall t \geq 0;$$

(vii) $\mathcal{H}_5$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_5 > 1$ *such that*

$$(2.11) \qquad\qquad \int_0^t h(s)^{\alpha-1} ds \leq H_5 h(t)^\alpha, \ \forall \alpha \in (0,1), \ t \geq 0;$$

(viii) $\mathcal{H}_6$ *is the set of all growth rates* $h : \mathbb{R}_+ \longrightarrow [1, \infty)$ *with the property that there is* $H_6 > 1$ *such that*

$$(2.12) \qquad\qquad \int_0^t h(s)^\alpha ds \leq H_6 h(t)^\alpha, \ \forall \alpha \in (0,1), \ t \geq 0.$$

REMARK 2.13. Let $f, g : \mathbb{R}_+ \longrightarrow [1, \infty)$, $f(t) = \xi^t$, $g(t) = (\eta t + 1)^\zeta$, where $\xi \geq e$ and $\eta, \zeta \geq 1$. Then we have the following assertions:

(i) $f, g \in \mathcal{H}_0$;
(ii) $f \in (\mathcal{H}_2 \cap \mathcal{H}_3) \subset (\mathcal{H}_1 \cap \mathcal{H}_3)$;
(iii) $g \in \mathcal{H}_1 \setminus (\mathcal{H}_2 \cup \mathcal{H}_3)$;
(iv) $f \in (\mathcal{H} \cap \mathcal{H}_6) \subset (\mathcal{H} \cap \mathcal{H}_5)$;
(v) if $\zeta = 1$, then $g \in \mathcal{H}_4 \cup (\mathcal{H}_5 \setminus \mathcal{H}_6)$.

3. Datko type theorems for uniform h-instability

In this section we give some Datko type characterizations for the uniform h-instability of evolution families on Banach spaces.

THEOREM 3.1. *Let* $h \in \mathcal{H}_0 \cap \mathcal{H}_1 \cap \mathcal{H}_4$ *and* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform h-decay. Then $\mathcal{U}$ is uniformly h-unstable if and only if there are two constants $D > 1$ and $d \in (0,1)$ such that*

$$(3.1) \qquad \int_r^\infty \frac{h(t)^{d-1}}{\|U(t,s)x\|} dt \leq \frac{Dh(r)^d}{\|U(r,s)x\|}, \ \forall (r,s,x) \in \Delta \times (X \setminus \{0\}).$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Remark 2.9, there are $N > 1$ and $v \in (0,1)$ such that the relation (2.4) holds. Let $d \in (0,v)$. By (2.4) and (2.7) we have

$$\int_r^\infty \frac{h(t)^{d-1}}{\|U(t,s)x\|} dt \leq N \int_r^\infty \left(\frac{h(r)}{h(t)}\right)^v \frac{h(t)^{d-1}}{\|U(r,s)x\|} dt$$

$$= \frac{Nh(r)^v}{\|U(r,s)x\|} \int_r^\infty h(t)^{d-v-1} dt \leq \frac{Nh(r)^v}{\|U(r,s)x\|} \cdot H_1 h(r)^{d-v} = \frac{Dh(r)^d}{\|U(r,s)x\|},$$

for all $(r,s,x) \in \Delta \times (X \setminus \{0\})$, where $D = NH_1$.

Sufficiency. Let $(t,r,s) \in T$ and $x \in X \setminus \{0\}$.

If $h(t) \geq 2r$, then by (2.2), (2.5), (2.10) and (3.1), we have

$$\frac{h(t)^d}{\|U(t,s)x\|} = \frac{2}{h(t)} \int_{\frac{h(t)}{2}}^{h(t)} \frac{h(t)^d}{\|U(t,s)x\|} d\tau = \frac{2}{h(t)} \int_{\frac{h(t)}{2}}^{h(t)} \frac{h(t)^d}{\|U(t,\tau)U(\tau,s)x\|} d\tau$$

$$\leq 2M \int_{\frac{h(t)}{2}}^{h(t)} \left(\frac{h(t)}{h(\tau)}\right)^{\omega+d} \frac{h(\tau)}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2M \int_{\frac{h(t)}{2}}^{h(t)} \left(\frac{h(t)}{\tau}\right)^{\omega+d} \frac{h(\tau)}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2M \int_{\frac{h(t)}{2}}^{h(t)} \left(\frac{h(t)}{2^{-1}h(t)}\right)^{\omega+d} \frac{h(\tau)}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2^{\omega+d+1} M \int_{\frac{h(t)}{2}}^{h(t)} \frac{h(h(t))}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2^{\omega+d+1} M \int_{\frac{h(t)}{2}}^{h(t)} \frac{h(H_4 h(t))}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2^{\omega+d+1} M \int_{\frac{h(t)}{2}}^{h(t)} \frac{(H_4)^2 h(t)}{h(t)} \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau$$

$$\leq 2^{\omega+d+1} M (H_4)^2 \int_r^\infty \frac{h(\tau)^{d-1}}{\|U(\tau,s)x\|} d\tau \leq \frac{2^{\omega+d+1} M (H_4)^2 Dh(r)^d}{\|U(r,s)x\|}.$$

This implies that

(3.2) $2^{\omega+d+1}M(H_4)^2Dh(r)^d\,\|U(t,s)x\| \geq h(t)^d\,\|U(r,s)x\|$, if $h(t) \geq 2r$.

If $h(t) < 2r$, then by (2.2) and (2.5) we have

$$\frac{h(t)^d}{\|U(t,s)x\|} \leq M\left(\frac{h(t)}{h(r)}\right)^{\omega+d}\frac{h(r)^d}{\|U(r,s)x\|} \leq M\left(\frac{h(t)}{r+1}\right)^{\omega+d}\frac{h(r)^d}{\|U(r,s)x\|}$$

$$\leq M\left(\frac{h(t)}{r}\right)^{\omega+d}\frac{h(r)^d}{\|U(r,s)x\|} \leq M\left(\frac{2r}{r}\right)^{\omega+d}\frac{h(r)^d}{\|U(r,s)x\|}$$

$$= \frac{2^{\omega+d}Mh(r)^d}{\|U(r,s)x\|}.$$

It follows that

(3.3) $2^{\omega+d}Mh(r)^d\,\|U(t,s)x\| \geq h(t)^d\,\|U(r,s)x\|$, if $h(t) < 2r$.

Using (3.2) and (3.3), we have that there exist $N = 2^{\omega+d+1}M(H_4)^2D$ and $v = d$ such that relation (2.4) holds for all $(t,r,s,x) \in T \times X$. By Remark 2.9 we conclude that $\mathcal{U}$ is u.h.us. ☐

COROLLARY 3.2 ([6]). *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be an evolution family with uniform polynomial decay. Then it is uniformly polynomially unstable if and only if there are two constants* $D > 1$ *and* $d \in (0,1)$ *such that*

$$\int_r^\infty \frac{(t+1)^{d-1}}{\|U(t,s)x\|}dt \leq \frac{D(r+1)^d}{\|U(r,s)x\|}, \quad \forall(r,s,x) \in \Delta \times (X \setminus \{0\}).$$

PROOF. It follows immediately from Theorem 3.1 for $h(t) = t+1$. ☐

THEOREM 3.3. *Let* $h \in \mathcal{H} \cap \mathcal{H}_2$ *and* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be an evolution family with uniform h-decay. Then* $\mathcal{U}$ *is uniformly h-unstable if and only if there are two constants* $D > 1$ *and* $d \in (0,1)$ *such that*

(3.4) $$\int_r^\infty \frac{h(t)^d}{\|U(t,s)x\|}dt \leq \frac{Dh(r)^d}{\|U(r,s)x\|}, \quad \forall(r,s,x) \in \Delta \times (X \setminus \{0\}).$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Remark 2.9, there are $N > 1$ and $v \in (0,1)$ such that the relation (2.4) holds. Let $d \in (0,v)$. By (2.4) and (2.8) we have

$$\int_r^\infty \frac{h(t)^d}{\|U(t,s)x\|}dt \leq N\int_r^\infty \left(\frac{h(r)}{h(t)}\right)^v\frac{h(t)^d}{\|U(r,s)x\|}dt$$

$$= \frac{Nh(r)^v}{\|U(r,s)x\|}\int_r^\infty h(t)^{d-v}dt \leq \frac{Nh(r)^v}{\|U(r,s)x\|}\cdot H_2h(r)^{d-v} = \frac{Dh(r)^d}{\|U(r,s)x\|},$$

for all $(r,s,x) \in \Delta \times (X \setminus \{0\})$, where $D = NH_2$.

Sufficiency. Let $(t,r,s) \in T$ and $x \in X \setminus \{0\}$.

If $t \geq r + 1$, then by (2.1), (2.6) and (3.4), we have

$$\frac{h(t)^d}{\|U(t,s)x\|} = \int_{t-1}^{t} \frac{h(t)^d}{\|U(t,\tau)U(\tau,s)x\|} d\tau \leq M \int_{t-1}^{t} \left(\frac{h(t)}{h(\tau)}\right)^{\omega} \frac{h(t)^d}{\|U(\tau,s)x\|} d\tau$$

$$= M \int_{t-1}^{t} \left(\frac{h(t)}{h(\tau)}\right)^{\omega+d} \frac{h(\tau)^d}{\|U(\tau,s)x\|} d\tau$$

$$\leq M \int_{t-1}^{t} \left(\frac{h(t)}{h(t-1)}\right)^{\omega+d} \frac{h(\tau)^d}{\|U(\tau,s)x\|} d\tau$$

$$\leq MH^{\omega+d} \int_{t-1}^{t} \frac{h(\tau)^d}{\|U(\tau,s)x\|} d\tau \leq MH^{\omega+d} \int_{r}^{\infty} \frac{h(\tau)^d}{\|U(\tau,s)x\|} d\tau$$

$$\leq \frac{MDH^{\omega+d}h(r)^d}{\|U(r,s)x\|}.$$

The above inequalities imply that

(3.5) $MDH^{\omega+d}h(r)^d \|U(t,s)x\| \geq h(t)^d \|U(r,s)x\|$, if $t \geq r + 1$.

If $t \in [r, r+1)$, then by (2.2) and (2.6) we have

$$\frac{h(t)^d}{\|U(t,s)x\|} \leq M \left(\frac{h(t)}{h(r)}\right)^{\omega+d} \frac{h(r)^d}{\|U(r,s)x\|}$$

$$\leq M \left(\frac{h(r+1)}{h(r)}\right)^{\omega+d} \frac{h(r)^d}{\|U(r,s)x\|} \leq \frac{MH^{\omega+d}h(r)^d}{\|U(r,s)x\|}.$$

It follows that

(3.6) $MH^{\omega+d}h(r)^d \|U(t,s)x\| \geq h(t)^d \|U(r,s)x\|$, if $t \in [r, r+1)$.

Set $N = MDH^{\omega+d}$ and $v = d$. Combining (3.5) with (3.6), we have

$$Nh(r)^v \|U(t,s)x\| \geq h(t)^v \|U(r,s)x\|, \ \forall(t,r,s,x) \in T \times X.$$

So $\mathcal{U}$ is u.h.us.   □

COROLLARY 3.4 ([6]). *Let* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform exponential decay. Then it is uniformly exponentially unstable if and only if there are two constants* $D > 1$ *and* $d \in (0,1)$ *such that*

$$\int_{r}^{\infty} \frac{e^{dt}}{\|U(t,s)x\|} dt \leq \frac{De^{dr}}{\|U(r,s)x\|}, \ \forall(r,s,x) \in \Delta \times (X \setminus \{0\}).$$

PROOF. It follows immediately from Theorem 3.3 for $h(t) = e^t$.   □

THEOREM 3.5. *Let* $h \in \mathcal{H} \cap \mathcal{H}_3$ *and* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform h-decay. Then* $\mathcal{U}$ *is uniformly h-unstable if and only if there are two constants* $D > 1$ *and* $d \in (0,1)$ *such that*

(3.7) $$\int_{s}^{t} \frac{\|U(r,s)x\|}{h(r)^d} dr \leq \frac{D \|U(t,s)x\|}{h(t)^d}, \ \forall(t,s,x) \in \Delta \times X.$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Remark (2.9), there are $N > 1$ and $v \in (0,1)$ such that the relation (2.4) holds. Let $d \in (0,v)$. By (2.4) and (2.8) we have

$$\int_s^t \frac{\|U(r,s)x\|}{h(r)^d} dr \leq N \int_s^t \left(\frac{h(r)}{h(t)}\right)^v \frac{\|U(t,s)x\|}{h(r)^d} dr$$

$$= \frac{N \|U(t,s)x\|}{h(t)^v} \int_s^t h(r)^{v-d} dr \leq \frac{N \|U(t,s)x\|}{h(t)^v} \int_0^t h(r)^{v-d} dr$$

$$\leq \frac{N \|U(t,s)x\|}{h(t)^v} \cdot H_3 h(t)^{v-d} = \frac{D \|U(t,s)x\|}{h(t)^d},$$

for all $(t,s,x) \in \Delta \times X$, where $D = NH_3$.

Sufficiency. Let $(t,r,s,x) \in T \times X$.

If $t \geq r+1$, then by (2.2), (2.6) and (3.7), we have

$$\frac{\|U(r,s)x\|}{h(r)^d} = \int_r^{r+1} \frac{\|U(r,s)x\|}{h(r)^d} d\tau \leq M \int_r^{r+1} \left(\frac{h(\tau)}{h(r)}\right)^\omega \frac{\|U(\tau,s)x\|}{h(r)^d} d\tau$$

$$= M \int_r^{r+1} \left(\frac{h(\tau)}{h(r)}\right)^{\omega+d} \frac{\|U(\tau,s)x\|}{h(\tau)^d} d\tau$$

$$\leq M \int_r^{r+1} \left(\frac{h(r+1)}{h(r)}\right)^{\omega+d} \frac{\|U(\tau,s)x\|}{h(\tau)^d} d\tau$$

$$\leq MH^{\omega+d} \int_r^{r+1} \frac{\|U(\tau,s)x\|}{h(\tau)^d} d\tau \leq MH^{\omega+d} \int_s^t \frac{\|U(\tau,s)x\|}{h(\tau)^d} d\tau$$

$$\leq \frac{MDH^{\omega+d} \|U(t,s)x\|}{h(t)^d},$$

which is equivalent to

(3.8) $MDH^{\omega+d} h(r)^d \|U(t,s)x\| \geq h(t)^d \|U(r,s)x\|$, if $t \geq r+1$.

If $t \in [r, r+1)$, then by (2.2) and (2.6) we have

$$h(t)^d \|U(r,s)x\| \leq M \left(\frac{h(t)}{h(r)}\right)^{\omega+d} h(r)^d \|U(t,s)x\|$$

(3.9)
$$\leq M \left(\frac{h(r+1)}{h(r)}\right)^{\omega+d} h(r)^d \|U(t,s)x\|$$

$$\leq MH^{\omega+d} h(r)^d \|U(t,s)x\|.$$

From inequalities (3.8) and (3.9), we get that there are $N = MDH^{\omega+d}$ and $v = d$ such that the relation (2.4) holds for all $(t,r,s,x) \in T \times X$. By Remark 2.9 we conclude that $\mathcal{U}$ is u.h.us. $\square$

COROLLARY 3.6 ([6]). *Let* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform exponential decay. Then it is uniformly exponentially unstable if and*

only if there are two constants $D > 1$ and $d \in (0,1)$ such that

$$\int_s^t \frac{\|U(r,s)x\|}{e^{dr}} dr \le \frac{D\|U(t,s)x\|}{e^{dt}}, \quad \forall (t,s,x) \in \Delta \times X.$$

PROOF. It follows immediately from Theorem 3.5 for $h(t) = e^t$.   □

THEOREM 3.7. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_4 \cap \mathcal{H}_5$ and $\mathcal{U} = \{U(t,s)\}_{t \ge s \ge 0}$ be an evolution family with uniform h-decay. Then $\mathcal{U}$ is uniformly h-unstable if and only if there are two constants $D > 1$ and $d \in (0,1)$ such that*

$$(3.10) \qquad \int_s^t \frac{\|U(r,s)x\|}{h(r)^{d+1}} dr \le \frac{D\|U(t,s)x\|}{h(t)^d}, \quad \forall (t,s,x) \in \Delta \times X.$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Remark 2.9, there are $N > 1$ and $v \in (0,1)$ such that the relation (2.4) holds. Let $d \in (0,v)$. By (2.4) and (2.11) we have

$$
\begin{aligned}
\int_s^t \frac{\|U(r,s)x\|}{h(r)^{d+1}} dr &\le N \int_s^t \left(\frac{h(r)}{h(t)}\right)^v \frac{\|U(t,s)x\|}{h(r)^{d+1}} dr \\
&= \frac{N\|U(t,s)x\|}{h(t)^v} \int_s^t h(r)^{v-d-1} dr \le \frac{N\|U(t,s)x\|}{h(t)^v} \int_0^t h(r)^{v-d-1} dr \\
&\le \frac{N\|U(t,s)x\|}{h(t)^v} \cdot H_5 h(t)^{v-d} = \frac{D\|U(t,s)x\|}{h(t)^d},
\end{aligned}
$$

for all $(t,s,x) \in \Delta \times X$, where $D = NH_5$.

Sufficiency. Let $(t,r,s,x) \in T \times X$.

Take $t \ge H_4 h(r)$, where H_4 is given by Definition 2.12 (vi). Then by (2.2), (2.10) and (3.10), we have

$$
\begin{aligned}
\frac{\|U(r,s)x\|}{h(r)^d} &= \frac{1}{(H_4 - 1)h(r)} \int_{h(r)}^{H_4 h(r)} \frac{\|U(r,s)x\|}{h(r)^d} d\tau \\
&\le \frac{M}{(H_4 - 1)h(r)} \int_{h(r)}^{H_4 h(r)} \left(\frac{h(\tau)}{h(r)}\right)^\omega \frac{\|U(\tau,s)x\|}{h(r)^d} d\tau \\
&= \frac{M}{H_4 - 1} \int_{h(r)}^{H_4 h(r)} \left(\frac{h(\tau)}{h(r)}\right)^{\omega+d+1} \frac{\|U(\tau,s)x\|}{h(\tau)^{d+1}} d\tau \\
&\le \frac{M}{H_4 - 1} \int_{h(r)}^{H_4 h(r)} \left(\frac{h(H_4 h(r))}{h(r)}\right)^{\omega+d+1} \frac{\|U(\tau,s)x\|}{h(\tau)^{d+1}} d\tau \\
&\le \frac{M H_4^{2(\omega+d+1)}}{H_4 - 1} \int_{h(r)}^{H_4 h(r)} \frac{\|U(\tau,s)x\|}{h(\tau)^{d+1}} d\tau \\
&\le \frac{M H_4^{2(\omega+d+1)}}{H_4 - 1} \int_r^{H_4 h(r)} \frac{\|U(\tau,s)x\|}{h(\tau)^{d+1}} d\tau
\end{aligned}
$$

$$\leq \frac{MH_4^{2(\omega+d+1)}}{H_4-1}\int_s^t \frac{\|U(\tau,s)x\|}{h(\tau)^{d+1}}d\tau$$

$$\leq \frac{MDH_4^{2(\omega+d+1)}\|U(t,s)x\|}{(H_4-1)h(t)^d}.$$

Thus, we obtain

(3.11) $$\frac{MDH_4^{2(\omega+d+1)}}{H_4-1}h(r)^d\|U(t,s)x\| \geq h(t)^d\|U(r,s)x\|,\ \text{if}\ t \geq H_4h(r).$$

If $t \in [r, H_4h(r))$, then by (2.2) and (2.10), we have

(3.12)
$$h(t)^d\|U(r,s)x\| \leq M\left(\frac{h(t)}{h(r)}\right)^{\omega+d}h(r)^d\|U(t,s)x\|$$

$$\leq M\left(\frac{h(H_4h(r))}{h(r)}\right)^{\omega+d}h(r)^d\|U(t,s)x\|$$

$$\leq MH_4^{2(\omega+d)}h(r)^d\|U(t,s)x\|.$$

Based on (3.11) and (3.12) we have that there exist

$$N = \max\left\{MDH_4^{2(\omega+d+1)}(H_4-1)^{-1}, MH_4^{2(\omega+d)}\right\}$$

and $v = d$ such that the relation (2.4) holds for all $(t,s,x) \in \Delta \times X$. By Remark 2.9 we conclude that $\mathcal{U}$ is u.h.us. □

COROLLARY 3.8 ([6]). *Let* $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ *be an evolution family with uniform polynomial decay. Then it is uniformly polynomially unstable if and only if there are two constants* $D > 1$ *and* $d \in (0,1)$ *such that*

$$\int_s^t \frac{\|U(r,s)x\|}{(r+1)^{d+1}}dr \leq \frac{D\|U(t,s)x\|}{(t+1)^{d+1}},\ \forall(t,s,x) \in \Delta \times X.$$

PROOF. It follows immediately from Theorem 3.7 for $h(t) = t + 1$. □

REMARK 3.9. Theorems 3.1, 3.3, 3.5 and 3.7 are the versions of the classical stability theorems and instability theorems due to Datko ([10]), Boruga et al. ([6, 8]), for uniform h-instability of evolution families.

4. BARBASHIN TYPE THEOREMS FOR UNIFORM h-INSTABILITY

In this section we give some Barbashin type characterizations for the uniform h-instability of evolution families. Throughout this section we assume that the mapping $s \mapsto \|U(t,s)^*x^*\|$ is measurable on $[0,t]$, for all $(t,x^*) \in \mathbb{R}_+ \times X^*$.

THEOREM 4.1. *Let $h \in \mathcal{H}_0 \cap \mathcal{H}_4 \cap \mathcal{H}_5$ and $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be an evolution family with uniform h-decay. Then $\mathcal{U}$ is uniformly h-unstable if and only if there are two constants $B > 1$ and $b \in (0,1)$ such that*

$$(4.1) \quad \int_0^t \frac{d\tau}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} \leq \frac{B}{h(t)^b \|x^*\|}, \quad \forall (t, x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\}).$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Definition 2.7, there are $N > 1$ and $v \in (0,1)$ such that the relation (2.3) holds. Let $b \in (0, v)$. By (2.3) and (2.11) we have

$$\int_0^t \frac{d\tau}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} \leq N \int_0^t \left(\frac{h(\tau)}{h(t)} \right)^v \frac{1}{h(\tau)^{b+1} \|x^*\|} d\tau$$

$$= \frac{N}{h(t)^v \|x^*\|} \int_0^t h(\tau)^{v-b-1} d\tau \leq \frac{N}{h(t)^v \|x^*\|} \cdot H_5 h(t)^{v-b} = \frac{B}{h(t)^b \|x^*\|},$$

for all $(t, x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\})$, where $B = NH_5$.

Sufficiency. Let $(t, s, x^*) \in \Delta \times (X^* \setminus \{0\})$.

Take $t \geq H_4 h(s)$, where H_4 is given by Definition 2.12 (vi). Then by (2.1), (2.10) and (4.1), we have

$$\frac{1}{h(s)^b \|U(t,s)^* x^*\|}$$

$$= \frac{1}{(H_4 - 1)h(s)} \int_{h(s)}^{H_4 h(s)} \frac{d\tau}{h(s)^b \|U(\tau,s)^* U(t,\tau)^* x^*\|}$$

$$\leq \frac{M}{(H_4 - 1)h(s)} \int_{h(s)}^{H_4 h(s)} \left(\frac{h(\tau)}{h(s)} \right)^\omega \frac{1}{h(s)^b \|U(t,\tau)^* x^*\|} d\tau$$

$$= \frac{M}{H_4 - 1} \int_{h(s)}^{H_4 h(s)} \left(\frac{h(\tau)}{h(s)} \right)^{\omega + b + 1} \frac{1}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} d\tau$$

$$\leq \frac{M}{H_4 - 1} \int_{h(s)}^{H_4 h(s)} \left(\frac{h(H_4 h(s))}{h(s)} \right)^{\omega + b + 1} \frac{1}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} d\tau$$

$$\leq \frac{M}{H_4 - 1} H_4^{2(\omega + b + 1)} \int_{h(s)}^{H_4 h(s)} \frac{1}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} d\tau$$

$$\leq \frac{M}{H_4 - 1} H_4^{2(\omega + b + 1)} \int_0^t \frac{1}{h(\tau)^{b+1} \|U(t,\tau)^* x^*\|} d\tau$$

$$\leq \frac{M B H_4^{2(\omega + b + 1)}}{(H_4 - 1)h(t)^b \|x^*\|}.$$

Thus, we get that

$$(4.2) \quad \frac{M B H_4^{2(\omega + b + 1)}}{H_4 - 1} h(s)^b \|U(t,s)^* x^*\| \geq h(t)^b \|x^*\|, \text{ if } t \geq H_4 h(s).$$

If $t \in [s, H_4 h(s))$, then by (2.1) and (2.10), we have

$$\frac{1}{h(s)^b \, \|U(t,s)^* x^*\|} \leq M \left(\frac{h(t)}{h(s)} \right)^{\omega+b} \frac{1}{h(t)^b \, \|x^*\|}$$

$$\leq M \left(\frac{h(H_4 h(s))}{h(s)} \right)^{\omega+b} \frac{1}{h(t)^b \, \|x^*\|} \leq \frac{M H_4^{2(\omega+b)}}{h(t)^b \, \|x^*\|}.$$

It follows that

(4.3) $$M H_4^{2(\omega+b)} h(s)^b \, \|U(t,s)^* x^*\| \geq h(t)^b \, \|x^*\| , \text{ if } t \in [s, H_4 h(s)).$$

From (4.2) and (4.3), it results that

$$N h(s)^b \, \|U(t,s)^* x^*\| \geq h(t)^b \, \|x^*\| , \ \forall (t,s,x^*) \in \Delta \times X^*,$$

where $N = \max \left\{ M B H_4^{2(\omega+b+1)} (H_4 - 1)^{-1}, M H_4^{2(\omega+b)} \right\}$. Hence $\mathcal{U}$ is u.h.us. □

COROLLARY 4.2 ([23]). *Let* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform polynomial decay. Then it is uniformly polynomially unstable if and only if there are two constants $B > 1$ and $b \in (0,1)$ such that*

$$\int_0^t \frac{d\tau}{(\tau+1)^{b+1} \, \|U(t,\tau)^* x^*\|} \leq \frac{B}{(t+1)^b \, \|x^*\|}, \ \forall (t,x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\}).$$

PROOF. It follows immediately from Theorem 4.1 for $h(t) = t + 1$.   □

THEOREM 4.3. *Let* $h \in \mathcal{H} \cap \mathcal{H}_6$ *and* $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ *be an evolution family with uniform h-decay. Then $\mathcal{U}$ is uniformly h-unstable if and only if there are two constants $B > 1$ and $b \in (0,1)$ such that*

(4.4) $$\int_0^t \frac{d\tau}{h(\tau)^b \, \|U(t,\tau)^* x^*\|} \leq \frac{B}{h(t)^b \, \|x^*\|}, \ \forall (t,x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\}).$$

PROOF. Necessity. If $\mathcal{U}$ is u.h.us., then by Definition 2.7, there are $N > 1$ and $v \in (0,1)$ such that the relation (2.3) holds. Let $b \in (0,v)$. By (2.3) and (2.12) we have

$$\int_0^t \frac{d\tau}{h(\tau)^b \, \|U(t,\tau)^* x^*\|} \leq N \int_0^t \left(\frac{h(\tau)}{h(t)} \right)^v \frac{1}{h(\tau)^b \, \|x^*\|} d\tau$$

$$= \frac{N}{h(t)^v \, \|x^*\|} \int_0^t h(\tau)^{v-b} d\tau \leq \frac{N}{h(t)^v \, \|x^*\|} \cdot H_6 h(t)^{v-b} = \frac{B}{h(t)^b \, \|x^*\|},$$

for all $(t,x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\})$, where $B = N H_6$.

Sufficiency. Let $(t,s,x) \in \Delta \times (X^* \setminus \{0\})$.

If $t \geq s + 1$, then by (2.1), (2.6) and (4.4), we have

$$\frac{1}{h(s)^b \left\| U(t,s)^* x^* \right\|} = \int_s^{s+1} \frac{d\tau}{h(s)^b \left\| U(\tau,s)^* U(t,\tau)^* x^* \right\|}$$

$$\leq M \int_s^{s+1} \left(\frac{h(\tau)}{h(s)} \right)^\omega \frac{1}{h(s)^b \left\| U(t,\tau)^* x^* \right\|} d\tau$$

$$= M \int_s^{s+1} \left(\frac{h(\tau)}{h(s)} \right)^{\omega+b} \frac{1}{h(\tau)^b \left\| U(t,\tau)^* x^* \right\|} d\tau$$

$$\leq M \int_s^{s+1} \left(\frac{h(s+1)}{h(s)} \right)^{\omega+b} \frac{1}{h(\tau)^b \left\| U(t,\tau)^* x^* \right\|} d\tau$$

$$\leq M H^{\omega+b} \int_s^{s+1} \frac{1}{h(\tau)^b \left\| U(t,\tau)^* x^* \right\|} d\tau$$

$$\leq M H^{\omega+b} \int_0^t \frac{1}{h(\tau)^b \left\| U(t,\tau)^* x^* \right\|} d\tau \leq \frac{M B H^{\omega+b}}{h(t)^b \left\| x^* \right\|}.$$

Thus, we get that

(4.5) $M B H^{\omega+b} h(s)^b \left\| U(t,s)^* x^* \right\| \geq h(t)^b \left\| x^* \right\|,$ if $t \geq s + 1.$

If $t \in [s, s+1)$, then by (2.1) and (2.6), we have

$$\frac{1}{h(s)^b \left\| U(t,s)^* x^* \right\|} \leq M \left(\frac{h(t)}{h(s)} \right)^{\omega+b} \frac{1}{h(t)^b \left\| x^* \right\|}$$

$$\leq M \left(\frac{h(s+1)}{h(s)} \right)^{\omega+b} \frac{1}{h(t)^b \left\| x^* \right\|} \leq \frac{M H^{\omega+b}}{h(t)^b \left\| x^* \right\|}.$$

It follows that

(4.6) $M H^{\omega+b} h(s)^b \left\| U(t,s)^* x^* \right\| \geq h(t)^b \left\| x^* \right\|,$ if $t \in [s, s+1).$

In virtue of (4.5) and (4.6), it results that

$$N h(s)^b \left\| U(t,s)^* x^* \right\| \geq h(t)^b \left\| x^* \right\|, \ \forall (t,s,x^*) \in \Delta \times X^*,$$

where $N = M B H^{\omega+b}$. Hence $\mathcal{U}$ is u.h.us. $\square$

COROLLARY 4.4 ([23]). *Let $\mathcal{U} = \{ U(t,s) \}_{t \geq s \geq 0}$ be an evolution family with uniform exponential decay. Then it is uniformly exponentially unstable if and only if there are two constants $B > 1$ and $b \in (0,1)$ such that*

$$\int_0^t \frac{d\tau}{e^{b\tau} \left\| U(t,\tau)^* x^* \right\|} \leq \frac{B}{e^{bt} \left\| x^* \right\|}, \ \forall (t,x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\}).$$

PROOF. It follows immediately from Theorem 4.3 for $h(t) = e^t$. $\square$

REMARK 4.5. Theorems 4.1 and 4.3 are the versions of the classical stability theorems and instability theorems due to Barbashin ([1]), Boruga et al. ([8, 23]), for uniform h-instability of evolution families.

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