GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 281 – 290

UNIFORM REGULARITY FOR THE NONISENTROPIC MHD SYSTEM

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Abstract. In this work, we prove the uniform regularity of smooth solutions to the full compressible MHD system in $\mathbb{T}^3$. Here our result is obtained by using the bilinear commutator and product estimates.

1. Introduction

Magnetic fields influence many natural and artificial flows. The study of these flows is called magnetohydrodynamics (MHD). The viscous compressible MHD model has a very wide range of applications in physical models, ranging from liquid metals to plasma. The MHD model is so important that it has been studied both from a theoretical and numerical perspective. In this paper, we consider the following MHD system:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\begin{aligned} &\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \mu \Delta u \\ &\quad - (\lambda+\mu)\nabla \operatorname{div} u = b \cdot \nabla b - \frac{1}{2}\nabla |b|^2, \end{aligned} \tag{1.2}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u + b \operatorname{div} u - \eta \Delta b = 0, \operatorname{div} b = 0, \tag{1.3}$$

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) + p \operatorname{div} u - k\Delta\theta = Q(\nabla u, \nabla b) \quad \text{in} \quad \mathbb{T}^3 \times (0,\infty), \tag{1.4}$$

$$(\rho, u, b, \theta)(\cdot, 0) = (\rho_0, u_0, b_0, \theta_0)(\cdot) \quad \text{in} \quad \mathbb{T}^3. \tag{1.5}$$

Here ρ denotes the density, u the velocity field, b the magnetic field, and $e := C_V \theta$ the specific internal energy, respectively. $p := R\rho\theta$ is the pressure.

2020 *Mathematics Subject Classification.* 35Q30, 35Q35, 76D03.
Key words and phrases. MHD, compressible, uniform regularity.
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λ and μ are two viscosity constants satisfying

$$\mu > 0 \text{ and } \lambda + \frac{2}{3}\mu \geq 0.$$

$\eta > 0$ is the resistivity, $k > 0$ is the heat conductivity coefficient. We will denote

$$Q := (\mu(\nabla u + \nabla u^t) + \lambda \text{div}\, u\mathbb{I}) : \nabla u + \eta |\text{rot}\, b|^2. \tag{1.6}$$

The system (1.1)-(1.6) describes the macroscopic behavior of MHD flow with dissipative mechanisms. It is obtained by combining the full Navier-Stokes equations with Maxwell's equation in free space and Ohm's law. In MHD flows, magnetic field can not only induce currents in a moving conductive fluid, but also change the magnetic filed itself. Therefore, there is a complex interaction between the magnetic and fluid dynamic phenomena, which brings more serious conundrums than Navier-Stokes equations. Compared with compressible Navier-Stokes equations, the mathematical analysis of MHD is much more complicated, as the oscillation of the density and the coupling interaction of hydrodynamics with magnetic field. In spite of these, there is a vast literature dedicated to existence, blow-up and asymptotic behavior of solutions, see [7, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 17, 18] and the reference cited therein. More precisely, for one-dimensional case, Hoff and Tsyganove ([9]) obtained the global existence and uniqueness of weak solutions with small initial energy. For multi-dimensional case, Fan and Yu ([5]) obtained the local existence of strong solutions to 3D compressible MHD equations when the initial density may contain vacuum. With regard to weak solutions, Fan and Yu ([6]), Ducomet and Feireisl ([3]), Hu and Wang ([10, 11]) proved the existence of global weak solutions. Wang ([17, 18]) showed the blow-up criterion. On the other hand, Nečasová and her coauthors ([7, 2, 4]) studied some models coupled with magnetohydrodynamic effort. Since the system (1.1)-(1.5) is a parabolic-hyperbolic one, we can deduce the the local existence of smooth solutions and uniqueness from the results in [16].

Proposition 1.1 ([16]). *Let $s > \frac{5}{2}$ be an integer and assume that the initial data satisfy*

$$\rho_0, u_0, b_0, \theta_0 \in H^s \quad \textit{and} \quad 0 < \inf \rho_0$$

for a positive constant C_0. Then the problem (1.1)-(1.5) *has a unique smooth solution (ρ, u, b, θ) satisfying*

$$\begin{aligned} &\rho \in C^\ell([0,T); H^{s-\ell}), u, b, \theta \in C^\ell([0,T); H^{s-2\ell}), \ell = 0, 1;\\ &\quad 0 < \inf \rho, \end{aligned}$$

for some $0 < T \leq \infty$.

To the best knowledge of the authors', the global existence of strong solutio for MHD system is still an important question. Moreover, it is well known that the uniform regularity plays an important role in the global existence of strong solutions. Here the aim of this paper is to prove uniform regularity estimates in (η, k) which is helpful in the process of proving the global existence. We will prove the following theorem.

THEOREM 1.2. *Let* $0 < \eta < 1, 0 < k < 1, 0 < \frac{1}{C_0} \le \rho_0 \le C_0, \rho_0, u_0, b_0, 0 \le \theta_0 \in H^s(\mathbb{T}^3)$ *with* $s > \frac{5}{2}$ *and* $\operatorname{div} b_0 = 0$ *in* $\mathbb{T}^3$. *Let* (ρ, u, b, θ) *be the unique local smooth solutions to the problem* (1.1)-(1.5) *on* $[0, T]$. *Then*

$$\|(\rho, u, b, \theta)(\cdot, t)\|_{H^s} \le C \quad in \quad [0, T_0] \tag{1.7}$$

holds true for some positive constants C *and* T_0 $(\le T)$ *independent of* η *and* k.

Let

$$\begin{aligned} M(t) := 1 + \sup_{0 \le \tau \le t} \bigg\{ & \|(\rho, u, b, \theta)(\cdot, \tau)\|_{H^s} \\ & + \|\partial_t u(\cdot, \tau)\|_{L^2} + \|\partial_t \theta(\cdot, \tau)\|_{L^2} + \left\| \frac{1}{\rho}(\cdot, \tau) \right\|_{L^\infty} \bigg\}. \end{aligned} \tag{1.8}$$

THEOREM 1.3. *For any* $t \in [0, T)(T \le 1)$, *we have that*

$$M(t) \le C_0(M_0) \exp(tC(M)) \tag{1.9}$$

for some nondecreasing continuous functions $C_0(\cdot)$ *and* $C(\cdot)$.

It follows from (1.9) and [1, 15] that:

$$M(t) \le C, \tag{1.10}$$

thus we only need to show Theorem 1.3.

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce ([13, 14]):

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|g\|_{L^{p_2}} \|\Lambda^s f\|_{L^{q_2}}), \tag{1.11}$$

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}), \tag{1.12}$$

with $s > 0, \Lambda := (-\Delta)^{\frac{1}{2}}$ and $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2}$.

2. Proof of Theorem 1.3

First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}t} \int \rho^q \mathrm{d}x = \left(1 - \frac{1}{q}\right) \int \rho^q \operatorname{div} u \mathrm{d}x \le \|\operatorname{div} u\|_{L^\infty} \int \rho^q \mathrm{d}x,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\rho\|_{L^q} \le \|\operatorname{div} u\|_{L^\infty} \|\rho\|_{L^q},$$

which gives

$$\|\rho\|_{L^q} \le \|\rho_0\|_{L^q} \exp\left(\int_0^t \|\mathrm{div}\, u\|_{L^\infty} \mathrm{d}\tau\right). \tag{2.1}$$

Taking $q \to +\infty$, we get

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty} \exp(tC(M)). \tag{2.2}$$

It follows from (1.1) that

$$\partial_t \frac{1}{\rho} + u \cdot \nabla \frac{1}{\rho} - \frac{1}{\rho} \mathrm{div}\, u = 0. \tag{2.3}$$

Testing (2.3) by $\left(\frac{1}{\rho}\right)^{q-1}$, we find that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{\rho}\right)^q \mathrm{d}x = \left(1+\frac{1}{q}\right)\int \left(\frac{1}{\rho}\right)^q \mathrm{div}\, u \mathrm{d}x \le \left(1+\frac{1}{q}\right)\left\|\frac{1}{\rho}\right\|_{L^q}^q \|\mathrm{div}\, u\|_{L^\infty},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{\rho}\right\|_{L^q} \le \left(1+\frac{1}{q}\right)\left\|\frac{1}{\rho}\right\|_{L^q} \|\mathrm{div}\, u\|_{L^\infty},$$

which gives

$$\left\|\frac{1}{\rho}\right\|_{L^q} \le \left\|\frac{1}{\rho_0}\right\|_{L^q} \exp\left(\left(1+\frac{1}{q}\right)\int_0^t \|\mathrm{div}\, u\|_{L^\infty}\mathrm{d}\tau\right)$$

and we have

$$\left\|\frac{1}{\rho}\right\|_{L^\infty} \le \left\|\frac{1}{\rho_0}\right\|_{L^\infty} \exp(tC(M)) \tag{2.4}$$

by sending $q \to +\infty$.

Testing (1.4) by θ^{q-1} and using (1.1), we get

$$\begin{aligned} &\frac{C_V}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho\theta^q \mathrm{d}x + k\int \nabla\theta\cdot\nabla\theta^{q-1}\mathrm{d}x \\ &\quad = \int Q\theta^{q-1}\mathrm{d}x - \int p\theta^{q-1}\mathrm{div}\, u\mathrm{d}x \\ &\quad \le C(M)\|Q\|_{L^q}\|\rho^{\frac{1}{q}}\theta\|_{L^q}^{q-1} + C\|\mathrm{div}\, u\|_{L^\infty}\|\rho^{\frac{1}{q}}\theta\|_{L^q}^q, \end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho^{\frac{1}{q}}\theta\|_{L^q} \le C(M)\|Q\|_{L^q} + C\|\mathrm{div}\, u\|_{L^\infty}\|\rho^{\frac{1}{q}}\theta\|_{L^q},$$

which, similarly to (2.2), implies

$$\|\theta\|_{L^\infty} \le C_0(M_0)\exp(tC(M)). \tag{2.5}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x = 2\int u\partial_t u\mathrm{d}x \le 2\|u\|_{L^2}\|\partial_t u\|_{L^2} \le C(M),$$

which implies

$$\|u\|_{L^2} \le C_0(M_0)\exp(tC(M)). \tag{2.6}$$

Testing (1.3) by b, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |b|^2\mathrm{d}x + \int |\nabla b|^2\mathrm{d}x = -\int (u\cdot\nabla b - b\cdot\nabla u + b\operatorname{div} u) b\mathrm{d}x$$
$$= -\int \left(\frac{1}{2}|b|^2\operatorname{div} u - b\cdot\nabla u\cdot b\right)\mathrm{d}x \le C\|\nabla u\|_{L^\infty}\|b\|_{L^2}^2 \le C(M),$$

which leads to

$$\|b\|_{L^2}^2 + \int_0^t\int |\nabla b|^2\mathrm{d}x\mathrm{d}\tau \le C_0(M_0)\exp(tC(M)). \tag{2.7}$$

Taking Λ^s to (1.1), testing by $\Lambda^s\rho$, using (1.11) and (1.12), we compute

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}}\int (\Lambda^s\rho)^2\mathrm{d}x &= -\int (\Lambda^s(u\cdot\nabla\rho) - u\cdot\nabla\Lambda^s\rho)\Lambda^s\rho\mathrm{d}x \\
&\quad + \frac{1}{2}\int (\Lambda^s\rho)^2\operatorname{div} u\mathrm{d}x - \int \Lambda^s(\rho\operatorname{div} u)\Lambda^s\rho\mathrm{d}x \\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}^2 + C\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}u\|_{L^2}\|\Lambda^s\rho\|_{L^2} \\
&\quad + C\|\rho\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2}\|\Lambda^s\rho\|_{L^2} \\
&\le C(M) + C(M)\|\Lambda^{s+1}u\|_{L^2} \\
&\le \frac{\mu}{16}\|\Lambda^{s+1}u\|_{L^2}^2 + C(M).
\end{aligned} \tag{2.8}$$

It is obvious that

$$\int_0^t\int |\partial_t u|^2\mathrm{d}x\mathrm{d}\tau \le t\sup\int |\partial_t u|^2\mathrm{d}x \le tC(M). \tag{2.9}$$

Applying Λ^{s-1} to (1.2), testing by $\Lambda^{s-1}\partial_t u$, using (1.11) and (1.12), we obtain

$$\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^s u|^2\mathrm{d}x + \frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\Lambda^{s-1}\operatorname{div} u)^2\mathrm{d}x + \int \rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x \\
&= -\int \Lambda^{s-1}\nabla p\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x - \int \Lambda^{s-1}(\rho u\cdot\nabla u)\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x \\
&\quad - \int [\Lambda^{s-1}(\rho\partial_t u) - \rho\Lambda^{s-1}\partial_t u]\Lambda^{s-1}\partial_t u\mathrm{d}x \\
&\quad + \int \Lambda^{s-1}\left(b\cdot\nabla b - \frac{1}{2}\nabla|b|^2\right)\Lambda^{s-1}\partial_t u\mathrm{d}x \\
&\le C\|\Lambda^s p\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2} + C\|\rho\|_{H^{s-1}}\|u\|_{H^s}^2\|\Lambda^{s-1}\partial_t u\|_{L^2} \\
&\quad + C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t u\|_{L^2} + \|\partial_t u\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\Lambda^{s-1}\left(b\cdot\nabla b-\frac{1}{2}\nabla|b|^2\right)\right\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M)(\|\Lambda^{s-2}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M)\big(\|\partial_t u\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\partial_t u\|_{L^2}\\
&\quad+\|\partial_t u\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}}\big)\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le C(M)\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\quad+C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le\frac{1}{2}\int\rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x+C(M),
\end{aligned}
$$

which gives

$$
\int_0^t\int|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x\mathrm{d}\tau\le C_0(M_0)\exp(tC(M)). \tag{2.10}
$$

Applying Λ^s to (1.2), testing by $\Lambda^s u$, using (1.1), (1.11) and (1.12), we have

(2.11)

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\Lambda^s u|^2\mathrm{d}x+\mu\int|\Lambda^{s+1}u|^2\mathrm{d}x+(\lambda+\mu)\int(\Lambda^s\mathrm{div}\,u)^2\mathrm{d}x\\
&\quad+\int\rho\Lambda^s\nabla\theta\cdot\Lambda^s u\mathrm{d}x+\int\theta\nabla\Lambda^s\rho\cdot\Lambda^s u\mathrm{d}x\\
&=-\int(\Lambda^s(\rho\partial_t u)-\rho\Lambda^s\partial_t u)\Lambda^s u\mathrm{d}x-\int(\Lambda^s(\rho u\cdot\nabla u)-\rho u\cdot\nabla\Lambda^s u)\Lambda^s u\mathrm{d}x\\
&\quad-\int(\Lambda^s(\rho\nabla\theta)-\rho\nabla\Lambda^s\theta)\Lambda^s u\mathrm{d}x-\int(\Lambda^s(\theta\nabla\rho)-\theta\nabla\Lambda^s\rho)\Lambda^s u\mathrm{d}x\\
&\quad+\int(\Lambda^s(b\cdot\nabla b)-b\cdot\nabla\Lambda^s b)\Lambda^s u\mathrm{d}x\\
&\quad+\int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x+\frac{1}{2}\int\Lambda^s|b|^2\cdot\Lambda^s\mathrm{div}\,u\mathrm{d}x\\
&\le C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad+C(\|\nabla u\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2}+\|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad+C(\|\nabla\rho\|_{L^\infty}\|\Lambda^s\theta\|_{L^2}+\|\nabla\theta\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad+C(\|\nabla\theta\|_{L^\infty}\|\Lambda^s\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\Lambda^s\theta\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad+C\|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\quad+\int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x+C\|b\|_{L^\infty}\|\Lambda^s b\|_{L^2}\|\Lambda^{s+1}u\|_{L^2}\\
&\le C(M)+C(M)(\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})
\end{aligned}
$$

$$
\begin{aligned}
&+\int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x+C(M)\|\Lambda^{s+1}u\|_{L^2}\\
&\le C(M)+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2+\frac{\mu}{16}\|\Lambda^{s+1}u\|_{L^2}^2+\int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\mathrm{d}x.
\end{aligned}
$$

Applying Λ^s to (1.3), testing by $\Lambda^s b$, using (1.11) and (1.12), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s b|^2\mathrm{d}x+\eta\int|\Lambda^{s+1}b|^2\mathrm{d}x\\
&\quad=-\int(\Lambda^s(u\cdot\nabla b)-u\cdot\nabla\Lambda^s b)\Lambda^s b\mathrm{d}x-\int u\cdot\nabla\Lambda^s b\cdot\Lambda^s b\mathrm{d}x\\
&\qquad+\int(\Lambda^s(b\cdot\nabla u)-b\cdot\nabla\Lambda^s u)\Lambda^s b\mathrm{d}x+\int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x\\
&\qquad-\int(\Lambda^s(b\operatorname{div}u)-b\Lambda^s\operatorname{div}u)\Lambda^s b\mathrm{d}x-\int b\Lambda^s\operatorname{div}u\Lambda^s b\mathrm{d}x\\
&\quad\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2}+\|\nabla b\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s b\|_{L^2}\\
&\qquad+\int\frac{1}{2}|\Lambda^s b|^2\operatorname{div}u\mathrm{d}x+\int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x-\int b\Lambda^s\operatorname{div}u\Lambda^s b\mathrm{d}x\\
&\quad\le C(M)+\int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x+C(M)\|\Lambda^{s+1}u\|_{L^2}\\
&\quad\le C(M)+\int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\mathrm{d}x+\frac{\mu}{16}\|\Lambda^{s+1}u\|_{L^2}^2.
\end{aligned}
\tag{2.12}
$$

Applying Λ^{s-1} to (1.4), testing by $\Lambda^{s-1}\partial_t\theta$, using (1.11) and (1.12), we have

$$
\begin{aligned}
&\frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Lambda^s\theta)^2\mathrm{d}x+\int\rho|\Lambda^{s-1}\partial_t\theta|_{L^2}^2\mathrm{d}x\\
&\quad=-\int\Lambda^{s-1}(p\operatorname{div}u)\Lambda^{s-1}\partial_t\theta\mathrm{d}x-\int\Lambda^{s-1}(\rho u\cdot\nabla\theta)\Lambda^{s-1}\partial_t\theta\mathrm{d}x\\
&\qquad-\int[\Lambda^{s-1}(\rho\partial_t\theta)-\rho\Lambda^{s-1}\partial_t\theta]\Lambda^{s-1}\partial_t\theta\mathrm{d}x+\int\Lambda^{s-1}Q\cdot\Lambda^{s-1}\partial_t\theta\mathrm{d}x\\
&\qquad\text{(where we take } C_V=1)\\
&\quad\le\|\Lambda^{s-1}(p\operatorname{div}u)\|_{L^2}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+\|\Lambda^{s-1}(\rho u\cdot\nabla\theta)\|_{L^2}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
&\qquad+C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
&\qquad+\|\Lambda^{s-1}Q\|_{L^2}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
&\quad\le C(M)\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+C(M)(\|\Lambda^{s-2}\partial_t\theta\|_{L^2}+\|\partial_t\theta\|_{L^\infty})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}\\
&\quad\le C(M)\|\Lambda^{s-1}\partial_t\theta\|_{L^2}+C(M)(\|\partial_t\theta\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^{\frac{s-2}{s-1}}+\|\partial_t\theta\|_{L^2}\\
&\qquad+\|\partial_t\theta\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t\theta\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t\theta\|_{L^2}
\end{aligned}
$$

$$\leq \frac{1}{2}\int \rho|\Lambda^{s-1}\partial_t\theta|^2\mathrm{d}x + C(M),$$

which leads to

$$\int_0^t\int |\Lambda^{s-1}\partial_t\theta|^2\mathrm{d}x\mathrm{d}\tau \leq C_0(M_0)\exp(tC(M)). \tag{2.13}$$

Taking Λ^s to (1.4), testing by $\Lambda^s\theta$, using (1.1), (1.11) and (1.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho(\Lambda^s\theta)^2\mathrm{d}x + k\int(\Lambda^{s+1}\theta)^2\mathrm{d}x \\
&= -\int(\Lambda^s(\rho\partial_t\theta) - \rho\Lambda^s\partial_t\theta)\Lambda^s\theta\mathrm{d}x - \int(\Lambda^s(\rho u\cdot\nabla\theta) - \rho u\cdot\nabla\Lambda^s\theta)\Lambda^s\theta\mathrm{d}x \\
&\quad - \int\Lambda^s(p\mathrm{div}\, u)\cdot\Lambda^s\theta\mathrm{d}x + \int\Lambda^s Q\cdot\Lambda^s\theta\mathrm{d}x \\
&\leq C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t\theta\|_{L^2} + \|\partial_t\theta\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s\theta\|_{L^2} \\
&\quad + C(\|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + \|\nabla\theta\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2})\|\Lambda^s\theta\|_{L^2} \\
&\quad + C(\|p\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^s p\|_{L^2})\|\Lambda^s\theta\|_{L^2} + \|\Lambda^s Q\|_{L^2}\|\Lambda^s\theta\|_{L^2} \\
&\leq C(M)(\|\Lambda^{s-1}\partial_t\theta\|_{L^2} + \|\partial_t\theta\|_{L^\infty}) + C(M) \\
&\quad + C(M)\|\Lambda^{s+1}u\|_{L^2} + \eta C(M)\|\Lambda^{s+1}b\|_{L^2} \\
&\leq \frac{\mu}{16}\|\Lambda^{s+1}u\|_{L^2}^2 + \frac{\eta}{8}\|\Lambda^{s+1}b\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2 + C(M).
\end{aligned} \tag{2.14}$$

Summing up (2.8), (2.11), (2.12) and (2.14), using (2.10) and (2.13), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int((\Lambda^s\rho)^2 + \rho|\Lambda^s u|^2 + |\Lambda^s b|^2 + \rho(\Lambda^s\theta)^2)\mathrm{d}x + \frac{\mu}{2}\int(\Lambda^{s+1}u)^2\mathrm{d}x \\
&\quad + \frac{\lambda+\mu}{2}\int(\Lambda^s\mathrm{div}\, u)^2\mathrm{d}x + \frac{\eta}{2}\int(\Lambda^{s+1}b)^2\mathrm{d}x + \frac{k}{2}\int(\Lambda^{s+1}\theta)^2\mathrm{d}x \\
&\leq C(M) + \|\Lambda^{s+1}\partial_t u\|_{L^2}^2 + \|\Lambda^{s-1}\partial_t\theta\|_{L^2}^2.
\end{aligned} \tag{2.15}$$

Whence

$$\begin{aligned}
&\|\Lambda^s(\rho,u,b,\theta)(\cdot,t)\|_{L^2} + \|\Lambda^{s+1}u\|_{L^2(0,t;L^2)} + \sqrt{\eta}\|\Lambda^{s+1}b\|_{L^2(0,t;L^2)} \\
&\quad + \sqrt{k}\|\Lambda^{s+1}\theta\|_{L^2(0,t;L^2)} \leq C_0(M_0)\exp(tC(M)).
\end{aligned} \tag{2.16}$$

On the other hand, it follows from (1.2) that

$$\begin{aligned}
\|\partial_t u\|_{L^2} &= \left\|\frac{1}{\rho}\left(b\cdot\nabla b - \frac{1}{2}\nabla|b|^2 + \mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}\, u - \nabla p - \rho u\cdot\nabla u\right)\right\|_{L^2} \\
&\leq C_0(M_0)\exp(tC(M)).
\end{aligned} \tag{2.17}$$

Similarly, we have

$$\|\partial_t \theta\|_{L^2} \leq C_0(M_0) \exp(tC(M)). \tag{2.18}$$

Combining (2.4), (2.6), (2.7), (2.16), (2.17) and (2.18), we conclude that (1.9) holds true.

This completes the proof.

ACKNOWLEDGEMENTS.

The authors are grateful to the referees and the editor whose comments and suggestions greatly improved the presentation of this paper. Tong Tang is partially supported by NSF of Jiangsu Province Grant No. BK20221369.

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Received: 8.2.2022.
Revised: 20.5.2022.