GLASNIK MATEMATIČKI
Vol. 57(77)(2022), 313 – 319

A NOTE ON MAXIMAL FOURIER RESTRICTION FOR SPHERES IN ALL DIMENSIONS

Marco Vitturi
University College Cork, Ireland

Abstract. We prove a maximal Fourier restriction theorem for hypersurfaces in $\mathbb{R}^d$ for any dimension $d \geq 3$ in a restricted range of exponents given by the Tomas-Stein theorem (spheres being the most canonical example). The proof consists of a simple observation. When $d = 3$ the range corresponds exactly to the full Tomas-Stein one, but is otherwise a proper subset when $d > 3$. We also present an application regarding the Lebesgue points of functions in $\mathcal{F}(L^p)$ when p is sufficiently close to 1.

1. Introduction

In 2016 a new line of investigation in the field of Fourier restriction studies has been opened by Müller, Ricci and Wright in [7], namely that of maximal Fourier restriction theorems. The goal of such line of investigation is to study the Lebesgue points of the Fourier transform $\widehat{f}$ of a generic function $f \in L^p$ when $1 \leq p \leq 2$ and sufficiently close to 1. In the aforementioned paper they prove that for the case of curves in $\mathbb{R}^2$ the following holds:

Theorem 1.1 ([7]). *Let Γ be a C^2 curve in $\mathbb{R}^2$ and let $f \in L^p$ with $1 \leq p < 8/7$. Then, with respect to arclength measure, a.e. point of Γ where the curvature does not vanish is a Lebesgue point of $\widehat{f}$.*

In particular, if $\mathcal{R}$ is the Fourier restriction operator associated with Γ, one has from the above that for all $f \in L^p$ with $1 \leq p < 8/7$

$$\mathcal{R}f = \widehat{f}\big|_{\Gamma}$$

for a.e. point of Γ where the curvature is non zero (a.e. with respect to arclength measure). Theorem 1.1 is the consequence of a clever trick (which we

2020 *Mathematics Subject Classification.* 42B10, 42B25.
Key words and phrases. Fourier restriction, maximal operators.

have included in the proof of Proposition 2.3, for the reader's convenience) and the following

Theorem 1.2 ([7]). *Let Γ be the graph of a C^2 function $\gamma : I \to \mathbb{R}$, where I is a bounded interval, and let μ denote the affine measure on Γ, which for Γ in this form is given by*

$$d\mu((\xi, \gamma(\xi))) = |\gamma''(\xi)|^{1/3}\, d\xi.$$

Let $\chi \in \mathscr{S}(\mathbb{R})$ be a fixed Schwartz function with $\int \chi = 1$ and define the maximal Fourier restriction operator

$$\mathcal{M}f(\xi) := \sup_{\varepsilon,\delta>0} \Big| \int\int_{\mathbb{R}^2} \widehat{f}(\xi + s, \gamma(\xi) + t)\chi_\epsilon(s)\chi_\delta(t)\, ds\, dt \Big|.$$

Then the estimate

$$\|\mathcal{M}f\|_{L^q(\Gamma,\, d\mu)} \lesssim_{p,q} \|f\|_{L^p(\mathbb{R}^2)} \tag{1.1}$$

holds for all $f \in L^p$ with $1 \le p < 4/3$ and $p' \ge 3q$.

Some clarifications about the notation used in the statement: given a single-variable function χ we have let

$$\chi_\epsilon(x) := \frac{1}{\epsilon}\chi\big(\frac{x}{\epsilon}\big)$$

denote the L^1 rescaling; p' denotes the conjugate exponent of p, that is $1/p + 1/p' = 1$; finally, we write $A \lesssim B$ when there exists a constant $C > 0$ such that $A \le CB$, and moreover if the constant C depends on some list of parameters L we highlight this by writing $A \lesssim_L B$.

Observe that the range of exponents for which (1.1) holds is the same as that for the usual operator of Fourier restriction to Γ. The proof of the above theorem follows the lines of Sjölin's proof of the Fourier restriction conjecture for curves in the plane as given in [11].

In this short note we consider the case of Fourier restriction to a compact hypersurface Σ immersed in d-dimensional euclidean space and of non-vanishing Gaussian curvature, where $d \ge 3$. Let $d\sigma$ denote the surface measure of Σ. Define, analogously to the above, the maximal Fourier restriction operator for Σ

$$\mathscr{M}f(\omega) := \sup_{\epsilon>0} \Big| \int \widehat{f}(\omega + y)\chi_\epsilon(y)\, dy \Big|,$$

where ω ranges over Σ and $\chi \in \mathscr{S}(\mathbb{R}^d)$ is a fixed Schwartz function with $\int \chi = 1$, $\chi_\epsilon(y) = \epsilon^{-d}\chi(y/\epsilon)$. Then we have the following

Theorem 1.3. *Let $d \ge 3$ and let $\Sigma \subset \mathbb{R}^d$ be a compact hypersurface with non-vanishing Gaussian curvature. The operator $\mathscr{M}$ satisfies*

$$\|\mathscr{M}f\|_{L^q(\Sigma,\, d\sigma)} \lesssim_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)} \tag{1.2}$$

for $1 \le p \le 4/3$, $p' \ge \frac{d+1}{d-1}q$.

Moreover, for $1 \leq p \leq 8/7$*, if* $f \in L^p(\mathbb{R}^d)$ *then* σ*-a.e. point of* Σ *is a Lebesgue point for* $\widehat{f}$*.*

Observe that, when $d = 3$, the range of exponents for which (1.2) holds has endpoint $L^{4/3} \to L^2$, and thus corresponds precisely to the Tomas-Stein range for this dimension. For larger values of d, the stated range is however only a subset of the full Tomas-Stein range, which is

$$1 \leq p \leq \frac{2(n+1)}{n+3}, \quad p' \geq \frac{d+1}{d-1} q.$$

It is precisely the fact that the adjoint estimate to $L^{4/3} \to L^q$ is $L^{q'} \to L^4$ allows for a simple proof of the theorem, since 4 is an even integer and thus the restriction estimate can be restated in cancellation-free form as in (2.2) below. Indeed, one can prove the Tomas-Stein theorem in $d = 3$ just by the coarea formula (see [8]).

Remark 1.4. This article was first circulated as a preprint in 2017.[1] Since then, many authors have contributed to the problem: see [6, 9, 10, 3, 2, 4]. The main question posed by the authors of [7] has been answered by V. Kovač in [5], in which he provided a general procedure to deduce maximal Fourier restriction estimates from the corresponding Fourier restriction ones.

2. Proof of the result

We divide the proof of Theorem 1.3 in two by proving separately two propositions. First we prove

Proposition 2.1. *Let* $d \geq 3$*. The operator* $\mathscr{M}$ *satisfies*

$$\|\mathscr{M} f\|_{L^q(\Sigma,\, d\sigma)} \lesssim_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)}$$

for $1 \leq p \leq 4/3$*,* $p' \geq \frac{d+1}{d-1} q$*.*

Proof. It suffices to prove the endpoint, that is $p = 4/3$ and

$$q_d := 4 \frac{d+1}{d-1}.$$

By the Tomas-Stein theorem (see e.g. [12]) one has that for the surface Σ with non-vanishing Gaussian curvature it holds that for every $f \in L^{4/3}(\mathbb{R}^d)$

$$\|\widehat{f}\|_{L^{q_d}(\Sigma,\, d\sigma)} \lesssim \|f\|_{L^{4/3}(\mathbb{R}^d)}. \tag{2.1}$$

By duality this is equivalent to the estimate

$$\|\widehat{g\, d\sigma}\|_{L^4(\mathbb{R}^d)} \lesssim \|g\|_{L^{q_d'}(\Sigma,\, d\sigma)}.$$

[1] arXiv:1703.09495.

The numerology here is particularly fortunate since 4 is an even exponent, which allows us to multilinearise and use Plancherel to write

$$\|\widehat{g\,d\sigma}\|^2_{L^4(\mathbb{R}^d)} = \|(\widehat{g\,d\sigma})^2\|_{L^2(\mathbb{R}^d)} = \|g\,d\sigma * g\,d\sigma\|_{L^2(\mathbb{R}^d)}$$

(here of course the L^2 norm on the right hand side has to be interpreted as the operator norm of the linear operator given by $h \mapsto \langle g\,d\sigma * g\,d\sigma, h\rangle$). Thus the Tomas-Stein type estimate (2.1) can be stated equivalently in this case as

$$\|g\,d\sigma * g\,d\sigma\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|^2_{L^{q'_d}(\Sigma,\,d\sigma)},$$

which means

$$\Big|\int_\Sigma \int_\Sigma g(\omega)g(\omega')h(\omega'-\omega)\,d\sigma(\omega)\,d\sigma(\omega')\Big| \lesssim \|g\|^2_{L^{q'_d}(\Sigma,\,d\sigma)}\|h\|_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

We linearise the maximal operator $\mathscr{M}$ by defining

$$\mathscr{A}_{\epsilon(\cdot)}f(\omega) := \int_{\mathbb{R}^d} \widehat{f}(\omega+y)\chi_{\epsilon(\omega)}(y)\,dy,$$

where $\epsilon(\cdot)$ an arbitrary measurable function that takes positive values. To bound $\mathscr{M}$ it suffices to bound $\mathscr{A}_{\epsilon(\cdot)}$ in the same range independently of $\epsilon(\cdot)$. The desired inequality

$$\|\mathscr{A}_{\epsilon(\cdot)}f\|_{L^{q_d}(\Sigma,\,d\sigma)} \lesssim \|f\|_{L^{4/3}(\mathbb{R}^d)}$$

is equivalent by duality to the inequality

$$\|\mathscr{A}^*_{\epsilon(\cdot)}g\|_{L^4(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'_d}(\Sigma,\,d\sigma)},$$

where $\mathscr{A}^*_{\epsilon(\cdot)}$ is the formal adjoint of $\mathscr{A}_{\epsilon(\cdot)}$, which is given by

$$\mathscr{A}^*_{\epsilon(\cdot)}g(x) := \int_\Sigma g(\omega)e^{i\omega\cdot x}\widehat{\chi}(\epsilon(\omega)x)\,d\sigma(\omega).$$

As before, this is equivalent to establishing

$$\|\widehat{\mathscr{A}^*_{\epsilon(\cdot)}g} * \widehat{\mathscr{A}^*_{\epsilon(\cdot)}g}\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|^2_{L^{q'_d}(\Sigma,\,d\sigma)}.$$

First of all, observe that by Fubini's theorem

$$\begin{aligned}\widehat{\mathscr{A}^*_{\epsilon(\cdot)}g}(\xi) &= \int e^{-ix\cdot\xi}\int_\Sigma g(\omega)e^{i\omega\cdot x}\widehat{\chi}(\epsilon(\omega)x)\,d\sigma(\omega)\,dx \\ &= \int_\Sigma g(\omega)\int e^{-ix\cdot(\xi-\omega)}\widehat{\chi}(\epsilon(\omega)x)\,dx\,d\sigma(\omega) \\ &= \int_\Sigma g(\omega)\chi_{\epsilon(\omega)}(\xi-\omega)\,d\sigma(\omega)\end{aligned}$$

(with a little abuse of notation). Let then $h \in L^2(\mathbb{R}^d)$, so that by the above observation and multiple applications of Fubini's theorem we have the following chain of equalities:

$$
\begin{aligned}
&\langle \widehat{\mathscr{A}^*_{\epsilon(\cdot)} g} * \widehat{\mathscr{A}^*_{\epsilon(\cdot)} g}, h \rangle = \int\int \widehat{\mathscr{A}^*_{\epsilon(\cdot)} g}(\xi-\eta)\widehat{\mathscr{A}^*_{\epsilon(\cdot)} g}(\eta)\overline{h(\xi)}\, d\eta\, d\xi \\
&= \int\int\int_\Sigma\int_\Sigma g(\omega)g(\omega')\chi_{\epsilon(\omega)}(\xi-\eta-\omega)\chi_{\epsilon(\omega')}(\eta-\omega')\, d\sigma(\omega)\, d\sigma(\omega')\overline{h(\xi)}\, d\eta\, d\xi \\
&= \int\int_\Sigma\int_\Sigma g(\omega)g(\omega')\overline{h(\xi)}\Big(\int \chi_{\epsilon(\omega)}(\xi-\eta-\omega) \\
&\qquad\qquad\qquad \cdot\chi_{\epsilon(\omega')}(\eta-\omega')\, d\eta\Big)\, d\sigma(\omega)\, d\sigma(\omega')\, d\xi \\
&= \int\int_\Sigma\int_\Sigma g(\omega)g(\omega')\overline{h(\xi)}(\chi_{\epsilon(\omega)} * \chi_{\epsilon(\omega')})(\xi+\omega'-\omega)\, d\sigma(\omega)\, d\sigma(\omega')\, d\xi \\
&= \int_\Sigma\int_\Sigma g(\omega)g(\omega')\int \tilde{h}(-\xi)(\chi_{\epsilon(\omega)} * \chi_{\epsilon(\omega')})(\xi+\omega'-\omega)\, d\xi\, d\sigma(\omega)\, d\sigma(\omega') \\
&= \int_\Sigma\int_\Sigma g(\omega)g(\omega')(\tilde{h} * \chi_{\epsilon(\omega)} * \chi_{\epsilon(\omega')})(\omega'-\omega)\, d\sigma(\omega)\, d\sigma(\omega'),
\end{aligned}
$$

where $\tilde{h}(\xi) = \overline{h(-\xi)}$. But then we have that pointwise

$$|(\tilde{h} * \chi_{\epsilon(\omega)} * \chi_{\epsilon(\omega')})(\omega'-\omega)| \lesssim M^2\tilde{h}(\omega'-\omega),$$

with constant depending only on the choice of χ, where M is the Hardy-Littlewood maximal function; therefore by the Tomas-Stein restriction estimate (2.2) we have

$$
\begin{aligned}
|\langle \widehat{\mathscr{A}^*_{\epsilon(\cdot)} g} * \widehat{\mathscr{A}^*_{\epsilon(\cdot)} g}, h \rangle| &\lesssim \int_\Sigma\int_\Sigma |g(\omega)||g(\omega')|M^2\tilde{h}(\omega'-\omega)\, d\sigma(\omega)\, d\sigma(\omega') \\
&\lesssim \|g\|^2_{L^{q'_d}(\Sigma,\, d\sigma)}\|M^2\tilde{h}\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|^2_{L^{q'_d}(\Sigma,\, d\sigma)}\|h\|_{L^2(\mathbb{R}^d)},
\end{aligned}
$$

which proves the desired estimate for $\mathscr{A}_{\epsilon(\cdot)}$. ∎

Remark 2.2. It is interesting to notice that the critical endpoint for Fourier restriction to curves in $\mathbb{R}^2$ is $L^{4/3} \to L^{4/3}$, and we know that the corresponding (even restricted) strong type estimate is false by work [1] of Beckner, Carbery, Semmes and Soria. Thus the proof above barely misses the case $d = 2$.

Finally, we prove the second half of Theorem 1.3, restated below.

Proposition 2.3. *Let $1 \leq p \leq 8/7$. If $f \in L^p(\mathbb{R}^d)$ then σ-a.e. point of Σ is a Lebesgue point for $\widehat{f}$.*

Proof. The proof that follows is taken from [7] and has been included only for the reader's convenience.

Let $\mathcal{R}$ denote the operator of Fourier restriction to the hypersurface Σ. Let $\mathscr{M}^+$ denote the positive maximal Fourier restriction operator associated with Σ, defined as

$$\mathscr{M}^+ f(\omega) := \sup_{\epsilon>0} \frac{1}{\epsilon^d} \int_{|y|\le\epsilon} |\widehat{f}(\omega+y)|\, dy.$$

To prove the proposition it suffices to show that

$$\|\mathscr{M}^+ f\|_{L^q(\Sigma,\, d\sigma)} \lesssim_{p,q} \|f\|_{L^p(\mathbb{R}^d)} \tag{2.3}$$

for $1 \le p \le 8/7$ and $p' \ge q\frac{d+1}{d-1}$. Indeed, assuming this holds, one can define

$$F(\omega) := \limsup_{\epsilon\to 0} \frac{1}{\epsilon^d} \int_{|y|\le\epsilon} |\widehat{f}(\omega+y) - \mathcal{R}f(\omega)|\, dy;$$

since $\mathcal{R}\varphi = \widehat{\varphi}|_\Sigma$ for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} F(\omega) &\le \limsup_{\epsilon\to 0} \frac{1}{\epsilon^d} \int_{|y|\le\epsilon} |\widehat{f-\varphi}(\omega+y)|\, dy + |\mathcal{R}(f-\varphi)(\omega)| \\ &\le \mathscr{M}^+(f-\varphi)(\omega) + |\mathcal{R}(f-\varphi)(\omega)|. \end{aligned}$$

By the Tomas-Stein estimate and (2.3) it follows then that

$$\|F\|_{L^q(\Sigma,\, d\sigma)} \lesssim \|f-\varphi\|_{L^p(\mathbb{R}^d)}$$

in the given range, and by taking φ to be an approximant of f in L^p norm we see that $\|F\|_{L^q(\Sigma,\, d\sigma)} = 0$ or equivalently that $F = 0$ σ-a.e., which proves the proposition. Thus it suffices to prove (2.3), and in particular it suffices to prove it under the assumption that $p' = q\frac{d+1}{d-1}$. This will follow from Proposition 2.1.

Observe that by Hölder's inequality we have

$$\frac{1}{\epsilon^d} \int_{|y|\le\epsilon} |\widehat{f}(\omega+y)|\, dy \lesssim \Big(\frac{1}{\epsilon^d} \int_{|y|\le\epsilon} |\widehat{f}(\omega+y)|^2\, dy\Big)^{1/2};$$

let then $h := f * \tilde{f}$, so that

$$\widehat{h} = |\widehat{f}|^2,$$

and we have

$$\mathscr{M}^+ f \lesssim (\mathscr{M} h)^{1/2}$$

pointwise. Let s be such that $s \le 4/3$ and

$$\frac{q}{2}\frac{d+1}{d-1} = s';$$

by Proposition 2.1 we have then

$$\begin{aligned} \|\mathscr{M}^+ f\|_{L^q(\Sigma,\, d\sigma)} &\lesssim \|\mathscr{M} h\|^{1/2}_{L^{q/2}(\Sigma,\, d\sigma)} \lesssim \|h\|^{1/2}_{L^s(\mathbb{R}^d)} \\ &\le \|f\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

where $1+\frac{1}{s} = \frac{2}{p}$ and the last inequality is an application of Young's inequality. Thus it follows that

$$p' = 2s' = q\frac{d+1}{d-1},$$

as desired. Since $s \leq 4/3$, we see that we can only afford $p \leq 8/7$, and this concludes the proof. ∎

Acknowledgements.

The author is indebted to Diogo Oliveira e Silva for the clever suggestion to look at the special exponents considered in this manuscript. He is also grateful to the reviewers and the editor for their kind and useful suggestions.

References

[1] W. Beckner, A. Carbery, S. Semmes and F. Soria, *A note on restriction of the Fourier transform to spheres,* Bull. London Math. Soc. **21** (1989), 394–398.

[2] C. Bilz, *Large sets without Fourier restriction theorems,* Trans. Amer. Math. Soc. **375** (2022), 6983–7000.

[3] M. Fraccaroli, *Uniform Fourier restriction for convex curves,* preprint, arXiv:2111.06874.

[4] M. Jerusum, *Maximal operators and Fourier restriction on the moment curve,* Proc. Amer. Math. Soc. **150** (2022), 3863–3873.

[5] V. Kovač, *Fourier restriction implies maximal and variational Fourier restriction,* J. Funct. Anal. **277** (2019), 3355–3372.

[6] V. Kovač and D. Oliveira e Silva, *A variational restriction theorem,* Arch. Math. (Basel) **117** (2021), 65–78.

[7] D. Müller, F. Ricci and J. Wright, *A maximal restriction theorem and Lebesgue points of functions in $\mathcal{F}(L^p)$,* Rev. Mat. Iberoam. **35** (2019), 693–702.

[8] D. Oberlin, *A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^3$,* Proc. Amer. Math. Soc. **132** (2004), 1195–1199.

[9] J. P. G. Ramos, *Maximal restriction estimates and the maximal function of the Fourier transform,* Proc. Amer. Math. Soc. **148** (2020), 1131–1138.

[10] J. P. G. Ramos, *Low-dimensional maximal restriction principles for the Fourier transform,* Indiana Univ. Math. J. **71** (2022), 339–357.

[11] P. Sjölin, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$,* Studia Math. **51** (1974), 169–182.

[12] E. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, 1993.

M. Vitturi
School of Mathematical Sciences, University College Cork
Western Gateway Building, Western Road, Cork, Ireland
E-mail: marco.vitturi@ucc.ie

Received: 4.10.2022.
Revised: 10.10.2022.