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Numerical approach to King's analytical study for circular loop antenna

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SUMMARY

A numerical technique based on the Pocklington's equation and the method of moments for the study of circular loop antennas is exhibited. The current distributions thus obtained are in total harmony with the corresponding analytical results of King.

Key words: *circular loop antenna, method of moments, Pocklington's equation.*

1. INTRODUCTION

The circular loop antenna consists of one or more turns of highly conducting wire around a circular shaped frame. Such antenna has its most important application for reception although it is also useful in transmission operation mode. The most common applications for this antenna include the measurement of magnetic fields and distributions of currents, and when its dimensions are electrically small the antenna is used as a valuable tool in direction finding.

The present paper shows a comparison between the analytical solution [1] for the current distribution around the antenna and a numerical solution based upon the method of moments and the Pocklington's equation [2-4]. It will be shown that the numerical solution provides a reliable and faster solution which produces excellent results compared with those of the analytical one. Such comparison shows the goodness of using the method of moments instead of the Fourier method both in the mathematical formulation and the computational procedure. Also, an objective expected in this paper is to show that the numerical solution is as good for linear structures as for curve structures which implies that the mathematical model used is suitable for arbitrary geometry wires.

2. ANALYTICAL SOLUTION FOR CIRCULAR LOOP ANTENNAS

The antenna to be analyzed is shown in Figure 1. It is built of a circular ring of perfectly conducting wire which is fed by a delta gap generator $\delta(\varphi)$ at $\varphi=0$. The radius of the ring is A, that of the wire a. In this analysis it is assumed that the antenna dimensions are electrically small compared with the wavelength in the medium so that:

$$a^2 \ll A^2, \quad |ka| \ll 1 \tag{1}$$

where $k=\beta-j\alpha$ is the complex propagation constant of the medium; when the medium is air, $\alpha=0$, $\beta=\omega\sqrt{\mu_0\varepsilon_0}=2\pi/\lambda$. In this paper the analysis will be carried out for air.

The boundary condition $E_\varphi=-V\delta(\varphi)/A$ around the antenna allows the formulation of the integral equation for the current $I(\varphi)$. The delta gap generator makes possible that the electric field exists only at $\varphi=0$ where it becomes infinite, and $E_\varphi=0$ elsewhere. The electric field is related with the generator's voltage V in such a way that:

$$\int_{-\pi}^{\pi} E_\varphi A d\varphi = -V \tag{2}$$

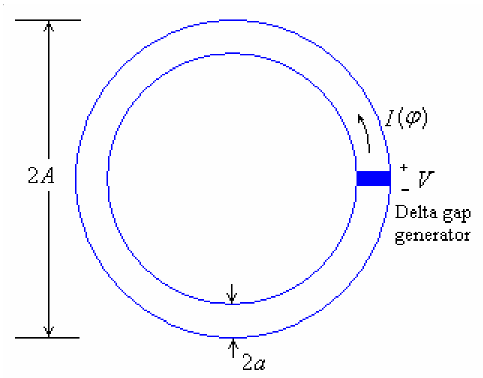

Fig. 1 Circular loop antenna

For getting the integral equation we make use of the electromagnetic potentials $\boldsymbol{A}$ and Φ on the surface of the wire:

$$\frac{V\delta(\varphi)}{\boldsymbol{A}} = \frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} j\omega\boldsymbol{A}_\varphi \tag{3}$$

The scalar and vector potentials are defined as follows:

$$\begin{aligned} \Phi &= \frac{1}{4\pi\varepsilon_0}\int_{-\pi}^{\pi} q(\varphi')W(\varphi-\varphi')d\varphi' \\ \boldsymbol{A}_\varphi &= \frac{\mu_0}{4\pi}\int_{-\pi}^{\pi} I(\varphi')W(\varphi-\varphi')\cos(\varphi-\varphi')d\varphi' \end{aligned} \tag{4}$$

where the kernel is:

$$\begin{aligned} W(\varphi-\varphi') &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkAR}}{R}d\psi \\ R &= 2\sqrt{\sin^2\left(\frac{\varphi-\varphi'}{2}\right)+\left(\frac{a}{A}\right)^2\sin^2\left(\frac{\psi}{2}\right)} \end{aligned} \tag{5}$$

The relation between the current $I(\varphi)$ and the charge $q(\varphi)$ is expressed by the equation of continuity:

$$\frac{1}{A}\frac{\partial I(\varphi)}{\partial\varphi} + j\omega q(\varphi) = 0 \tag{6}$$

which yields the following integral equation for the current in the loop:

$$\begin{aligned} V\delta(\varphi) &= \frac{j\zeta_0}{4\pi}\int_{-\pi}^{\pi}\left[kA\cos(\varphi-\varphi')+\frac{1}{kA}\frac{\partial^2}{\partial\varphi^2}\right]\cdot \\ &\cdot W(\varphi-\varphi')I(\varphi')d\varphi', \quad \zeta_0=\sqrt{\mu_0/\varepsilon_0} \end{aligned} \tag{7}$$

The solution for this equation is obtained by expanding both the kernel and the current in its Fourier series:

$$\begin{aligned} W(\varphi-\varphi') &= \sum_{-\infty}^{\infty} K_m e^{-jm(\varphi-\varphi')} \\ K_{\pm m} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\varphi-\varphi')e^{jm(\varphi-\varphi')}d\varphi \\ I(\varphi') &= \sum_{-\infty}^{\infty} I_n e^{-jn\varphi'} \\ I_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} I(\varphi')e^{jn\varphi'}d\varphi' \end{aligned} \tag{8}$$

The substitution of Eqs. (8) into Eq. (7) leaves the integral equation in the form:

$$\begin{aligned} V\delta(\varphi) &= \frac{j\zeta_0}{4\pi}\sum_{-\infty}^{\infty} a_n\int_{-\pi}^{\pi} e^{-jn(\varphi-\varphi')}I(\varphi')d\varphi' = \\ &= \frac{j\zeta_0}{2}\sum_{-\infty}^{\infty} a_n I_n e^{-jn\varphi} \\ a_{\pm n} &= \frac{kA}{2}(K_{n+1}+K_{n-1}) - \frac{n^2}{kA}K_n \end{aligned} \tag{9}$$

which is a Fourier series with the coefficients $j\zeta_0 a_n I_n/2$ which are evaluated in the common way:

$$\begin{aligned} \frac{j\zeta_0 a_n I_n}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} V\delta(\varphi)e^{jn\varphi}d\varphi = \frac{V}{2\pi} \\ I_n &= \frac{-jV}{\zeta_0\pi a_n} \end{aligned} \tag{10}$$

therefore the desired solution for the current distribution is:

$$I(\varphi) = \frac{-jV}{\zeta_0\pi}\left(\frac{1}{a_0}+2\sum_{1}^{\infty}\frac{\cos n\varphi}{a_n}\right) \tag{11}$$

In spite of the simple form in the solution of the current distribution, the evaluation of the coefficients a_n entails mathematical difficulties. There have been researchers interested in obtaining reliable approximations for K_n which is the major difficulty in the analysis of the circular loop antenna; among them the most important are Hallén [5], Storer [6], Wu [7] and Watson [8]. The last two have created a procedure which produces good results for a great range of the parameters involved. The results are the following:

$$\begin{aligned} K_0 &= \frac{1}{\pi}\ln\frac{8A}{a} - \frac{1}{2}\int_0^{2kA}\left[\Omega_0(x)+jJ_0(x)\right]dx \\ K_{\pm n} &= \frac{1}{\pi}\left[\mathcal{K}_0\left(\frac{na}{A}\right)I_0\left(\frac{na}{A}\right)+C_n\right] - \\ &-\frac{1}{2}\int_0^{2kA}\left[\Omega_{2n}(x)+jJ_{2n}(x)\right]dx \end{aligned} \tag{12}$$

where Ω_m are the Lommel-Weber functions:

$$\Omega_m(x)=\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta-m\theta)d\theta \tag{13}$$

and J_{2n} are the Bessel functions, I_0 and $\mathcal{K}_0$ are the modified Bessel functions of the first and second kind, respectively, and C_n are defined as:

$$C_n=\gamma-2\sum_{m=0}^{n-1}(2m+1)^{-1}+\ln(4n) \tag{14}$$

where $\gamma=0.5772...$ is the Euler-Mascheroni constant. Although the solution provides an analytical expression, the evaluation of each K_n involves the calculation of several functions, integrals and sums which leads to computational difficulties, which are increased by the number of terms in the expansion (11). In this way the numerical solution shown next provides a simpler way for finding the current distribution with less computational effort.

3. NUMERICAL SOLUTION FOR THE CIRCULAR LOOP ANTENNA

The numerical solution is based upon the well known Pocklington's equation [2-4] for wires with arbitrary geometry:

$$E_S^I=-\frac{j}{\omega\varepsilon}\int_{s'}I(s')\left[k^2\mathbf{s}\bullet\mathbf{s}'+\frac{\partial^2}{\partial s\partial s'}\right]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds' \tag{15}$$

where E_S^I is the tangential impressed electric field on the wire's surface and s, s', are the arc length along the axis and current filament, respectively. The wire's geometry is expressed by the dot product $\mathbf{s}\bullet\mathbf{s}'$, where $\mathbf{s}(s)$ is the unit tangential vector for the wire's axis and $\mathbf{s}'(s')$ is the unit tangential vector for the parallel curve which represents the current filament, as it is shown in Figure 2. The equations which describe the antenna geometry are the following:

$$\begin{aligned}
\mathbf{r}(s)&=A\cos\left(\frac{s}{A}\right)\mathbf{i}+A\sin\left(\frac{s}{A}\right)\mathbf{j}+a\mathbf{k}\\
\mathbf{r}'(s')&=A\cos\left(\frac{s'}{A}\right)\mathbf{i}+A\sin\left(\frac{s'}{A}\right)\mathbf{j}+a\mathbf{k}\\
\mathbf{s}(s)&=-\sin\left(\frac{s}{A}\right)\mathbf{i}+\cos\left(\frac{s}{A}\right)\mathbf{j}\\
\mathbf{s}'(s')&=-\sin\left(\frac{s'}{A}\right)\mathbf{i}+\cos\left(\frac{s'}{A}\right)\mathbf{j}
\end{aligned} \tag{16}$$

The numerical solution consists in the application of the method of moments which expands the current as a series of N basis functions, where the coefficients of such a series must be determined:

$$I(s')=\sum_{n=1}^{N}c_nI_n(s') \tag{17}$$

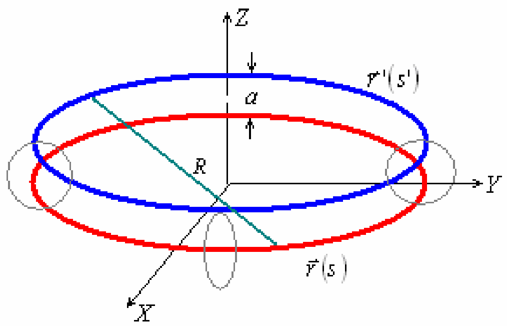

Fig. 2 Circular loop antenna and its associated vectors

By substituting Eq. (17) into Eq. (15) it results in one equation with N unknowns:

$$E_S^I=-\frac{j}{\omega\varepsilon}\sum_{n=1}^{N}c_n\int_{s'}I_n(s')\left[k^2\mathbf{s}\bullet\mathbf{s}'+\frac{\partial^2}{\partial s\partial s'}\right]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds' \tag{18}$$

The coefficients c_n are obtained by building a set of N linear equations from Eq. (18) by taking the inner product with a set of N weighting functions $w_m(s)$:

$$\begin{aligned}
&\int_s w_mE_S^Ids=-\frac{j}{\omega\varepsilon}\sum_{n=1}^{N}c_n\int_s w_m\int_{s'}I_n(s')\cdot\\
&\qquad\cdot\left[k^2\mathbf{s}\bullet\mathbf{s}'+\frac{\partial^2}{\partial s\partial s'}\right]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds'\,ds\\
&m=1,2,\cdots,N
\end{aligned} \tag{19}$$

The set of linear equations could be expressed in matrix form as follows:

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N}\\ Z_{21} & Z_{22} & \cdots & Z_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ Z_{N1} & Z_{N2} & \cdots & Z_{NN}\end{bmatrix}\begin{pmatrix}c_1\\c_2\\\vdots\\c_N\end{pmatrix}=\begin{pmatrix}v_1\\v_2\\\vdots\\v_N\end{pmatrix} \tag{20}$$

$$[Z_{mn}](c_n)=(v_m)$$

where the matrix's elements are expressed as:

$$\begin{aligned}
Z_{mn}&=-\frac{j}{\omega\varepsilon}\int_s w_m\int_{s'}I_n(s')\left[k^2\mathbf{s}\bullet\mathbf{s}'+\frac{\partial^2}{\partial s\partial s'}\right]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds'ds\\
v_m&=\int_s w_mE_S^Ids
\end{aligned} \tag{21}$$

The basis functions used in this paper are pulse functions and the employed weighting functions are Dirac's delta functions, in order to model a delta gap generator which fed the antenna. In this way the matrix elements (21) become:

$$\begin{aligned}
Z_{mn}&=-\frac{j}{\omega\varepsilon}\int_{s'}\left[k^2\mathbf{s}\bullet\mathbf{s}'+\frac{\partial^2}{\partial s\partial s'}\right]\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}ds'\Bigg|_{s=s_m}\\
v_m&=\begin{cases}V/\Delta & s=s_m\\ 0 & \text{elsewhere}\end{cases}
\end{aligned} \tag{22}$$

where Δ is the length of a segment in which the antenna is divided. The numerical solution for the current is given then by:

$$(c_n) = [Z_{mn}]^{-1}(v_m) \tag{23}$$

4. NUMERICAL RESULTS

The method of moments procedure was programmed in order to reproduce the results obtained by King [1] for several circular antennas. The results obtained agree with those shown by him. The following plots, Figures 3 to 8, represent the real and imaginary parts for the current distribution along the length of the antenna. The antennas were fed by a delta gap generator with *1 V* amplitude which is working at *1 GHz*. Each antenna was divided into 149 segments. The wire's radius of each antenna is determined by the relation:

$$\Omega = 2\,ln\left(\frac{2\pi A}{a}\right) \tag{24}$$

with $\Omega=10$ for each one.

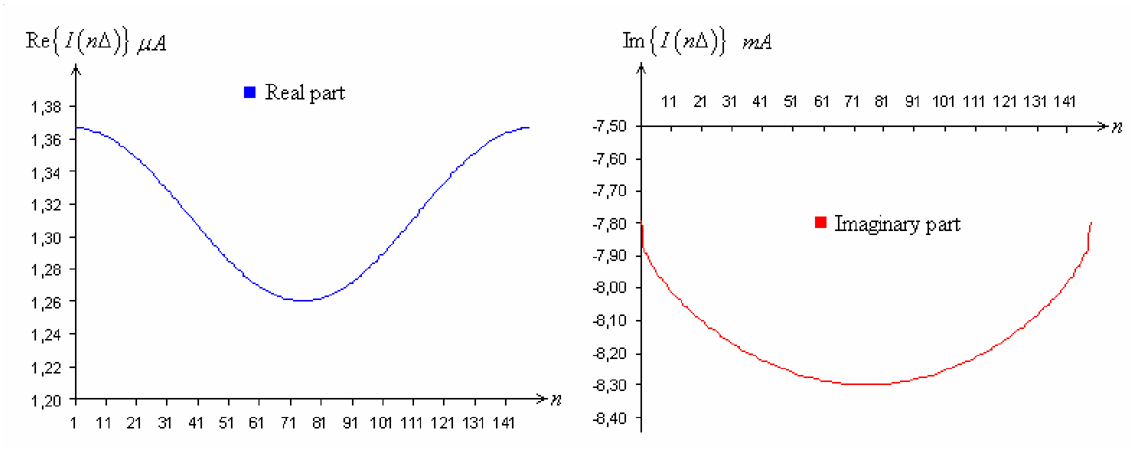

Fig. 3 Current distribution for kA=0.1

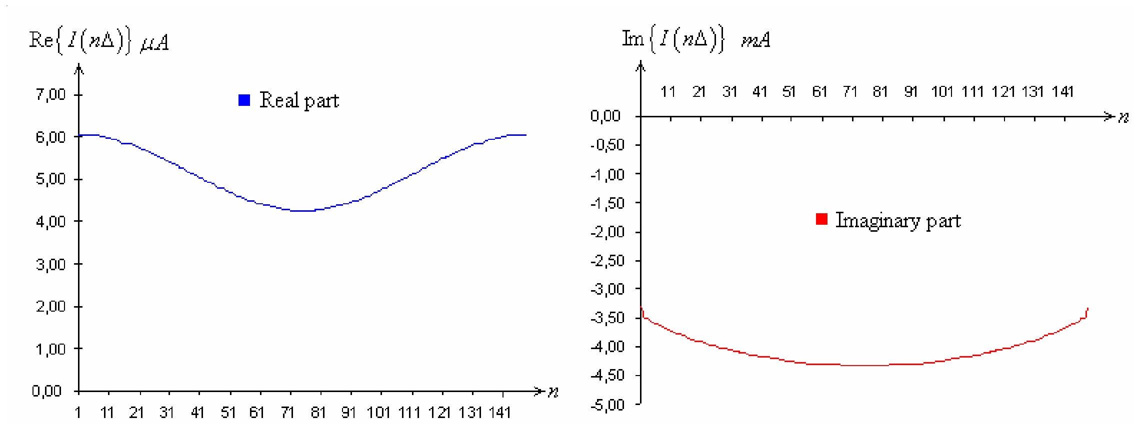

Fig. 4 Current distribution for kA=0.2

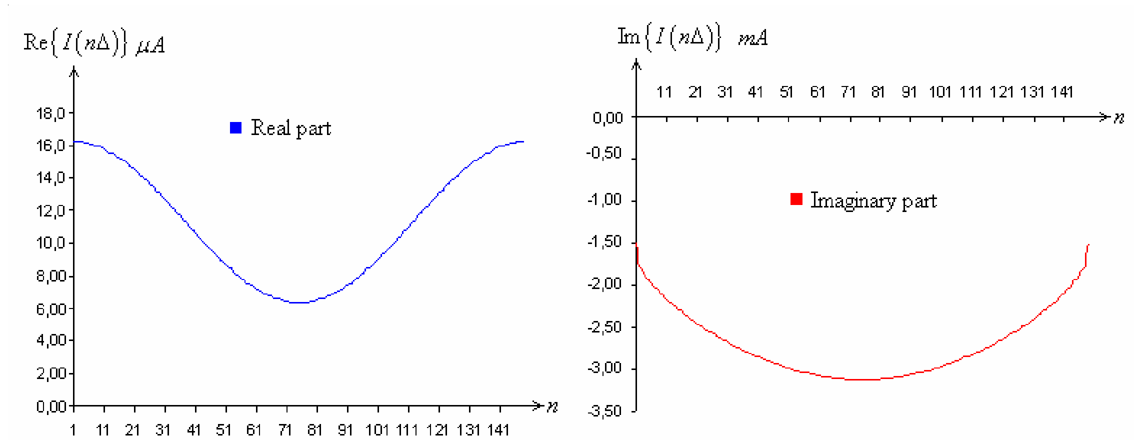

Fig. 5 Current distribution for kA=0.3

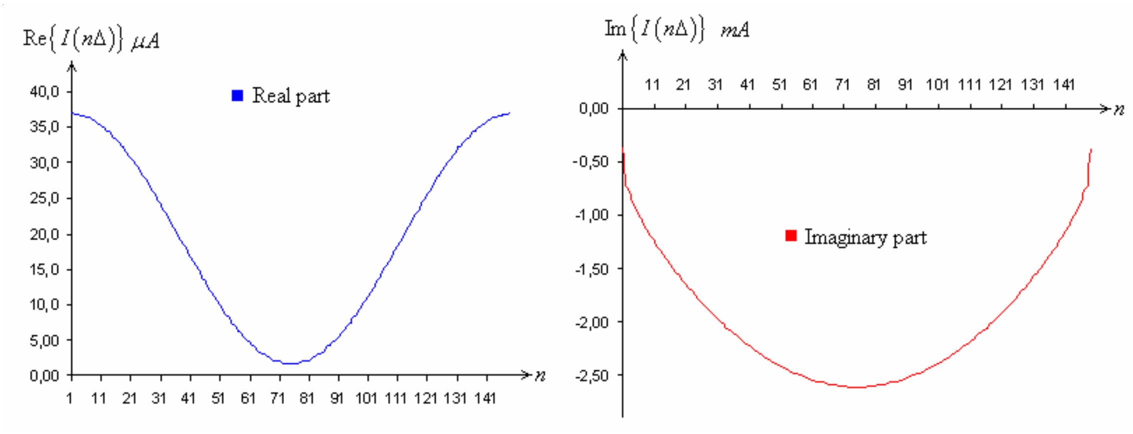

Fig. 6 Current distribution for kA=0.4

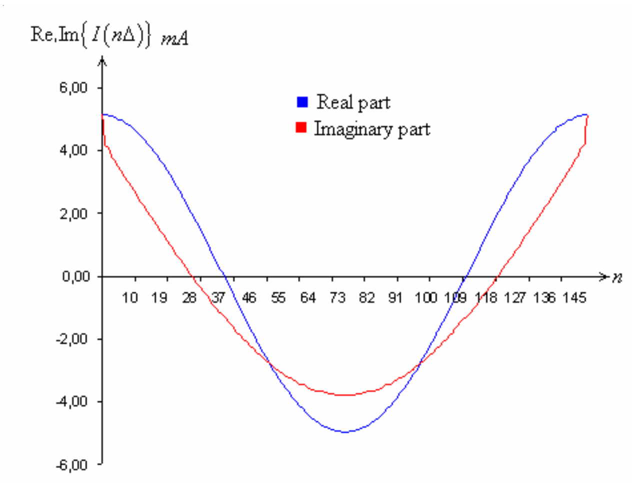

Fig. 7 Current distribution for kA=1

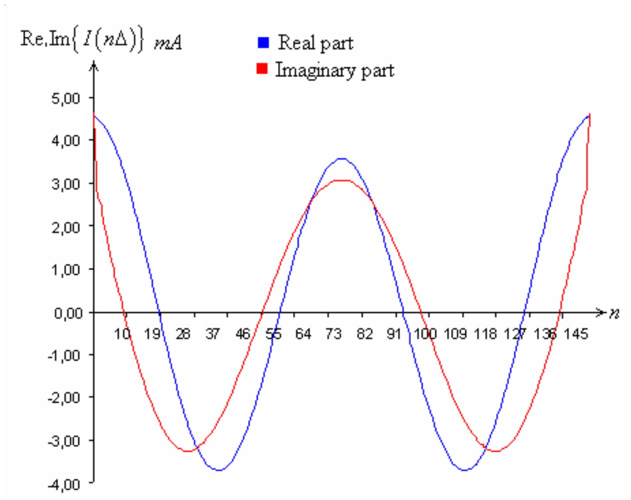

Fig. 8 Current distribution for kA=2

The comparison between the analytical and these numerical results can be done in the original results given by King in Ref. [1] on p. 468-469, Figures 11.4 a,b. The results presented here agree correctly with those presented by him.

5. CONCLUSIONS

Although the analytical analysis provides a mathematical tool for the electrical behaviour in the circular loop antenna, it has shown to be difficult for developing its algorithm computationally. Mainly it is due to the intricate shape for the coefficients K_n which involves the calculation of several Bessel functions in its different forms. However it is necessary to say that although Eq. (12) provides a mathematical formula for each K_n, such a formula still is an approximation for the correct value of each coefficient.

The numerical choice presented here solves the complexity associated with King's algorithm in a way that Pocklington's equation represents correctly the circular loop antenna. The method of moments brings a suitable solution for the integral equation which models the current distribution in the antenna, providing a matrix solution which can be easily programmed with the aid of the standard algorithms for sets of linear equations. Although the obtained solution is an approximation for the current distribution, its exactness can be increased by the way the number N of segments is increased.

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NUMERIČKI PRISTUP KING-OVOJ ANALITIČKOJ STUDIJI ZA KRUŽNU ANTENU

SAŽETAK

U ovom radu prikazana je numerička tehnika za proučavanje kružnih antena koja se temelji na Pocklington-ovoj jednadžbi i metodi momenata. Pokazano je da se tako dobivena distribucija struje u potpunosti slaže s odgovarajućim analitičkim rezultatima koje je postavio King.

Ključne riječi*: kružna antena, metoda momenata, Pocklingtonova jednadžba.*