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On the evolution of overtaking collision of solitons

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SUMMARY

The classical problem of two solitons with different amplitudes moving along the same direction is reinvestigated in this paper. The well-known solution, which has been investigated by many studies, is that, after the interaction, the two solitons will gradually regain their original forms with the phase shifts respectively. Besides two regimes which classify the collision types according to the number of peaks while interaction occurs strongly, we present the third regime for describing the interaction more exactly and completely.

Key words: *overtaking collision, problem of two solitons, collision of solitons, coastal engineering, KdV equation, peak regimes.*

1. INTRODUCTION

In 1895, Korteweg and de Vries [1] derived a classical and important equation, i.e. the so-called KdV equation, for describing the nonlinear waves propagating along the same direction. Among the wave motions governed by the KdV equation, the interaction of two unidirectional solitons is most frequently investigated not only in physics, but in coastal engineering. This problem was first numerically solved by Zabusky and Kruskal [2] who concluded that the two solitons after collision will gradually recover their original wave forms respectively except for a forward phase shift for the larger (faster) soliton and a backward phase shift for the smaller (slower) one. Gardner et al. [3] provided an analytical method to solve the initial-value problem of the KdV equation, especially for the case of soliton motions. Hirota [4] also gave the exact solution of the KdV equation for multiple-soliton collisions and his results were discussed in detail by Whitham [5]. More recently, Wu and Zhang [6] elucidated the transient rates of mass and energy transfer between two unidirectional solitons throughout their overtaking interaction. They also analytically determined the criterion separating the single-peak and double-peak regimes which was also noted by many researchers, for example, Zabusky and Kruskal.

In our present study, we will stress on the number of peaks of the entire wave form while strong interaction occurs. Based on the transform method derived by Hirota, we found that, for describing the peak number precisely, three regimes, which include the 2-peak, the 2-1-2-peak and the 2-1-2-1-2-peak regimes, are necessary to classify the collision types. This classification is different from that concluded by most previous investigations.

2. MATHEMATICAL FORMULATION

We start our reinvestigation with the *KdV* equation:

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0 \tag{1}$$

The well-known solution of Eq. (1), the *sech*^{2} type soliton, can be obtained by introducing the following logarithmic transform:

$$\eta = 2\frac{\partial^2}{\partial x^2} \ln F \tag{2}$$

Inserting Eq. (2) into Eq. (1) yields a nonlinear differential equation:

$$F_{xt}F - F_xF_t + F_{xxxx}F - 4F_{xxx}F_x + 3\left(F_{xx}\right)^2 = 0 \tag{3}$$

The solution of Eq. (3) was first found by Hirota and takes a concise form $F=1+exp(\varsigma)$ where $\varsigma = ax - a^3t + \theta$. Obviously, this solution exactly describes a single soliton propagating along the positive *x*-axis. Hirota also derived the solution for the case of unidirectional multisolitons. The enhanced form of *F* for the case of *N* solitons is as follows:

$$F = \sum_{\mu=0,1} exp\left(\sum_{j=1}^{N} \mu_j \varsigma_j + \sum_{j>k}^{N} \mu_j \mu_k A_{jk}\right) \tag{4}$$

with:

$$\varsigma_i = a_i x - a_i^3 t + \theta_i \tag{5}$$

$$exp\left(A_{jk}\right) = \left(\frac{a_j - a_k}{a_j + a_k}\right)^2 \tag{6}$$

where $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1=0,1,\ \mu_2=0,1,\ ...,\ \mu_N=0,1$ and $\sum_{j>k}^{N}$ means the summation over all possible combinations of N elements under the condition $j>k$. Since our goal is to analyze the two-soliton behaviors, one can simplify and rewrite Eq. (4) as:

$$F = 1 + exp\left(\varsigma_1\right) + exp\left(\varsigma_2\right) + \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 exp\left(\varsigma_1 + \varsigma_2\right) \tag{7}$$

where $a_1>a_2$ is assumed. The values of $a_i^2/2$, a_i^2 and θ_i indicate the wave amplitude, the wave speed and the initial phase of the *i-th* soliton without any interactions with other solitons, respectively. For convenient descriptions of real wave properties, we denote the original amplitude of the *i-th* soliton by α_i which is equal to $a_i^2/2$. According to Eq. (7), the position of the peak of each soliton before and after the interaction is at:

$$\begin{cases} a_1 x = a_1^3 t - \theta_1 & for\ \varsigma_1 \\ a_2 x = a_2^3 t - \theta_2 + 2\ln\left(\dfrac{a_1 + a_2}{a_1 - a_2}\right) & for\ \varsigma_2 \end{cases} \quad as\ t \to \infty \tag{8}$$

and:

$$\begin{cases} a_1 x = a_1^3 t - \theta_1 + 2\ln\left(\dfrac{a_1 + a_2}{a_1 - a_2}\right) & for\ \varsigma_1 \\ a_2 x = a_2^3 t - \theta_2 & for\ \varsigma_2 \end{cases} \quad as\ t \to \infty \tag{9}$$

Equations (8) and (9) imply that, after the larger soliton overtakes the smaller one, the phase of the larger soliton will be pushed forward and the smaller soliton will have a phase lag. Since the phase shifts after interaction have been solved, assigning the suitable values to θ_1 and θ_2 is crucial to observe the variation of the wave profile theoretically. Therefore, we choose:

$$\theta_1 = \theta_2 = \ln\left(\frac{a_1 + a_2}{a_1 - a_2}\right) \tag{10}$$

in order to make the wave profile a fore-and-aft symmetry with respect to $x=0$ and $t=0$, namely, $\eta(x,t)=\eta(-x,-t)$. Thus, the wave profile is readily obtained by inserting all components of *F* into Eq. (2). The result is:

$$\eta = 2 \cdot \frac{F_{xx}F - F_x^2}{F^2} \tag{11}$$

3. DISCUSSIONS OF THE COLLISION TYPES

In this section, we focus on the evolution of the wave profile and the collision types. Since we have chosen suitable values of initial phases to make the wave motion symmetrical with respect to $x=t=0$, the elevation at $x=0$ and $t=0$, namely at the center plane, will be first examined. The elevation at the center plane can be directly obtained by setting both *x* and *t* in Eq. (11) to be zero and the result is:

$$\eta\left(x=0,\ t=0\right) = \alpha_1 - \alpha_2 \tag{12}$$

Equation (12) implies that a run-down phenomenon takes place more obviously as the wave amplitudes of solitons are much closer to each other. Besides the run-down phenomenon occurs at the center plane, it is of interest to observe the number of peaks during the duration of the strong interaction. By differentiating Eq. (11) twice with respect to *x*, it gives:

$$\eta_{xx}\left(x=0,\ t=0\right) = -\left(\alpha_1 - \alpha_2\right)\cdot\left(\alpha_1 - 3\alpha_2\right) \tag{13}$$

Equation (13) infers the concavity of the wave profile at the strongest interaction point. The negative value of Eq. (13), which leads the profile to maintain a single-peak status, will happen under the condition $0<1/3<R$. It means that the larger soliton will merge the smaller soliton to a single soliton at $t=0$. On the other side, for the positive value of Eq. (13), i.e. under the condition of $1/3<R<1$, the wave profile will maintain two peaks at $t=0$. That is to say that the larger and the smaller solitons will gradually shrink and grow respectively and the amplitudes finally become identical at $t=0$. Naturally, the case of $R=1/3$ indicates the critical status between the single-peak and the double-peak regimes at $t=0$. Up to now, this consequence is deemed an important principle to classify the collision types into two regimes. The wave profiles with various amplitude ratios $R(=\alpha_2/\alpha_1)$ at $t=0$ are plotted in Figure 1. It is noted again that the above result is obtained merely for a specific time, $t=0$. In other words, the further investigation is still needed to determine whether the classification can be applied to the whole duration of interaction or not.

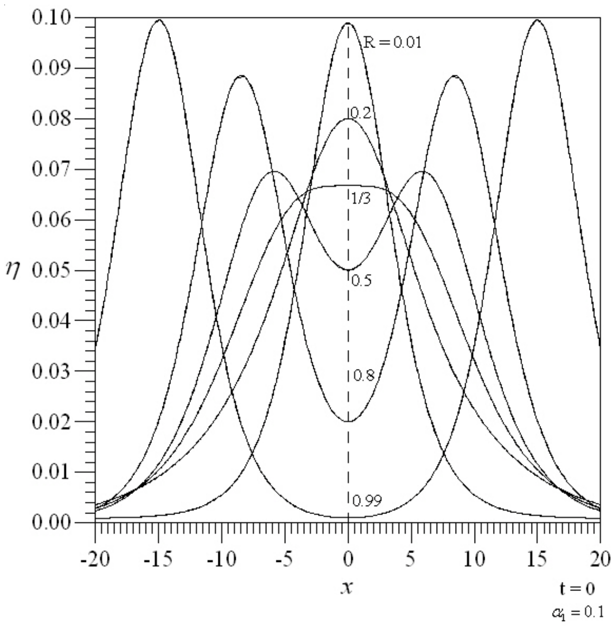

Fig. 1 The wave profiles at t=0 with different amplitude ratios (α_1=0.1)

Therefore, we start to observe the wave profile at the interval which is close to $t=0$. Consider the following equation:

$$F_{xxx}F^2-3F_{xx}F_xF+2F_x^3=0 \tag{14}$$

The left side of Eq. (14) is the numerator of the differentiation of Eq. (11) with respect to x. The number of solutions of Eq. (14) will determine the number of wave peaks. Since F is a function of x and t, we can solve Eq. (14) numerically for any specific time to obtain the number of solutions. Three solutions and one solution of Eq. (14) indicate the wave profiles with two peaks and one peak respectively. Certainly, the critical case occurs while there exist two solutions. Figure 2 shows that, by taking the number of solutions into account, the collision types are then divided into three regimes which are slightly different from the conclusion shown in Figure 1 that there are only two regimes at $t=0$. The shadow and the white areas indicate the single-peak and the two-peak states respectively. As the amplitude ratio is smaller than 1/3, the larger soliton will gradually catch up with the smaller one and merge to a single soliton during the strongest interaction. The smaller amplitude ratio leads to the longer duration of the single peak status. This regime is named the 2-1-2-peak regime. As $R>0.383$, the wave profile always keeps a two-peak status throughout the collision. This is so-called the 2-peak regime. The third regime, to author's best knowledge, which has never been pointed out, will appear under the condition $1/3<R<0.383$. This regime is the 2-1-2-1-2-peak regime that the single peak status emerges twice throughout the interaction. Actually, if one observes the center plane shown in Figure 2, the result will be completely the same as that of Figure 1, namely it needs two regimes to classify the collision types.

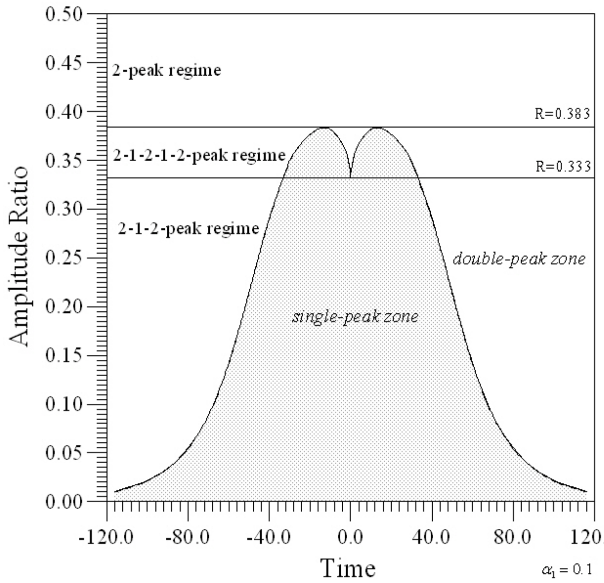

Fig. 2 Three regimes for classifying the collision types

Figures 3, 4 and 5 show the evolutions of these three regimes with the amplitude ratio *0.2*, *0.35* and *0.5*, respectively.

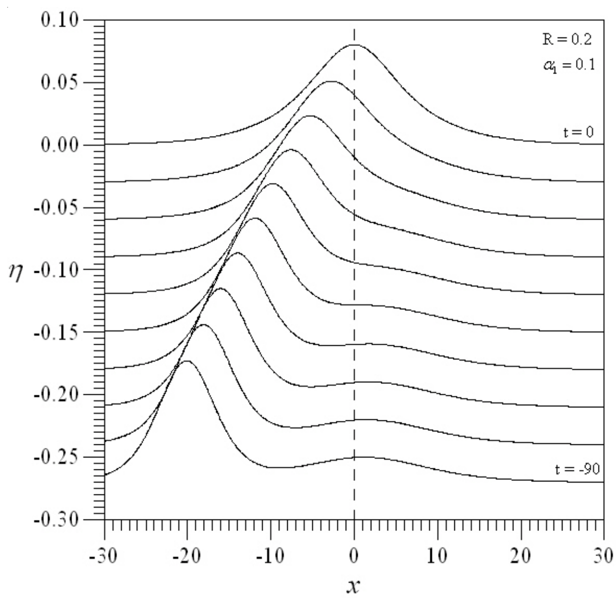

Fig. 3 The wave evolution of the 2-1-2-peak regime (R=0.2)

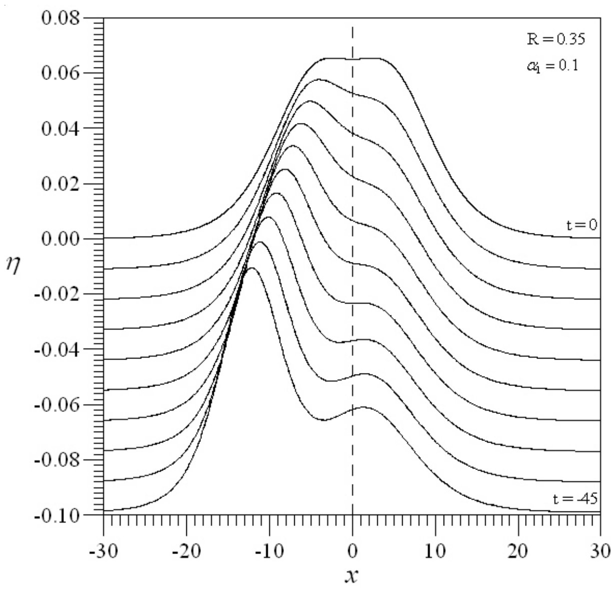

Fig. 4 The wave evolution of the 2-1-2-1-2-peak regime (R=0.35)

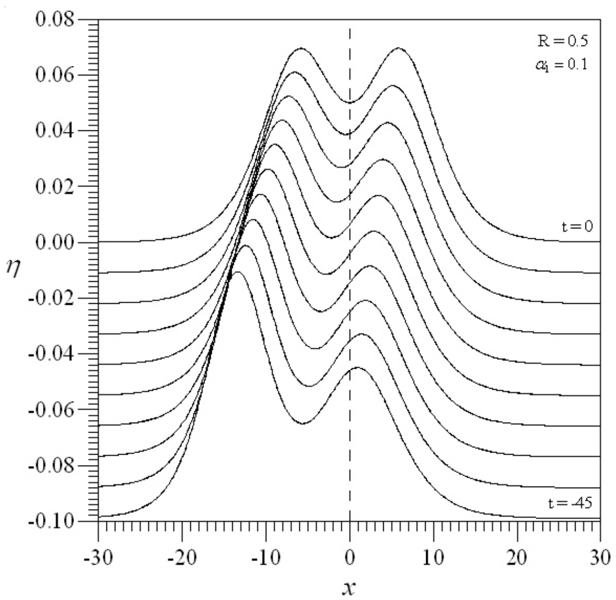

Fig. 5 The wave evolution of the 2-peak regime (R=0.5)

4. CONCLUDING REMARKS

We reinvestigate the classical overtaking collision of two solitons moving along the same direction. The well-known physical phenomenon is that the larger soliton will overtake the smaller one and two solitons will eventually recover their original wave forms. In the most previous studies, the evolution of the wave profile was classified into two regimes according to the number of peaks at $t=0$. Though it is correct at that point, however, a more detailed classification is provided in our present study. Observing the wave evolution at the neighbouring time interval of $t=0$, one needs three regimes, which include the 2-peak, 2-1-2-peak and 2-1-2-1-2-peak regimes, to precisely classify the collision types. The critical amplitude ratios between these regimes are 1/3 and 0.383.

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DOPRINOS RAZVOJU PREKLAPANJA VELIKIH VALOVA

SAŽETAK

Ovaj članak nastavlja istraživanje klasičnog problema dvaju velikih valova različitih amplituda koji se kreću u istom smjeru. Dobro poznato rješenje a koje je analizirano u mnogim studijama, je da će nakon interakcije, ta dva velika vala postepeno poprimati izvorne oblike s faznim pomakom. Osim dva režima koji klasificiraju vrste sudara prema broju vrhova pri izraženoj interakciji, pretpostavljamo i treći režim za točnije i potpunije opisivanje te interakcija.

Ključne riječi: *preklapanje, problem dva velika vala, sudar velikih valova, obalno inženjerstvo, KdV jednadžba, vršni režimi.*